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ON PERTURBATIONS OF FRAMES IN HILBERT SPACES

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Abstract: Perturbation of frames in Hilbert spaces by a non-zero element has been considered. Examples has been given to establish that such a perturbation need not be a frame. It has been proved that the perturbation of a near exact frame by a non-zero element is not a frame. Also perturbation of a frame $\{x_n\}$ of the type $\{x_n + c_n x_0\}$, where x_0 is a non-zero element and $\{c_n\}$ is any sequence of scalars has been considered and a sufficient condition for this type of perturbation to be a frame has been given. Finally, perturbation of a frame by a finite set of linearly independent elements has been considered and a sufficient condition for the same has been given.

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1. Introduction

Frames are main tool for use in signal processing, image processing, data compression, etc. In 1946, Dannis Gabor formulated a fundamental approach to signal decomposition in terms of elementary signals. On the basis of this devel-

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opment, in 1952, Duffin and Schacffer [5] introduced frames for Hilbert spaces to study some deep problem in non harmonic Fourier series. In fact, they abstracted the fundamental notion of Gabor for studying signal processing. The idea of Duffin and Schacffer did not generate much interest outside non harmonic Fourier series. But after the landmark paper of Daubechies, Grossmann and Meyer [4], in 1986, the theory of frames began to be more widely studied.

Holub [10], [11] studied perturbation of bases in Banach spaces. Christensen [3] studied perturbation of frames and obtained Paylay-wiener type stability conditions. Favier and Zalik [7] observed that a frame is stable under small perturbation Stability and perturbation of frames was further studied in [1], [2], [7], [8], [9], [12].

In the present paper, we considered perturbation of frames by a non-zero element and gave examples to show that such a perturbation need not be a frame. It has been proved that the perturbation of a near exact frame by a non-zero element is not a frame. Perturbation of the type $\{x_n + c_n x_0\}$, where $\{x_n\}$ is a frame for H, $\{c_n\}$ is any sequence of scalars and x_0 is a non-zero element in H, has also been considered and a sufficient condition for this type of perturbation to be a frame for H has been obtained. Finally, perturbation of a frame by a finite set of linearly independent elements has been considered and a sufficient condition for the same has been obtained.

2. Basic Notation and Definitions

Through out the paper E will denote an infinite dimensional Hilbert space.

Definition 2.1. A sequence $\{f_n\} \subset H$ is called a frame for H if there exists constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad f \in H\,. \tag{2.1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds of the frame $\{f_n\}$. The inequality (2.1) is called the *frame inequality.*

The frame $\{f_n\} \subset H$ is called *tight* if it is possible to choose A, B satisfying inequality (2.1) with $A = B$ and is called *normalized tight* if $A = B = 1$. The frame $\{f_n\} \subset H$ is called *exact* if removal of one f_n renders the collection $\{f_n\}$ no longer a frame for H. The frame $\{f_n\} \subset H$ is called *near exact* if it can be made exact by removing finitely many elements from it.

3. Main Results

Let $\{x_n\}$ be a frame for H and let x_0 be a non-zero element in H. Then the perturbed sequence $\{x_n + x_0\}$ is not a frame for H. In fact in some cases it is not even a Bessel sequence and in some cases it even losses completeness (Example 3.1).

Example 3.1. Let $\{x_n\}$ be the sequence of unit vector in H.

(a) Note that $x_1 = (1, 0, 0, \ldots) \neq 0$ and the perturbed sequence $\{x_n + x_1\}$ is complete in H. But since $\left\{\sum_{n=1}^{k} |\langle x_1, x_n + x_1\rangle|^2\right\}_k$ is not bounded above, $\{x_n + x_1\}$ is not a frame for H.

(b) For $x_0 = -x_1 \neq 0$, the perturbed sequence $\{x_n + x_0\}$ is not complete in H. Further, since $\left\{\sum_{n=1}^{k} |\langle x_0, x_n + x_0\rangle|^2\right\}_k$ is not bounded above, $\{x_n + x_0\}$ is not a frame for H.

In view of Example 3.1, it is natural to ask for conditions on the frame $\{x_n\}$ under which the perturbed sequence $\{x_n + x_0\}$ is not a frame for any non-zero x_0. The following theorem answers this query.

Theorem 3.2. *Let $\{x_n\}$ be a near exact frame for H. Then there exists no non-zero element x_0 in H such that $\{x_n + x_0\}$ is a frame for H.*

Proof. Let $\{c_n\}$ be a sequence of scalars such that $x_0 = \sum_{n=1}^{\infty} c_n x_n$. Since $x_0 \neq 0$, there exists some $c_n \neq 0$. Let $c_1 \neq 0$. If $x_1 \notin [x_n]_{n\neq 1}$, the $\langle x_1, x_n\rangle = 0$, for all $n > 1$. Therefore

$$\langle x_1, x_n + x_0\rangle = \langle x_1, x_n\rangle + \left\langle x_1, \sum_{n=1}^{\infty} c_n x_n \right\rangle = \langle x_1, x_n\rangle + \sum_{n=1}^{\infty} c_n \langle x_1, x_n\rangle.$$

Since $c_1 \neq 0$, $\{x_n + x_0\}$ is not a Bessel sequence and so it cannot be a frame for H.

If $x_1 \in [x_n]_{n>1}$, then for scalars $\{\alpha_n\}_{n\geq 2}$, $x_1 = \sum_{n=2}^{\infty} \alpha_n x_n$. Then $x_0 = \sum_{n=2}^{\infty} (c_n + c_1\alpha_n)x_n$. Again, for some $n(= 2$ say$)$, $c_n + c_1\alpha_n \neq 0$. If $x_2 \notin [x_n]_{n>2}$, then by using above arguments, $\{x_n + x_0\}$ is not a Bessel sequence and hence not a frame for H. Since $\{x_n\}$ is near exact, this process will terminate after finite number of steps say n which in turn will give some x_n such that $c_n \neq 0$ and $x_n \notin [x_i]_{i>n}$. Then by arguments used above $\{x_n + x_0\}$ is not a frame for H. ☐

We shall now consider perturbation of a frame of the type considered by Holub [10], [11] for bases and we shall prove the following result in this direction.

Theorem 3.3. *Let $\{x_n\}$ be a frame for a real Hilbert space H and let $\{y_1, y_2, \ldots, y_m\}$ be a linearly independent set in H. Let $\{z_1, z_2, \ldots, z_m\}$ be a set of vectors in H with $\langle z_i, x_n\rangle = c_i^{(n)}$, $i = 1, 2, \ldots, m$ and $n \in \mathbb{N}$ such that $\left\{x_n + \sum_{i=1}^{m} c_i^{(n)} y_i\right\}$ is a frame for H. Then -1 is not an eigenvalue of the matrix*

$$A_m = \begin{pmatrix} \langle z_1, y_1\rangle & \langle z_2, y_1\rangle & \ldots & \langle z_m, y_1\rangle \\ \langle z_1, y_2\rangle & \ldots & \ldots & \ldots \\ \vdots & & & \vdots \\ \langle z_1, y_m\rangle & \ldots & \ldots & \langle z_m, y_m\rangle \end{pmatrix}.$$

Proof. It is enough to prove the result for $m = 2$.

$$A_2 = \begin{pmatrix} \langle z_1, y_1\rangle & \langle z_2, y_1\rangle \\ \langle z_1, y_2\rangle & \langle z_2, y_2\rangle \end{pmatrix}.$$

Let -1 be an eigenvalue of A_2. Then

$$\begin{pmatrix} \langle z_1, y_1\rangle + 1 & \langle z_2, y_1\rangle \\ \langle z_1, y_2\rangle & \langle z_2, y_2\rangle + 1 \end{pmatrix} = 0.$$

Therefore there exist scalars α, β (both not zero)

$$\alpha \begin{pmatrix} \langle z_1, y_1\rangle + 1 \\ \langle z_1, y_2\rangle \end{pmatrix} + \beta \begin{pmatrix} \langle z_2, y_1\rangle \\ \langle z_2, y_2\rangle + 1 \end{pmatrix} = 0\,.$$

This gives

$$\alpha\langle z_1, y_1\rangle + \beta\langle z_2, y_1\rangle = -\alpha\,, \text{ and } \alpha\langle z_1, y_2\rangle + \beta\langle z_2, y_2\rangle = -\beta\,.$$

Let $z = -\alpha z_1 - \beta z_2$. Then $z \neq 0$ because otherwise $\langle z, y_1\rangle = 0 = \langle z, y_2\rangle$ gives $\alpha = 0 = \beta$.

Thus

$$\langle z, x_n\rangle = \langle -\alpha z_1 - \beta z_2, x_n\rangle = -\alpha c_1^{(n)} - \beta c_2^{(2)}\,,$$

and so

$$\langle z, x_n + c_1^{(n)} y_1 + c_2^{(n)} y_2\rangle = \langle z, x_n\rangle + c_1^{(n)}\langle z, y_1\rangle + c_2^{(n)}\langle z, y_2\rangle = 0, \text{ for all } \; n\,.$$

Therefore $\{x_n + c_1^{(n)} y_1 + c_2^{(n)} y_2\}$ is not a frame for H. This is contradiction. Hence -1 cannot be eigenvalue of A_2. □

Now consider the perturbation of the form $\{x_n + c_n x_0\}$, where $\{x_n\}$ is a frame for H, $\{c_n\}$ is any sequence of scalars and $x_0 \in H$ be any element. If $x_0 = 0$, then $\{x_n + c_n x_0\}$ is a frame for H. If $x_0 \neq 0$, then $\{x_n + c_n x_0\}$ need not be a frame for H.

Example 3.4. Let $\{x_n\}$ be he sequence of unit vectors in H.

(a) If $c_n = 1$ for all n, then $\{x_n + c_n x_1\}$ is not a frame for H as it is not even a Bessel sequence.

(b) If $c_n = \frac{1}{2^n}$, $n \in \mathbb{N}$, then $\{x_n + c_n x_1\}$ is a frame for H.

The following result gives a sufficient condition for the perturbed sequence of the above type to be a frame.

Theorem 3.5. *Let $\{x_n\}$ be a frame for H with bounds A and B. Let $x_0 \neq 0$ be any element in H and $\{c_n\}$ be any sequence of scalars. Then the perturbed sequence $\{x_n + c_n x_0\}$ is a frame for H if $\sum_{n=1}^{\infty} |c_n|^2 < \frac{A}{\|x_0\|^2}$.*

Proof. Let $y_n = x_n + c_n x_0$, $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 = \|x_0\|^2 \sum_{n=1}^{\infty} |c_n|^2 .$$

Therefore, by Theorem 1 in [3], $\{x_n + c_n x_0\}$ is a frame for H if $\sum_{n=1}^{\infty} |c_n|^2 \|x_0\|^2 < A$, i.e., if $\sum_{n=1}^{\infty} |c_n|^2 < \frac{A}{\|x_0\|^2}$. □

Remark. It may be possible that $\sum_{n=1}^{\infty} |c_n|^2 \geq \frac{A}{\|x_0\|^2}$ and still the perturbed sequence $\{x_n + c_n x_0\}$ is a frame for H. Indeed, if $\{x_n\}$ is a sequence of unit vectors in H and $c_n = \frac{1}{2^{n-1}}$, for all n. Then $\{x_n + c_n x_1\}$ is a frame for H and $\sum_{n=1}^{\infty} |c_n|^2 > 1 = A$.

Observation. (i) Let $\{x_n\}$ be a tight frame for H and x_0 be any non-zero element in H. Let $\{c_n\}$ be any sequence of scalars which is not identically equal to zero. If $\{x_n + c_n x_0\}$ is a frame for H, then it cannot be tight. Indeed, by Theorem 3.5, $\|x_0\|^2 \sum_{n=1}^{\infty} |c_n|^2 < A$. Then, by Theorem 1 in [3], the frame $\{x_n + c_n x_0\}$ would have bounds $A\left(1 - \sqrt{\frac{R}{A}}\right)^2$ and $B\left(1 + \sqrt{\frac{R}{B}}\right)^2$, where

$R = \|x_0\|^2 \sum\limits_{n=1}^{\infty} |c_n|^2$. Since $R < A$, the tightness of the frame $\{x_n + c_n x_0\}$ got disturbed.

(ii) Let $\{x_n\}$ be an exact frame for H and x_0 be any non zero element in H. Let $\{c_n\}$ be any sequence of scalars such that $\{x_n + c_n x_0\}$ is a frame for H. Then $\{x_n + c_n x_0\}$ will also be an exact frame.

Finally, we give a sufficient condition for a frame to be near exact.

Theorem 3.6. *A frame $\{f_n\}$ in H is near exact if for every infinite sequence of indices $\{\sigma_n\}$, $[f_n]_{n \neq \sigma_1, \sigma_2, \ldots} \neq [f_n]$.*

Proof. Suppose that $\{f_n\}$ is not near exact. Then $\{f_n\}$ is not exact. Therefore there exists a σ_1 such that

$$f_{\sigma_1} \in [f_n]_{n \neq \sigma_1} = [f_n].$$

Also, there exists $m_1 \geq \sigma_1$ such that

$$\operatorname{dist}\left(f_{\sigma_1}, [f_n]_{\substack{n=1 \\ n \neq \sigma_1}}^{m_1}\right) < \frac{1}{2}.$$

Now $\{f_n\}_{n=m_1+1}^{\infty}$ is not exact, therefore there exists a σ_2 such that

$$f_{\sigma_2} \in [f_n]_{\substack{n=m_1+1 \\ n \neq \sigma_2}}^{\infty} = [f_n]_{n=m_1+1}^{\infty},$$

and an $m_2 \geq \sigma_2$ such that

$$\operatorname{dist}\left(f_{\sigma_2}, [f_n]_{\substack{n=1 \\ n \neq \sigma_1, \sigma_2}}^{m_2}\right) < \frac{1}{4}.$$

Furthermore, since

$$[f_n] = [f_n]_{n \neq \sigma_1} = [f_n]_{\substack{n=1 \\ n \neq \sigma_1}}^{m_1} + [f_n]_{\substack{n=m_1+1 \\ n \neq \sigma_1}}^{\infty},$$

and

$$[f_n]_{\substack{n=m_1+1 \\ n \neq \sigma_2}}^{\infty} = [f_n]_{n=m_1+1}^{\infty},$$

We have

$$[f_n]_{n \neq \sigma_1, \sigma_2} = [f_n]_{\substack{n=1 \\ n \neq \sigma_1}}^{m_1} + [f_n]_{\substack{n=m_1+1 \\ n \neq \sigma_2}}^{\infty} = [f_n].$$

Continuing this way, we get a sequence of indices $\{\sigma_n\}$ and an increasing sequence $\{m_n\}$ such that

$$\operatorname{dist}\left(f_{\sigma_j}, [f_n]_{\substack{n=1 \\ n\neq\sigma_1,\sigma_2,\ldots,\sigma_k}}^{m_k}\right) < \frac{1}{2^k}\,.$$

Therefore

$$f_{\sigma_j} \in [f_n]_{n\neq\sigma_1,\sigma_2,\ldots} \quad \text{for all } j = 1, 2, \ldots\,.$$

Thus, we get a sequence of indices $\{\sigma_n\}$ such that $[f_n]_{n\neq\sigma_1,\sigma_2,\ldots} = [f_n]$. This is a contradiction. Hence $\{f_n\}$ is near exact. □

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