International Journal of Pure and Applied Mathematics
Volume 37 No. 3 2007, 297-319

THE FUNCTIONAL ANALYTIC SETTING OF HUM
PART I: GENERAL THEORY

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Abstract: The question of controllability for partial differential evolution equations of hyperbolic and parabolic nature has been studied intensively over the last decade, motivated and inspired by numerous applications in science and technology. The problems to be considered here are the boundary control problems that can be formulated as follows:

Let $u(x,t)$ denote the state of the "system" as a function of space $x \in \Omega$, and of time t. We are allowed to act on the system by "control variables". These are functions κ which are applied on Γ, the boundary of Ω, or merely on parts of the boundary. The basic aim is to achieve exact controllability, i.e., *given a time* $T > 0$ *and initial data, find a control* κ *that drives the system to a certain state at time* T. In this paper we analyze the Hilbert Uniqueness Method - HUM - in terms of functional analysis and provide a unified setting in terms of reconstruction and controllability operators. In Part II of this paper we will then apply the functional analytic methods to the Mindlin-Timoshenko plate system.

AMS Subject Classification: 35B37, 35L35
Key Words: functional analysis, Hilbert uniqueness method, reconstruction and controllability operators, Mindlin-Timoshenko plate system

1. Introduction

There are two major problems involved in obtaining exact controllability,

Received: March 12, 2007

one is to find the control κ and the other is to characterize in a suitable fashion the function spaces in which the initial data may lie in order to achieve exact controllability. These kinds of controllability problems are highly relevant in a number of engineering disciplines, e.g., in the control of flexible space structures, plates, beams, etc.

There exists a vast literature on these problems. One of the highlights of the theory is the Hilbert Uniqueness Method due to J.-L. Lions [11], [12], further developed and perfected in particular by E. Zuazua. We will refer to the papers [21], [22], [23], [24], [25], [26], [27], [28], that represents a selection of the variety of problems the method can be applied to. The HUM-method is based on uniqueness results for the PDE-systems under consideration, and appropriate Hilbert space structures are then constructed on the space of initial data for these systems. This makes it possible to characterize the spaces in which the initial data must lie in order to guarantee exact controllability, and identify a suitable set of controls κ. The controls found are optimal in the sense that the method minimizes the energy required to bring the system to rest. Since the method is in principle systematic, it can be applied to a variety of problems. The method works equally well for the study of approximate controllability. We refer to the excellent survey by Zuazua [28].

A seemingly different approach to the boundary control problem comes from the field of linear systems, providing a very general setting for control systems operating with what is usually called the "control system" and the "observability system" which are dual in a sense to be explained later. With its roots in electrical engineering it is natural in this framework to formulate all problems as first order systems in a general Banach- or Hilbert space setting. Following this approach it is natural that the controllability questions are answered in terms of properties of the observability systems – that is, the dual. This boils down to, essentially, an observability inequality and are formulated in terms of properties of the so-called "controllability operator" and the "observability operator".

The abstract formulation of PDE-control problems in this setting was first formulated by Russell in [4]. There are many approaches to linear system theory, here we will only need some basic and well-known results as presented by El Jay and Pritchard in [5] or Curtain and Zwart in [3]. This is a semigroup based approach and many of the results on the underlying semigroup dates back to pioneering work of Fattorini [6], [7], [8]. The link between HUM and linear systems is complicated by the fact that the linear systems setting typically operates with the control acting as an interior term in the operator equation, that is, like Bu in $\dot{z}(t) = Az(t) + Bu(t)$, and this is typically the analogue to an "internal control" in the PDE-setting. But the connection between the controllability operator and observability operator from linear systems theory and the controllability operator from HUM becomes more apparent when we apply tools from the pseudodifferential operator based Boutet de Monvel-calculus method, developed by Grubb in [9], to the boundary control system. Applying this theory, it is shown by the author in [14], [18], [16], [13], that this provides a general way to transform the boundary control systems into interior control systems, with precise estimates and mapping properties so much of the classical linear systems approach is immediately applicable. The so-called Λ-operator from HUM – the invertibility of which is the key issue in HUM – is decomposed in a natural way analogous to the finite dimensional reconstruction operator. The basic tools are provided by the Boutet de Monvel-calculus giving the natural framework where the "adjoints" of Poisson operators like the Greens operator are the trace operators, like the conormal boundary derivative or "complimentary boundary operator" appearing in HUM. The force of these tools are the precise composition rules and estimates available. Applying the pseudo-differential transformations introduced by the author in [14], [16], [13], [15] and [19], it is possible to understand the "internal" nature of HUM completely in basic terms from linear systems theory.

The characterisation of the geometric aspects of the boundary control problem was clarified by Bardos, Lebeau and Rauch in the seminal paper [1], for the class of smooth domains and in [2] Burq extended the results to less smooth (C^3) domains. These approaches are based on microlocal analysis and uses the fact that the underlying semigroup propagates the energy along the rays of geometrical optics (the generalized bicharacteristics) that are reflected on the boundary, and explains in an intuitively clear way why the so called "observability inequality" (see eg. [28]) is necessary and sufficient for controllability.

2. Functional Analytic Setting of HUM

In order to investigate the internal nature of HUM we will now recapitulate the setting and introduce the fundamental operators. Then we will relate the control problems in a precise manner to those of observability, relying on the vast theory of linear systems. The result will be a modern formulation of the (classical) HUM method, relying exclusively on the positivity of a so-called *control functional* – which really is the Euler equation corresponding to the system we are considering. The well-posedness results used here can be found in Pedersen [20] and how to transform very general systems into this formulation have been discussed in detail in Pedersen [14], [18], [16], [13], [17], [15].

This modern formulation of HUM owes a lot to Zuazau, e.g. [29], [28]. We will develop the theory here with the purpose of applying it to the Mindlin-Timoshenko plate system in Part II of this paper.

3. Factorization of the Λ-Operator

We consider the *control system*:

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & (x,t) \in Q, \\ u = g 1_{\Sigma_0}, & (x,t) \in \Sigma, \\ u(0) = u^0, \quad \partial_t u(0) = u^1, & x \in \Omega. \end{cases} \tag{1}$$

Here, as usual, $Q = \Omega \times]0,T[$, where Ω is an open, bounded subset of $\mathbb{R}^n$, with sufficiently smooth boundary Γ (C^2 will do for the time being). Γ_0 denotes an open and nonempty subset of Γ, and we denote $\Gamma \times]0,T[= \Sigma$ and $\Gamma_0 \times]0,T[= \Sigma_0$. The function 1_{Σ_0} is the characteristic function of Σ_0 and is introduced here in order to simplify the notation.

We recall from [20] that we have the following existence and uniqueness results for the system (1):

Theorem 1. *Assume that* $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

Then, for all $g \in L^2(\Sigma_0)$, *the nonhomogeneous boundary value problem (1) has a unique (weak) solution*

$$(u, \partial_t u) \in C([0,T]; L^2(\Omega) \times H^{-1}(\Omega)). \tag{2}$$

The mapping $\{u^0, u^1, g\} \to \{u, \partial_t u\}$ *is linear and continuous*

$$L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma) \to C([0,T]; L^2(\Omega) \times H^{-1}(\Omega)), \tag{3}$$

and there exists a $c(T) > 0$ *(depending only on* T*) such that*

$$\|(u, \partial_t u)\|_{L^\infty((0,T); L^2(\Omega) \times H^{-1}(\Omega))} \leq c(T)(\|(u^0, u^1)\|_{L^2(\Omega) \times H^{-1}(\Omega)} + \|g\|_{L^2(\Sigma_0)}). \tag{4}$$

To the control system (1) we associate the adjoint $\varphi - system$:

$$\begin{cases} \partial_t^2 \varphi - \Delta \varphi = 0, & (x,t) \in Q, \\ \varphi = 0, & (x,t) \in \Sigma, \\ \varphi(0) = \varphi^0, \quad \partial_t \varphi(0) = \varphi^1, & x \in \Omega, \end{cases} \tag{5}$$

for $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega) = (L^2(\Omega) \times H^{-1}(\Omega))'$.

3.1. Observability and Reconstruction Operators

To simplify the notation we introduce the (energy) spaces

$$E = H_0^1(\Omega) \times L^2(\Omega), \quad B = L^2(\Sigma), \tag{6}$$

so

$$E' = L^2(\Omega) \times H^{-1}(\Omega), \quad B' = B. \tag{7}$$

Then we have that $(\varphi, \partial_t \varphi) \in C([0,T], E)$ for the solution to the adjoint system (5), and we define, for $(\varphi^0, \varphi^1) \in E$, the *observability operator* Φ:

$$\Phi : E \to B \tag{8}$$

defined by

$$\Phi(\varphi^0, \varphi^1) = \frac{\partial \varphi}{\partial n} 1_{\Sigma_0}, \tag{9}$$

where φ is the corresponding solution to the problem (5).

Remark 2. Recall that Φ is continuous, i.e.:

$$\|\Phi(\varphi^0, \varphi^1)\|_B \le C\|(\varphi^0, \varphi^1)\|_E; \tag{10}$$

sometimes referred to as the "hidden regularity" of the wave equation but actually a consequence of the fact that the wave group propagates $H^s \times H^{s-1}$-regularity along the bicharacteristics, such that in a neighborhood of $x \in \Gamma$, $\frac{\partial}{\partial n}$ (that in local coordinates $(x', x_n), x' \in \mathbb{R}^{n-1}$, where $\Gamma = (x', 0)$, is $\frac{\partial}{\partial x_n}$) is not worse than ∂_t. See Bardos, Lebeau and Rauch [1] or the original proof by Lions, Lasiecka and Triggiani in [10].

We then consider the $\psi - system$:

$$\begin{cases} \partial_t^2 \psi - \Delta \psi = 0, & (x,t) \in Q, \\ \psi = g 1_{\Sigma_0}, & (x,t) \in]0,T[\times \Gamma, \\ \psi(T) = 0, \quad \partial_t \psi(T) = 0, & x \in \Omega, \end{cases} \tag{11}$$

which is the nonhomogeneous, backwards wave equation. From the system (11) we define the *reconstruction operator*

$$\Psi : B \to E', \tag{12}$$

given by

$$\Psi(g) = (\psi(0), \partial_t \psi(0)), \tag{13}$$

which is well defined according to Theorem 1. We see immediately that (apart from possibly a notation and sign convention)

$$\Lambda = \Psi\Phi : E \to E', \tag{14}$$

where Λ is the operator introduced by Lions in [11]. We see, by construction, that if (φ^0, φ^1) is such that

$$\Lambda(\varphi^0, \varphi^1) = \Psi\Phi(\varphi^0, \varphi^1) = (u^0, u^1), \tag{15}$$

then $u(x, T) = \partial_t u(x, T) = 0$, so the system (1) with initial data $(u^0, u^1) \in E'$ is exactly controllable to 0 in time T with the control $\frac{\partial\varphi}{\partial n} 1_{\Sigma_0}$. This is possible in general whenever Λ is surjective.

For general open sets $\Gamma_0 \subset \Gamma$, the space

$$F' = R(\Psi) = \Psi(L^2(\Sigma_0)) \subset E', \tag{16}$$

is the subspace of intial data controllable from Γ_0, and we have exact controllability in E' for the system (1) when $F' = E'$.

4. Duality

Now, for $(\psi^0, \psi^1) \in E'$ and $(\varphi^0, \varphi^1) \in E$ we introduce the duality product $\langle \cdot, \cdot \rangle_{E',E}$ between E and E' by

$$\langle (\psi^0, \psi^1), (\varphi^0, \varphi^1) \rangle_{E',E} = \langle \psi^1, \varphi^0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - (\psi^0 \mid \varphi^1), \tag{17}$$

where the expressions on the right hand side are the usual duality products between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, resp. in $L^2(\Omega)$.

Now, for all solutions $\varphi(x, t)$ to (5) and all solutions $\psi(x, t)$ to (11) we have:

$$\begin{aligned} \langle (\psi(0), \partial_t\psi(0)), (\varphi(0), \partial_t\varphi(0)) \rangle_{E',E} - \langle (\psi(t), \partial_t\psi(t)), (\varphi(t), \partial_t\varphi(t)) \rangle_{E',E} \\ = \int_0^t \int_{\Gamma_0} \frac{\partial\varphi}{\partial n} g \, d\Gamma dt. \end{aligned} \tag{18}$$

This follows from the following calculation:

Assume that all data of the φ-system and ψ-system are smooth. Then we multiply the equation $\partial_t^2\psi - \Delta\psi = 0$ with a solution φ to the φ-system and integrate:

$$0 = \int_0^t \int_\Omega (\partial_t^2\psi - \Delta\psi)\varphi \, dx dt$$

$$= \int_\Omega (\partial_t\psi\varphi - \psi\partial_t\varphi)|_0^t dx + \int_0^t \int_\Omega \psi\partial_t^2\varphi dxdt - \int_0^t \int_\Omega \psi\Delta\varphi dxdt$$

$$- \int_0^t \int_\Gamma \frac{\partial\psi}{\partial n}\varphi d\Gamma dt + \int_0^t \int_\Gamma \frac{\partial\varphi}{\partial n}\psi d\Gamma dt$$

$$= \int_\Omega (\partial_t\psi\varphi - \psi\partial_t\varphi)|_0^t dx + \int_0^t \int_\Gamma \frac{\partial\varphi}{\partial n}\psi d\Gamma dt$$

$$= \int_\Omega (\partial_t\psi(t)\varphi(t) - \psi(t)\partial_t\varphi(t)) - (\partial_t\psi(0)\varphi(0) - \psi(0)\partial_t\varphi(0))dx$$

$$+ \int_0^t \int_\Gamma \frac{\partial\varphi}{\partial n}\psi d\Gamma dt.$$

Then we have (18) by density. Returning to the original problem (1) we see that we have the following theorem.

Theorem 3. *The initial data $(u^0, u^1) \in E'$ for the problem (1) can be steered to 0 in time $T > 0$ if and only if there exists $g \in L^2(\Sigma_0)$ such that the following identity holds:*

$$\langle (u^0, u^1), (\varphi^0, \varphi^1)\rangle_{E',E} = \int_0^T \int_{\Gamma_0} \frac{\partial\varphi}{\partial n} g d\Gamma dt, \tag{19}$$

for all $(\varphi^0, \varphi^1) \in E$ and φ the corresponding solution of the adjoint φ-system (5). Since the problem (1) is linear, this is equivalent to (1) being exactly controllable on E'.

Comparison of (18) and Theorem 3 gives us immediately the next theorem.

Theorem 4. *Assume that the control system (1) is exactly controllable. Then*

$$\Psi^* = \Phi : E \to B \tag{20}$$

is the adjoint of the operator $\Psi : B \to E'$.

Proof. Write the equation (19) as

$$\langle \Psi(g1_{\Sigma_0}), (\varphi^0, \varphi^1)\rangle_{E',E} = \int_0^T \int_{\Gamma_0} \Phi(\varphi^0, \varphi^1) g d\Gamma dt$$

$$= (g1_{\Sigma_0} \mid \Phi(\varphi^0, \varphi^1))_B. \qquad \square \tag{21}$$

5. Exact Controllability

Let us now investigate the operator Φ further. Since the operator $-\Delta$ generates a unitary group, $S(t)$, on E, we can write

$$\Phi(\varphi^0, \varphi^1) = \frac{\partial}{\partial n} S(\cdot)(\varphi^0, \varphi^1) \cdot 1_{\Sigma_0}, \tag{22}$$

which is exactly the so called "observability operator" from Linear Systems Theory (see e.g [3]). The adjoint,

$$\Phi^* : B \to E', \tag{23}$$

is the operator

$$\Phi^* g = \int_0^T S^*(s)(\frac{\partial}{\partial n} 1_{\Sigma_0})^* g(s) ds, \tag{24}$$

which we have identified above as the operator Ψ. From [3] we have the following theorem.

Theorem 5. *The system (1) is exactly controllable on E' if and only if the adjoint system (5) is exactly observable.*

Each of the following conditions are equivalent to the system (5) being exactly observable on E:

— $\langle \Phi^* \Phi(\varphi^0, \varphi^1), (\varphi^0, \varphi^1) \rangle_{E',E} \geq \gamma \|(\varphi^0, \varphi^1)\|_E^2, \quad \gamma > 0, \quad \forall (\varphi^0, \varphi^1) \in E,$
— $\int_0^T \int_{\Gamma_0} |\Phi(\varphi^0, \varphi^1)|^2 d\Gamma dt \geq \gamma \|(\varphi^0, \varphi^1)\|_E^2, \quad \gamma > 0, \quad \forall (\varphi^0, \varphi^1) \in E,$
— $\ker(\Phi) = \{(0,0)\}$ *and* $R(\Phi)$ *closed.*

We see that the second statement is the *observability inequality*:

$$\int_0^T \int_{\Gamma_0} |\frac{\partial \varphi}{\partial n}|^2 d\Gamma dt \geq \gamma \|(\varphi^0, \varphi^1)\|_E^2, \quad \gamma > 0, \quad (\varphi^0, \varphi^1) \in E, \tag{25}$$

which today is the mean to investigate the controllability properties of the system (1). We state this fact in a theorem.

Theorem 6. *The control system (1) is exactly controllable on E' if and only if the solution φ to the adjoint system (5) satisfies the observability inequality*

$$\int_0^T \int_{\Gamma_0} |\frac{\partial \varphi}{\partial n}|^2 d\Gamma dt \geq \gamma \|(\varphi^0, \varphi^1)\|_E^2, \quad \gamma > 0, \tag{26}$$

for all $(\varphi^0, \varphi^1) \in E$.

Now, comparing the results so far, we see that in the case of exact controllability of (1) we have the following information about the adjoint system:

$$\gamma\|(\varphi^0,\varphi^1)\|_E^2 \le \|\Phi(\varphi^0,\varphi^1)\|_B^2 \le C\|(\varphi^0,\varphi^1)\|_E^2, \tag{27}$$

for some positive constants γ and C. Hence the mapping $M : E \to \mathbb{R}$, given by

$$M(\varphi^0,\varphi^1) = \|\Phi(\varphi^0,\varphi^1)\|_B, \tag{28}$$

defines a norm on E, equivalent to the usual norm.

Now let us elaborate further on the connection to linear systems theory. It is well known how we can write the φ-system (5) as a first order system

$$W' = \mathcal{A}W, \quad W(0) = W^0, \tag{29}$$

in E. Here $W = (\varphi,\varphi')$, $W^0 = (\varphi^0,\varphi^1)$, and $\mathcal{A}W = (\varphi',\Delta\varphi)$ with $D(\mathcal{A}) = (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$. We can formulate the control system (1) as a first order system on E', dual to (5), in the following way:

$$U = -\mathcal{A}^*U + \Phi^*(g1_{\Sigma_0}), \quad U(0) = U^0. \tag{30}$$

Here $U = (-u',u)$, $U^0 = (-u^1,u^0)$. Then the duality relation (see (19)) can be written

$$\langle U(t),W(t)\rangle_{E',E} = \langle U(0),W(0)\rangle_{E',E} + \int_0^t (g1_{\Sigma_0} \mid \Phi W(t))_B dt \tag{31}$$

(recall that $\mathcal{A}$ is skew adjoint). Since $\mathcal{A}$ is the generator of the group of isometries $S(t)$ on E we have the following theorem.

Theorem 7. *Assume that we have for some $T > 0$ the observability inequality*

$$c\|W^0\|_E \le \|\Phi W^0\|_B. \tag{32}$$

Then, for all $U^0 \in E'$ there exists a control $g \in B$ such that the system (30) (or, equivalently (1)) satisfies $U(T) = 0$. The control function g can be chosen such that

$$\|g\|_B \le c(T)\|U^0\|_{E'}, \tag{33}$$

with a constant $c(T)$ only depending on T.

Proof. The mapping

$$(W^0,\tilde{W}^0) \to \int_0^T (\Phi S(t)W^0 \mid \Phi S(t)\tilde{W}^0)_B dt \tag{34}$$

from $E \to \mathbb{R}$ is symmetric, positive and coercive. Hence there exists a self adjoint and positive definite isomorphism $\Lambda : E \to E'$ such that

$$\langle \Lambda W^0, \tilde{W}^0 \rangle_{E',E} = \int_0^T (\Phi S(t) W^0 \mid \Phi S(t) \tilde{W}^0)_B dt, \tag{35}$$

for all $W^0, \tilde{W}^0 \in E$. Now, given arbitrary initial data $U^0 \in E'$, we define

$$g(t) = -\Phi S(t) \Lambda^{-1} U^0. \tag{36}$$

We claim that this control drives (30) to rest in time T. Let $W^0 \in E$. Then

$$\begin{aligned} \langle U(T), W(T) \rangle_{E',E} &= \langle U(0), W(0) \rangle_{E',E} + \int_0^T (g(t) \mid \Phi W(t))_B dt \\ &= \langle U(0), W(0) \rangle_{E',E} - \int_0^T (\Phi S(t) \Lambda^{-1} U^0 \mid \Phi S(t) W^0)_B dt \\ &= \langle U(0), W(0) \rangle_{E',E} - \langle \Lambda \Lambda^{-1} U^0, W^0 \rangle_{E',E} = 0. \end{aligned}$$

Since $W(t) = S(t) W^0$, where $S(t)$ is an isomorphism, we see that $W(T)$ spans E as W^0 runs through E, hence $U(T) = 0$.

Since Φ is continuous from E to B we have immediately by the construction of Λ:

$$\|g\|_B = \|\Phi S(t) \Lambda^{-1} U^0\|_B \le c' \|\Lambda^{-1} U^0\| \le c(T) \|U^0\|_{E'}. \qquad \square \tag{37}$$

Remark 8. The operator $-\Phi S(t) \Lambda^{-1} = -\Phi S(t) (\Phi^* \Phi)^{-1}$ is the key operator in HUM. We see that this is actually an infinite dimensional version of the controllability operator from classical finite dimensional control theory.

5.1. The Control Functional

Now, consider the "control functional" $J : E \to \mathbb{R}$, associated to the control system (1) (see [29]):

$$J_T(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} |\frac{\partial \varphi}{\partial n}|^2 d\Gamma dt - \langle (u^0, u^1), (\varphi^0, \varphi^1) \rangle_{E',E}. \tag{38}$$

The following theorem shows how minimization of J_T is linked to control of (1).

Theorem 9. *Assume that* $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in E$ *is a minimizer of* J_T. *Then*

$$g = \Phi(\tilde{\varphi}^0, \tilde{\varphi}^1) \tag{39}$$

is a control for the system (1) that steers the initial data $(u^0, u^1) \in E'$ *to zero in time* T.

Proof. Since $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in E$ is assumed to be a minimizer for J_T, we have, for all $(\varphi^0, \varphi^1) \in E$:

$$\begin{aligned} 0 &= \lim_{h\to 0} \frac{1}{h}(J_T((\tilde{\varphi}^0, \tilde{\varphi}^1) + h(\varphi^0, \varphi^1)) - J_T(\varphi^0, \varphi^1)) \\ &= \lim_{h\to 0} \frac{1}{h}(\|\Phi(\tilde{\varphi}^0, \tilde{\varphi}^1) + h\Phi(\varphi^0, \varphi^1)\|_B^2 - \langle (u^0, u^1), (\tilde{\varphi}^0, \tilde{\varphi}^1) + h(\varphi^0, \varphi^1)\rangle_{E',E} \\ &\qquad -(\|\Phi(\tilde{\varphi}^0, \tilde{\varphi}^1)\|_B^2 - \langle (u^0, u^1), (\tilde{\varphi}^0, \tilde{\varphi}^1)\rangle_{E',E}) \\ &= \int_0^T \int_{\Gamma_0} \Phi(\tilde{\varphi}^0, \tilde{\varphi}^1)\Phi(\varphi^0, \varphi^1) d\Gamma dt - \langle (u^0, u^1), (\varphi^0, \varphi^1)\rangle_{E',E}. \end{aligned}$$

From Theorem 3 we see that $g = \Phi(\tilde{\varphi}^0, \tilde{\varphi}^1)$ is a control that steers the initial data $(u^0, u^1) \in E'$ to zero in time T. This ends the proof. □

So in order to control the system (1) we see that it is relevant to study the minimization problem for the control functional J_T. Since continuity is obvious, a unique minimizer is guaranteed to exist if J_T is coercive and stricly convex. On the other hand, this will be seen to be the case when the adjoint φ-system is exactly observable. Since exact observability of the φ-system is equivalent to exact controllability of the original control system, we have the following theorem.

Theorem 10. *The control system (1) is exactly controllable on E' if and only if J_T is coercive and strictly convex on E.*

Proof. Assume J_T is coercive and stricly convex. Then there exists a unique minimizer and according to Theorem 9 the system (1) is exactly controllable.

Assume now that (1) is exactly controllable. Then the observability inequality from Theorem 5 holds, so for $(\varphi^0, \varphi^1), (\rho^0, \rho^1) \in E$ and $0 < \lambda < 1$ we have:

$$\begin{aligned} J_T(\lambda(\varphi^0, \varphi^1) + (1-\lambda)(\rho^0, \rho^1)) &= \lambda J_T(\varphi^0, \varphi^1) + (1-\lambda) J_T(\rho^0, \rho^1) \\ &\quad -\frac{1}{2}\lambda(1-\lambda)\int_0^T \int_{\Gamma_0} |\Phi(\varphi^0, \varphi^1) - \Phi(\rho^0, \rho^1)|^2 d\Gamma dt. \end{aligned} \tag{40}$$

But from the observability inequality we have that

$$\int_0^T \int_{\Gamma_0} |\Phi(\varphi^0, \varphi^1) - \Phi(\rho^0, \rho^1)|^2 d\Gamma dt \geq \gamma \|(\varphi^0, \varphi^1) - (\rho^0, \rho^1)\|_E^2, \tag{41}$$

hence, for all $(\varphi^0, \varphi^1) \neq (\rho^0, \rho^1)$ we have

$$J_T(\lambda(\varphi^0, \varphi^1) + (1-\lambda)(\rho^0, \rho^1)) < \lambda J_T(\varphi^0, \varphi^1) + (1-\lambda) J_T(\rho^0, \rho^1), \tag{42}$$

so J_T is strictly convex. Coercivity of J_T is ensured in this case if we can show that

$$J_T(\varphi^0, \varphi^1) \to \infty \quad \text{for} \quad \|(\varphi^0, \varphi^1)\|_E \to \infty. \tag{43}$$

But since

$$\begin{aligned} J_T(\varphi^0, \varphi^1) &\geq \frac{1}{2}\Big(\int_0^T \int_{\Gamma_0} |\Phi(\varphi^0, \varphi^1)|^2 - \|(u^0, u^1)\|_{E'} \|(\varphi^0, \varphi^1)\|_E\Big) \\ &\geq \frac{\gamma}{2}\|(\varphi^0, \varphi^1)\|_E^2 - \frac{1}{2}\|(u^0, u^1)\|_{E'}\|(\varphi^0, \varphi^1)\|_E, \end{aligned} \tag{44}$$

where $(u^0, u^1) \in E'$ is fixed, we see that (43) holds. Hence J_T is strictly convex and there exists a unique minimizer in E, and the proof is completed. □

As one might expect, the control obtained by

$$g = \Phi(\tilde{\varphi}^0, \tilde{\varphi}^1) \tag{45}$$

where $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in E$ is the minimizer of J_T, is the control of minimal B-norm that steers the initial data $(u^0, u^1) \in E'$ to zero. Indeed, let g_1 be another control in B that steers $(u^0, u^1) \in E'$ to zero in time T. Then, from Theorem 3,

$$\langle (u^0, u^1), (\varphi^0, \varphi^1) \rangle_{E', E} = \int_0^T \int_{\Gamma_0} \Phi(\varphi^0, \varphi^1) g_1 d\Gamma dt, \tag{46}$$

for all $(\varphi^0, \varphi^1) \in E$. Hence, in particular,

$$\langle (u^0, u^1), (\tilde{\varphi}^0, \tilde{\varphi}^1) \rangle_{E', E} = \int_0^T \int_{\Gamma_0} \Phi(\tilde{\varphi}^0, \tilde{\varphi}^1) g_1 d\Gamma dt. \tag{47}$$

But

$$\begin{aligned} \|g\|_B^2 &= \int_0^T \int_{\Gamma_0} |\Phi(\tilde{\varphi}^0, \tilde{\varphi}^1)|^2 d\Gamma dt = \langle (u^0, u^1), (\tilde{\varphi}^0, \tilde{\varphi}^1) \rangle_{E', E} \\ &= \int_0^T \int_{\Gamma_0} \Phi(\tilde{\varphi}^0, \tilde{\varphi}^1) g_1 d\Gamma dt \leq \|\Phi(\tilde{\varphi}^0, \tilde{\varphi}^1)\|_B \|g_1\|_B = \|g\|_B \|g_1\|_B\,, \end{aligned}$$

hence $\|g\|_B \leq \|g_1\|_B$.

6. Approximative Controllability

The problem of approximative controllability in E' (from the boundary) of the system (1) can be stated as follows:

Let $\epsilon > 0$, $T > 0$ and $(z_T^0, z_T^1) \in E'$ be given. Find, if possible, $g \in L^2(\Sigma_0)$, such that the corresponding solution $(u, \partial_t u) \in E'$ satisfies

$$\|(u(T), \partial_t u(T)) - (z_T^0, z_T^1)\|_{E'} \leq \epsilon. \tag{48}$$

When this is possible for any $\epsilon > 0$ and $(u^0, u^1), (z_T^0, z_T^1)$ in E', we say that (1) is *approximately controllable*. Again we turn to linear systems theory where we have the following equivalent conditions for approximative observability of the adjoint system – equivalent to approximative controllability of the control system.

Theorem 11. *The system (1) is approximately controllable on E' if and only if the adjoint system (5) is approximately observable on E.*

Each of the following conditions are equivalent to the system (5) being approximately observable:

— $\Phi^*\Phi > 0$,

— $\ker(\Phi) = \{(0,0)\}$,

— $\Phi(\varphi^0, \varphi^1) = 0$ *on* $[0,T] \Rightarrow (\varphi^0, \varphi^1) = (0,0)$.

Since $\varphi(x,t) = S(t)(\varphi^0, \varphi^1)$ where $S(t)$ is a unitary group, we see that the third condition above is equivalent to the following unique continuation result.

Theorem 12. *The control system (1) is approximately controllable on E' if and only if the adjoint system (5) has the following unique continuation property with respect to Γ_0:*

$$\Phi(\varphi^0, \varphi^1) = 0 \quad on \quad [0,T] \Rightarrow \varphi = 0. \tag{49}$$

6.1. Duality

As it was the case with exact controllability, a variational approach will prove to be fruitful. By the linearity of the systems we see that we only need to consider the case where the initial data (u^0, u^1) equals $(0,0)$ in E'. Now it is convenient to consider the *backwards* Φ-operator, $\Phi_b : E \to B$, given by

$$\Phi_b(\varphi_T^0, \varphi_T^1) = \frac{\partial}{\partial n} S(T - \cdot)(\varphi_T^0, \varphi_T^1) 1_{\Sigma_0}, \tag{50}$$

where $\varphi(x,t) = S(T-t)(\varphi_T^0, \varphi_T^1)$ is the solution to the backwards problem

$$\begin{cases} \partial_t^2 \varphi - \Delta\varphi = 0, & (x,t) \in Q, \\ \varphi = 0, & (x,t) \in \Sigma, \\ \varphi(T) = \varphi_T^0, \quad \partial_t\varphi(T) = \varphi_T^1, & x \in \Omega. \end{cases}$$

Recall that the backward solvability results for the wave equation with time-reversible boundary conditions are equivalent to the forward solvability results, and in particular for homogeneous boundary conditions where the solution is propagated by the unitary wave-group on E.

Again, by the duality relation, we see that a control $g \in B$ will steer $(u^0, u^1) = (0,0)$ to $(u(T), \partial_t u(T))$ if and only if

$$-\langle (u(T), \partial_t u(T)), (\varphi_T^0, \varphi_T^1)\rangle_{E',E} = \int_0^T \int_{\Gamma_0} \Phi_b(\varphi_T^0, \varphi_T^1) g \, d\Gamma dt, \tag{51}$$

for all $(\varphi_T^0, \varphi_T^1) \in E$. In order to achieve approximative controllability we must be able to steer $(u(T), \partial_t u(T))$ arbitrarily close to a given $(z_T^0, z_T^1) \in E'$. Obviously

$$\|(u(T) - z_T^0, \partial_t u(T) - z_T^1)\|_{E'} \leq \epsilon \tag{52}$$

provided

$$|\langle (u(T) - z_T^0, \partial_t u(T) - z_T^1), (\varphi_T^0, \varphi_T^1)\rangle_{E',E}| \leq \epsilon \|(\varphi_T^0, \varphi_T^1)\|_E \tag{53}$$

for all $(\varphi_T^0, \varphi_T^1) \in E$. This indicates how it is now natural to study the functional $J_\epsilon : E \to \mathbb{R}$ given by

$$\begin{aligned} J_\epsilon(\varphi_T^0, \varphi_T^1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} & |\Phi_b(\varphi_T^0, \varphi_T^1)|^2 d\Gamma dt \\ & + \langle (z_T^0, z_T^1), (\varphi_T^0, \varphi_T^1)\rangle_{E',E} + \epsilon \|(\varphi_T^0, \varphi_T^1)\|_E, \end{aligned} \tag{54}$$

for fixed terminal data – the target – $(z_T^0, z_T^1) \in E'$. The following theorem is now proved in exactly the same way as Theorem 9.

Theorem 13. *Let $\epsilon > 0$ and $(z_T^0, z_T^1) \in E'$ be given, and assume that $(\tilde{\varphi}_T^0, \tilde{\varphi}_T^1) \in E$ is a minimizer of J_ϵ. Then the control*

$$g = \Phi_b(\tilde{\varphi}_T^0, \tilde{\varphi}_T^1) \tag{55}$$

will drive the solution to (1) with initial data $(u^0, u^1) = (0,0)$ to $(u(T), \partial_t u(T))$, such that

$$\|(u(T) - z_T^0, \partial_t u(T) - z_T^1)\|_{E'} \leq \epsilon. \tag{56}$$

We have then the following theorem (see e.g. [29]).

Theorem 14. *The control system (1) is approximately controllable on E' if and only if J_ϵ is continuous, strictly convex and coercive on E.*

Due to the equivalence between approximative observability and the unique continuation property, we will reformulate Theorem 14 in the following equivalent way.

Theorem 15. *The φ-system (5) has the unique continuation property with respect to $\Gamma_0 \subset \Gamma$ if and only if J_ϵ is continuous, strictly convex and coercive on E.*

Proof. If J_ϵ is continuous, strictly convex and coercive on E, a unique minimizer exists and according to Theorem 13 the system (1) is approximately controllable, which again implies that the φ-system (5) has the unique continuation property with respect to Γ_0.

So now assume that (1) is approximately controllable. Then unique continuation holds for the φ-system, and exactly as in the proof of Theorem 10 this implies that J_ϵ is strictly convex. Hence we will only have to show that it is coercive, that is,

$$J_\epsilon(\varphi_T^0, \varphi_T^1) \to \infty \quad \text{for} \quad \|(\varphi_T^0, \varphi_T^1)\|_E \to \infty. \tag{57}$$

So, let $\{(\varphi_T^{0j}, \varphi_T^{1j})\} \in E$ be a sequence such that $\|(\varphi_T^{0j}, \varphi_T^{1j})\|_E \to \infty$ for $j \to \infty$, and consider the corresponding normalized sequence

$$(\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j}) = \frac{(\varphi_T^{0j}, \varphi_T^{1j})}{\|(\varphi_T^{0j}, \varphi_T^{1j})\|_E}. \tag{58}$$

Then

$$\begin{aligned} \frac{J_\epsilon(\varphi_T^{0j}, \varphi_T^{1j})}{\|(\varphi_T^{0j}, \varphi_T^{1j})\|_E} = \frac{1}{2}\|(\varphi_T^{0j}, \varphi_T^{1j})\|_E \int_0^T \int_{\Gamma_0} |\Phi(\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j})|^2 d\Gamma dt \\ + \langle (z_T^0, z_T^1), (\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j}) \rangle_{E',E} + \epsilon. \end{aligned} \tag{59}$$

Now two situations can occur; if

$$\liminf_{j \geq 1} \int_0^T \int_{\Gamma_0} |\Phi(\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j})|^2 d\Gamma dt > 0, \tag{60}$$

then obviously

$$\frac{J_\epsilon(\varphi_T^{0j}, \varphi_T^{1j})}{\|(\varphi_T^{0j}, \varphi_T^{1j})\|_E} \to \infty \tag{61}$$

and we are done. If

$$\liminf_{j\geq 1} \int_0^T \int_{\Gamma_0} |\Phi(\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j})|^2 d\Gamma dt = 0, \tag{62}$$

we can, since $\{(\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j})\}$ is bounded in E, extract a weakly convergent subsequence denoted $\{(\hat{\varphi}_T^{0j_k}, \hat{\varphi}_T^{1j_k})$ that converges weakly to $(y^0, y^1) \in E$.

Let $y(x,t) = S(T-t)(y^0, y^1)$ denote the corresponding solution to the backward problem. If we denote

$$\hat{\varphi}_T^j(x,t) = S(T-t)(\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j}), \tag{63}$$

we have that

$$(\hat{\varphi}_T^{j_k}, \partial_t \hat{\varphi}_T^{j_k}) \rightharpoonup (y, \partial_t y), \tag{64}$$

weakly in $L^2(0,T;E) \times H^1(0,T;E')$. By lower semicontinuity we have

$$\int_0^T \int_{\Gamma_0} |\Phi_b(y^0, y^1)|^2 d\Gamma dt \leq \liminf_{j\geq 1} \int_0^T \int_{\Gamma_0} |\Phi_b(\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j})|^2 d\Gamma dt = 0. \tag{65}$$

Hence $\Phi_b(y^0, y^1) = 0$ on Γ_0 for $t \in [0,T]$ and by the unique continuation property we must have that $(y^0, y^1) = (0,0)$ and we see that $(\hat{\varphi}_T^{0j_k}, \hat{\varphi}_T^{1j_k}) \rightharpoonup (0,0)$ weakly in E. But then

$$\liminf_{j\geq 1} \frac{J_\epsilon(\varphi_T^{0j}, \varphi_T^{1j})}{\|(\varphi_T^{0j}, \varphi_T^{1j})\|_E} \geq \liminf_{j\geq 1} (0 + \langle (z_T^0, z_T^1), (\hat{\varphi}_T^{0j}, \hat{\varphi}_T^{1j}) \rangle_{E',E} + \epsilon) = \epsilon, \tag{66}$$

hence $J_\epsilon(\varphi_T^{0j}, \varphi_T^{1j}) \to \infty$ for $\|(\varphi_T^{0j}, \varphi_T^{1j})\|_E \to \infty$, and the proof is completed. □

7. General Higher Order Operators

The preceding theory applies almost immediately to the general $2m$ order case where boundary control for the system

$$\begin{cases} \partial_t^2 u + Au = 0, & (x,t) \in Q, \\ u = g 1_{\Sigma_0}, & (x,t) \in \Sigma, \\ u(0) = u^0, \quad \partial_t u(0) = u^1, & x \in \Omega, \end{cases} \tag{67}$$

is considered. Now we assume that $A = \sum_{|\alpha|,|\beta|\leq m} D^\beta(a_{\alpha\beta}(x) D^\alpha)$ is a formally selfadjoint, uniformly strongly elliptic differential operator of order $2m$ with

C^m-coefficients $a_{\alpha\beta} = a_{\beta\alpha}$ (we consider the real case only – the extension to the complex case is immediate).

We will here introduce the Dirichlet trace operator $\gamma = \{\gamma_j\}_{0\le j<m}$, where

$$\gamma_j u = (\frac{\partial}{\partial n})^j u|_\Gamma\,, \tag{68}$$

here n is the normal directed outward. We denote similarly $\nu = \{\gamma_j\}_{m\le j<2m}$, the Neumann trace operator and finally we define the Cauchy-data ρu as

$$\rho u = \{\gamma u, \nu u\}. \tag{69}$$

Now for the formally selfadjoint operator A we have

$$(Au \mid v) - (u \mid Av) = (\mathcal{A}\rho u \mid \rho v)_\Gamma, \tag{70}$$

where $\mathcal{A}$ is a skew-triangular $2m\times 2m$ matrix of differential operators over the boundary Γ, of the form

$$\mathcal{A} = \begin{pmatrix} S_1^0 & \dots & S_{2m-1} & S_{2m}^0 \\ S_2^0 & \dots & S_{2m}^0 & 0 \\ \vdots & \dots & \vdots & \vdots \\ S_{2m}^0 & \dots & 0 & 0 \end{pmatrix} + \begin{pmatrix} \text{lower order} & & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \tag{71}$$

the S_k^0 being differential operators on Γ of order $2m-k$, (see e.g., Grubb [9]). We usually write $\mathcal{A}$ in $m\times m$ blocks as

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^{00} & \mathcal{A}^{01} \\ \mathcal{A}^{10} & 0 \end{pmatrix} \tag{72}$$

and we have the following version of the Green formula

$$(Au \mid v) - (u \mid Av) = (\mathcal{A}^{01}\nu u + \mathcal{A}^{00}\gamma u \mid \gamma v)_\Gamma + (\mathcal{A}^{10}\gamma u \mid \nu v)_\Gamma, \tag{73}$$

for $u, v \in H^{2m}(\Omega)$. The coefficient matrices $\mathcal{A}^{ij}$ are uniquely determined from A.

The generalized version of Theorem 1 is now the following:

Theorem 16. *Assume that $(u^0, u^1) \in L^2(\Omega)\times H^{-m}(\Omega)$. Then, for all $g \in (L^2(\Sigma_0))^m$, the nonhomogeneous boundary value problem (67) has a unique (weak) solution*

$$(u, \partial_t u) \in C([0,T]; L^2(\Omega)\times H^{-m}(\Omega)). \tag{74}$$

The mapping $\{u^0, u^1, g\} \to \{u, \partial_t u\}$ *is linear and continuous*

$$L^2(\Omega) \times H^{-m}(\Omega) \times (L^2(\Sigma))^m \to C([0,T]; L^2(\Omega) \times H^{-m}(\Omega)), \tag{75}$$

and there exists a $c = c(T) > 0$ *(depending only on* T*) such that*

$$\|(u, \partial_t u)\|_{L^\infty((0,T);L^2(\Omega)\times H^{-m}(\Omega))} \leq c(\|(u^0, u^1)\|_{L^2(\Omega)\times H^{-m}(\Omega)} + \|g\|_{(L^2(\Sigma_0))^m}). \tag{76}$$

To this general control system we associate the adjoint system, the $\varphi -$ *system*:

$$\begin{cases} \partial_t^2 \varphi + A\varphi = 0, & (x,t) \in Q, \\ \varphi = 0, & (x,t) \in \Sigma, \\ \varphi(0) = \varphi^0, \quad \partial_t \varphi(0) = \varphi^1, & x \in \Omega, \end{cases} \tag{77}$$

for $(\varphi^0, \varphi^1) \in H_0^m(\Omega) \times L^2(\Omega) = (L^2(\Omega) \times H^{-m}(\Omega))'$.

Again, to simplify the notation, we introduce the spaces

$$E = H_0^m(\Omega) \times L^2(\Omega), \quad B = (L^2(\Sigma))^m, \tag{78}$$

so

$$E' = L^2(\Omega) \times H^{-m}(\Omega), \quad B' = B. \tag{79}$$

Then we have that $(\varphi, \partial_t \varphi) \in C([0,T], E)$ for the solution to (77), and similarly we define, for $(\varphi^0, \varphi^1) \in E$, the *observability operator* Φ:

$$\Phi : E \to B \tag{80}$$

defined by

$$\Phi(\varphi^0, \varphi^1) = \mathcal{A}^{10*} \nu \varphi 1_{\Sigma_0}, \tag{81}$$

where φ is the corresponding solution to the problem (77).

Recall that Φ is continuous, i.e.:

$$\|\Phi(\varphi^0, \varphi^1)\|_B \leq C \|(\varphi^0, \varphi^1)\|_E; \tag{82}$$

due to the "hidden regularity" of $+\frac{1}{2}$ for the normal derivative of the wave equation.

We then consider the associated $\psi -$ *system*:

$$\begin{cases} \partial_t^2 \psi + A\psi = 0, & (x,t) \in Q, \\ \psi = g 1_{\Sigma_0}, & (x,t) \in \Sigma, \\ \psi(T) = 0, \quad \partial_t \psi(T) = 0, & x \in \Omega, \end{cases} \tag{83}$$

which is the nonhomogeneous, backwards wave equation. From the system (83) we define the operator

$$\Psi : B \to E', \tag{84}$$

given by

$$\Psi(g) = (\psi(0), \partial_t \psi(0)), \tag{85}$$

which is well defined according to Theorem 16.

Now we will imitate the preceding construction and, for $(\psi^0, \psi^1) \in E'$ and $(\varphi^0, \varphi^1) \in E$ we introduce the duality product $\langle \cdot, \cdot \rangle_{E',E}$ between E and E' by

$$\langle (\psi^0, \psi^1), (\varphi^0, \varphi^1) \rangle_{E',E} = \langle \psi^1, \varphi^0 \rangle_{H^{-m}(\Omega), H_0^m(\Omega)} - (\psi^0 \mid \varphi^1), \tag{86}$$

where the expressions on the right hand side are the usual duality products between $H_0^m(\Omega)$ and $H^{-m}(\Omega)$, resp. in $L^2(\Omega)$.

Now, for all solutions $\varphi(x,t)$ to (77) and all solutions $\psi(x,t)$ to (83) we have the equality that holds the key to the entire construction:

$$\begin{aligned} \langle (\psi(0), \partial_t \psi(0)), (\varphi(0), \partial_t \varphi(0)) \rangle_{E',E} - \langle (\psi(t), \partial_t \psi(t)), (\varphi(t), \partial_t \varphi(t)) \rangle_{E',E} \\ = - \int_0^t \int_{\Gamma_0} (\mathcal{A}^{10*} \nu \varphi) g d\Gamma dt. \end{aligned} \tag{87}$$

This follow again from Greens formula, so assume that all data of the φ-system and ψ-system are smooth. Then we multiply the equation $\partial_t^2 \psi + A\psi = 0$ with a solution φ to the φ-system and integrate.

$$\begin{aligned} 0 &= \int_0^t \int_\Omega (\partial_t^2 \psi + A\psi) \varphi dx dt \\ &= \int_\Omega ((\partial_t \psi(t) \varphi(t) - \psi(t) \partial_t \varphi(t)) - (\partial_t \psi(0) \varphi(0) - \psi(0) \partial_t \varphi(0)))\, dx \\ &\qquad - \int_0^t \int_\Gamma (\mathcal{A}^{10*} \nu \varphi)(\gamma \psi) d\Gamma dt. \end{aligned}$$

Then we have (87) by density.

We see immediately that we have the following theorem:

Theorem 17. *The initial data $(u^0, u^1) \in E'$ for the problem (67) can be steered to 0 in time $T > 0$ if and only if there exists $g \in (L^2(\Sigma_0))^m$ such that the following identity holds:*

$$\langle (u^0, u^1), (\varphi^0, \varphi^1) \rangle_{E',E} = \int_0^T \int_{\Gamma_0} (\mathcal{A}^{10*} \nu \varphi) g d\Gamma dt, \tag{88}$$

for all $(\varphi^0, \varphi^1) \in E$ *and* φ *the corresponding solution of the* φ*-system (77). Since the problem (67) is linear, this is equivalent to (67) being exactly controllable on* E'*.*

Remark 18. Notice that if $R(\Psi) = F' \subsetneq E'$ then (u^0, u^1) must belong to F' in order for (88) to hold. The problem of identifying F', the space of data controllable from a given, open subset $\Gamma_0 \subset \Gamma$, is a key problem in HUM, intimately linked to the unique continuation property as we have seen already for the classical wave equation.

In view of remark (18) we will formulate the following generalized version of Theorem 5. Notice that the optimal case $F = E$ is included:

Theorem 19. *The system (67) is exactly controllable on* $F' \subset E'$ *if and only if the adjoint system (77) is exactly observable on* $F \supset E$*.*

Each of the following conditions are equivalent to the system (77) being exactly observable on F*:*

— $\langle \Phi^*\Phi(\varphi^0, \varphi^1), (\varphi^0, \varphi^1)\rangle_{E',E} \geq \gamma \|(\varphi^0, \varphi^1)\|_F^2, \quad \gamma > 0, \forall (\varphi^0, \varphi^1) \in F,$

— $\int_0^T \int_{\Gamma_0} |\Phi(\varphi^0, \varphi^1)|^2 d\Gamma dt \geq \gamma \|(\varphi^0, \varphi^1)\|_F^2, \quad \gamma > 0, \forall (\varphi^0, \varphi^1) \in F,$

— $\ker(\Phi) = \{(0, 0)\}$ *and* $R(\Phi)$ *closed.*

Now we have that

$$\Phi(\varphi^0, \varphi^1) = \mathcal{A}^{10^*} \nu S(t)(\varphi^0, \varphi^1) 1_{\Sigma_0} \tag{89}$$

for $(\varphi^0, \varphi^1) \in E$ where $S(t)$ is the generalized wave group generated by A on E, and we see that

$$\Phi^* g = \int_0^T S^*(s)(1_{\Sigma_0} \mathcal{A}^{10^*} \nu)^* g(s) ds, \tag{90}$$

for $g \in B = (L^2((0, T) \times \Gamma))^m$.

Like for the wave problem we will state the observability inequality in a theorem:

Theorem 20. *The control system (67) is exactly controllable on* E' *if and only if the solution* φ *to the adjoint system (77) satisfies the observability inequality*

$$\int_0^T \int_{\Gamma_0} |\mathcal{A}^{10^*} \nu \varphi|^2 d\Gamma dt \geq \gamma \|(\varphi^0, \varphi^1)\|_E^2, \quad \gamma > 0, \tag{91}$$

for all $(\varphi^0, \varphi^1) \in E$*.*

The proof is identical to the proof in the case of the classical wave equation. The observability inequality ensures that the operator $\Lambda = \Phi^*\Phi : E \to E'$ is an isomorphism. Then given initial data $(u^0, u^1) \in E'$ for the system (67), the control

$$g(t) = -\mathcal{A}^{10^*}\nu S(t)\Lambda^{-1}(u^0, u^1), \tag{92}$$

will drive the system to rest i time T.

Remark 21. Similar considerations for approximative controllability for general systems can be made, and we arrive again at the same conditions as in the case of the classical wave equation. The complimentary boundary operator $\mathcal{C} = \mathcal{A}^{10^*}\nu$ (which we also recognize as the generalized "conormal derivative") replaces $\frac{\partial}{\partial n}$ in all the calculations.

In the context of the remark above it is of some interest to point out the following theorem.

Theorem 22. *Assume that the subset $\Gamma_0 \subset \Gamma$ is chosen such that the observability operator $\Phi = \mathcal{C}S(t)$ is injective. Then $\mathcal{C}\varphi = 0$ on Γ_0 for $t \in [0, T]$ implies that $\varphi = 0$ in $Q = \Omega \times]0, T[$.*

References

[1] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, *SIAM J. Cont. Optim.*, **30**, No. 4 (1992), 1024-1065.

[2] N. Burq, Controlabilite exacte des ondes dans des ouverts peu reguliers, *Asymptotic Analysis*, **14** (1997), 157-191.

[3] R.F. Curtain, H.J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Texts in Applied Mathematics, **21**, Springer-Verlag (1995).

[4] S. Dolecki, D.L. Russel, A general theory of observation and control, *SIAM J. Cont. Optim.*, **15** (1977), 185-220.

[5] A. El Jay, A.J. Pritchard, *Sensors and Controls in the Analysis of Distributed Systems*, Ellis Horwood Series in Mathematics and its Applications (1988).

[6] H.O. Fattorini, Un theorema de perturbation para generatores de funciones coseno, *Rev. Un. Mat. Argentina*, **25** (1971), 199-211.

[7] H.O. Fattorini, Ordinary differential equations in linear topological spaces, I, *J. Differential Equations*, **5** (1968), 72-105.

[8] H.O. Fattorini, Ordinary differential equations in linear topological spaces, II, *J. Differential Equations*, **6** (1969), 50-70.

[9] G. Grubb, *Functional Calculus of Pseudo-Differential Boundary Problems*, Progress in Math., **65**, Birkhäuser, Boston, Basel, Stuttgart (1986).

[10] I. Lasiecka, J-L. Lions, R. Triggiani, Non homogeneous boundary value problems for second order hyperbolic operators, *J. Math. Pures. Appl.*, **65** (1986), 149-192.

[11] J.-L. Lions, Exact controllability, stabilizability and perurbations for distributed systems, *SIAM Rev.*, **30** (1988), 1-68.

[12] J.-L. Lions, *Controlabilite Exacte, Stabilisation et Perturbations de Systemes Distribues*, Tomes 1, 2, Masson, RMA 8,9, Paris (1988).

[13] M. Pedersen, Boundary feedback stabilization of distributed parameter systems: An application of pseudo-differential operators, In: *Proc. 27-th IEEE Conf. on Decision and Control*, Austin, Texas (1988), 366-368.

[14] M. Pedersen, Pseudo-differential perturbations and stabilization of distributed parameter systems: Dirichlet feedback control problems, *SIAM J. Cont. Optim.*, **29**, No. 1 (1991), 222-252.

[15] M. Pedersen, Symbolic boundary calculus and feedback operators, *Int. Series of Numerical Mathematics*, **100** (1991), 265-272.

[16] M. Pedersen, State space representation of non-local boundary value problems, *J. Math. Systems, Estimation and Control*, **2**, No. 2 (1992), 237-244.

[17] M. Pedersen, Regularity of pointwise boundary control systems, *J. Diff. and Int. Eq.*, **5**, No. 1 (1992), 137-148.

[18] M. Pedersen, A pseudo-differential approach to distributed parameter systems and stabilization: Neumann feedback control problems, *J. Math. Anal. Appl.*, **174**, No. 2 (1993), 589-613.

[19] M. Pedersen, G. Chen, Regularities and pseudodifferential symbols of boundary layer operators for elastostatic thin Kirchoff biharmonic plate, In: *Proc. Sino-Japanese Joint Workshop on Operator Theory and Differential Eq. Academica Sineca*, Tapei (1994).

[20] M. Pedersen, *Functional Analysis in Applied Matematics and Engineering*, Chapmann and Hall/CRC (1999).

[21] E. Zuazua, Exponential decay for semilinear wave equations with localized damping, *Comm. PDE.*, **15**, No. 2 (1990), 205-235.

[22] E. Zuazua, Exact controllability for the semilinear wave equation, *J. Math. Pures. Appl.*, **69** (1990), 33-55.

[23] E. Zuazua, Exact boundary controllability for the semilinear wave equation, *Nonlinear Partial Differential Equations and their Applications*, **X** (Ed-s: H. Brezis, J.-L. Lions), Pitman (1991), 357-391.

[24] E. Zuazua, Exact controllability for semilinear wave equations, *Ann. Inst. Henri Poincare*, **10**, No. 1 (1993), 109-129.

[25] E. Zuazua, Stability and decay for a class of of nonlinear hyperbolic problems, *Asymptotic Analysis*, **1**, No. 2 (1988), 1-28.

[26] E. Zuazua, Uniform stabilization of the wave equation by nonlinear boundary feedback, *SIAM J. Cont. Optim.*, **28**, No. 2 (1990), 466-477.

[27] E. Zuazua, Approximate controllability for semilinear heat equations with globally Lipschitz nonlinearities, *Control and Cybernetics*, **28**, No. 3 (1999), 665-683.

[28] E. Zuazua, Controllability of partial differential equations and its semi-discrete approximations, *Discrete and Continuous Dynamical Systems*, **8**, No. 2 (2002), 469-513.

[29] E. Zuazua, S. Micu, *An Introduction to the Controllability of Partial Differential Equations*, Lecture Notes (2004).