International Journal of Pure and Applied Mathematics

Volume 37 No. 3 2007, 321-348

THE p-OPTIMAL MARTINGALE MEASURE WHEN THERE EXIST INACCESSIBLE JUMPS

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Abstract: We consider the p-optimal martingale measure (the general importance of these tools is shortly described in the conclusion) in an incomplete financial market model with inaccessible jumps described by a random jump measure. Using a dynamic programming approach, we obtain a backward martingale equation (BME) with the property that if the BME has a solution, then the p-optimal martingale measure is equivalent to the original measure. Furthermore we give a description of the p-optimal martingale measure by the solution of the BME.

In a simple case similar to Jeanblanc, Kloeppel and Miyahara, see [14], we give an explicit solution of the BME. As an application, we consider the optimal utility of an investor with utility function $U(x) = -|1 - \frac{x}{k_0}|^q$, and explicitly derive the optimal strategy by the solution of the BME.

AMS Subject Classification: 90A09, 60H30, 60G44
Key Words: optimality principle, backward martingale equations, stochastic Riccati equation, p-optimal martingale measure

1. Introduction

We consider a financial market model, in which inaccessible jumps exist de-

Received: March 21, 2007

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scribed by a random measure (see e.g. Jacod, [13]). In this market there exists a risky asset whose discounted price process $S = (S_t)_{0\leq t\leq T}$ is a semimartingale with jumps. Since too large jumps are not allowed in most markets, it is reasonable to assume that the jumps of the price process are bounded. More precisely, assume that $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is a filtration satisfying the usual conditions on a complete probability space $(\Omega, \mathcal{F}, P)$ and the price process S is a semimartingale with the following decomposition

$$dS_t = S_0 + \int_0^t \alpha_u dA_u + \int_0^t \sigma_1(u) dM_u + \int_0^t \int_{\mathbb{R}} \sigma_2(u, y)\widetilde{N}(du, dy),$$

where:

— $A = (A_t)_{0\leq t\leq T}$ is an $\mathbb{F}$-adapted continuous increasing process with $A_0 = 0$ and $EA_T < \infty$;

— $M = (M_t)_{0\leq t\leq T}$ is a continuous local martingale with $\langle M, M\rangle_t = A_t$;

— $\mu = \{\mu(\omega; dt, dx) : \omega \in \Omega\}$ is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\nu(\omega; dt, dx) = dA_t(\omega)K(\omega, t; dx)$, where $K(\omega, t; dx)$ is a kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}, \mathcal{B})$. Let $\widetilde{N}(\omega; dt, dx) := \mu(\omega; dt, dx) - \nu(\omega; dt, dx)$.

We assume that α and σ_1 are two predictable processes with $\int_0^T \sigma_1(s)^2 dA_s < \infty$ a.s. and $\sigma_2(t, y)$ is a bounded $\widetilde{\mathcal{P}}$-measurable function.

When S is a Lévy process or a geometric Lévy process the model has been widely studied under different sets of assumptions in e.g. Chan [4], Cont and Tankov [5], Cont, Tankov and Voltchkova [6], Kunita [19], Jeanblanc, Kloeppel and Miyahara [14]; also see the references in the cited articles. It is commonly known that – as in our case the (jump-) market is incomplete- there exist many equivalent martingale measures. In a very general market model these martingale measures are signed measures. Unfortunately, these measures have not yet been extensively studied and there is little known about existence and properties, so that we will exclude this possibility. By $\mathcal{M}^a$ we denote the set of measures Q absolutely continuous with respect to P on $\mathcal{F}_T$ such that S is a local martingale under Q. Let $\mathcal{M}^e$ be a set of equivalent martingale measures, i.e., $\mathcal{M}^e := \{Q; Q \in \mathcal{M}^a$ and $Q \sim P\}$. For $p > 1$, let

$$\mathcal{M}_p^e := \left\{ Q \in \mathcal{M}^e : E\big\{\big(\frac{dQ}{dP}\big)^p\big\} < \infty \right\}.$$

As a no-arbitrage condition for our market we let $\mathcal{M}_p^e \neq \emptyset$, and our interest is mainly focused on the following optimization problem:

$$\min_{Q\in\mathcal{M}_p^e} E\left\{ \big(\frac{dQ}{dP}\big)^p \right\}, \quad p \geq 1. \tag{1}$$

We call the solution of the problem (1) the *p-optimal martingale measure*.

Mania, Santacroce and Tevzadze [21] considered the p-optimal martingale measure when every local martingale is continuous. Using a dynamic programming approach, they give a description of the p-optimal martingale measure in terms of the value process and show that this value process uniquely solves an appropriate semimartingale backward equation. In their paper, the p-optimal martingale measure Q^* is always equivalent to the original measure P, i.e., $Q \in \mathcal{M}_p^e$. However, this is NOT true when we turn to the case the market contains jumps, even when the price process S is continuous if $p \neq 2$ (when $p = 2$ and S is continuous, Delbaen and Schachermayer [7] showed that the variance optimal martingale measure Q^* is equivalent to P and by considering properties of the associated dual problem Grandits and Krawczyk [11]) prove existence of the $p(\neq 2)$-optimal martingale measure under further conditions like the Reverse Hölder condition. Therefore in the jump case under consideration here, it is necessary to look for appropriate conditions ensuring that Q^* is equivalent to P in the more general model above.

In this paper, we first introduce a new martingale measure $Q^0 \in \mathcal{M}_p^e$ whose density process $Z^0 = (Z_t^0)_{0 \leq t \leq T}$ is a continuous uniformly integrable martingale and $Z_T^0 \in L^p(P)$. Like in Kohlmann, Xiong and Ye [17], we then treat the problem under Q^0 as the basic underlying measure. By using the dynamic programming approach (see El Karoui and Quenez [15] or Laurent and Pham [20]) we establish a backward martingale equation (BME) to show that if the BME has a solution, then the p-optimal martingale measure Q^* is equivalent to P. We give a description of Q^* in terms of the solution of the BME. Especially, when the price process S is a continuous process (i.e. $\sigma_2(u, y) = 0$), we derive the result that "the BME has a unique solution" is equivalent to "$Q^* \sim P$". Similar to Mania, Santacroce and Tevzadze [21], we show in Theorem 3.6 that "the BME has a solution" is equivalent to "there exists an $l^*(u, y)$ such that

$$(Z_T^0)^{p-1} Z_T (l^*)^{p-1} = c + \int_0^T h_u dS_u$$

for a constant c and an X-integrable predictable process h such that $(\int_0^t h_u dS_u)_{t \in [0,T]}$ is a Q-martingale for every $Q \in \mathcal{M}_p^e$." As an example, we consider in Section 4 the simpler case similar to Jeanblanc, Kloeppel and Miyahara [14], for which we give an explicit form of the solution of the BME. As an application, we consider in Section 5 the optimal utility of an investor with utility function $U(x) = -|1 - \frac{x}{k_0}|^q$, and give an explicit form of the optimal strategy by means of the solution of the BME. This generalizes the basic results in Kohlmann and Niethammer [16] what can be used to study the convergence of the p-optimal measures and the convergence of the dual isoelastic control problems to the minimal entropy and the exponential control problem, respectively.

2. The Preliminaries

We start with a finite time horizon $T > 0$ and a complete probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ be a filtration satisfying the usual conditions with $\mathcal{F}_T = \mathcal{F}$. We assume that there exists a risk asset in the market, whose discounted price $S = (S_t)_{0\leq t\leq T}$ is described by a special semimartingale with bounded jumps. To describe the dynamics of S, we need the following notations:

— $A = (A_t)_{0\leq t\leq T}$ is an $\mathbb{F}$-adapted continuous increasing process with $A_0 = 0$ and $EA_T < \infty$;

— $M = (M_t)_{0\leq t\leq T}$ is a continuous local martingale with $\langle M, M\rangle_t = A_t$;

— $\mu = \{\mu(\omega; dt, dx) : \omega \in \Omega\}$ is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\nu(\omega; dt, dx) = dA_t(\omega)K(\omega, t; dx)$, where $K(\omega, t; dx)$ is a kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}, \mathcal{B})$. Let $\widetilde{N}(\omega; dt, dx) := \mu(\omega; dt, dx) - \nu(\omega; dt, dx)$.

Throughout the paper, we make use of the following representation property

Assumption 2.1. *Any local $(P, \mathbb{F})$-martingale m can be represented in the following form*

$$m_t = m_0 + \int_0^t \phi_s dM_s + \int_0^t \int_{\mathbb{R}} \psi(s, x)\widetilde{N}(ds, dx).$$

Remark. 1. When $\mathbb{F}$ is the filtration generated by a Wiener process and a marked point process as in Runggaldier(2002) or the filtration generated by a Lévy process as in Kunita(2004), Assumption 2.1 is satisfied.

2. As A is a continuous process, $a_t(\omega) := \nu(\omega, \{t\} \times \mathbb{R}) = 0$. Thus for all $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}$-measurable function $W(\omega, t, x)$, let

$$\widehat{W}_t(\omega) := \begin{cases} \displaystyle\int_{\mathbb{R}} W(\omega, t, x)\nu(\omega; \{t\} \times dx)\,, & \\ & \text{if } \displaystyle\int_{\mathbb{R}} |W(\omega, t, x)|\nu(\omega; \{t\} \times dx) < \infty\,, \\ +\infty\,, & \text{otherwise}\,, \end{cases}$$

then $\widehat{W} = 0$. Thus Assumption 2.1 implies that the filtration $\mathbb{F}$ is quasi-left continuous.

3. It is easy to see that there exists a $\mathbb{F}$-optional process $\beta = (\beta_t)_{0\leq t\leq T}$ and a sequence of stopping times $(\hat{\tau}_n)$ such that for all positive $\widetilde{\mathcal{P}}$-measurable function $W(\omega, t, x)$,

$$W * \mu_t := \int_0^t \int_{\mathbb{R}} W(\omega, u, x)\mu(\omega; du, dx) = \sum_{(n)} W(\hat{\tau}_n, \beta_{\hat{\tau}_n}) I_{\{\hat{\tau}_n \leq t\}}.$$

We now describe the market model: We assume the dynamics of the discounted price process S is given by

$$dS_t = S_0 + \int_0^t \alpha_u dA_u + \int_0^t \sigma_1(u)dM_u + \int_0^t \int_{\mathbb{R}} \sigma_2(u, y)\widetilde{N}(du, dy),$$

where α and σ_1 are two predictable processes with $\int_0^T \sigma_1(s)^2 dA_s < \infty$ a.s. and $\sigma_2(t, y)$ is a bounded $\widetilde{\mathcal{P}}$-measurable function. We assume that there exist two constants k and K such that

$$\int_0^T \alpha_u^2 dA_u \leq K < \infty \text{ and } \sigma_1(s) \geq k > 0. \tag{2}$$

Remark. If we let

$$\widehat{M}_t := \int_0^t \sigma_1(u)dM_u + \int_0^t \int_{\mathbb{R}} \sigma_2(u, y)\widetilde{N}(du, dy),$$

then by a simple calculation, we can see that

$$\langle \widehat{M} \rangle_t = \int_0^t \big\{\sigma_1(u)^2 + \int_{\mathbb{R}} \sigma_2(u, y)^2 K(u; dy)\big\} dA_u \ .$$

Thus S can be rewritten as

$$S_t = S_0 + \int_0^t \widehat{\lambda}_u d\langle \widehat{M} \rangle_u + \widehat{M}_u,$$

where

$$\widehat{\lambda}_u := \frac{\alpha_u}{\sigma_1(u)^2 + \displaystyle\int_{\mathbb{R}} \sigma_2(u, y)^2 K(u; dy)}.$$

So, we see that S satisfies the structure condition (SC) and the mean-variance tradeoff is given by

$$\begin{aligned} K_t : &= \int_0^t \widehat{\lambda}_u^2 d\langle \widehat{M} \rangle_u \\ &= \int_0^t \frac{\alpha_u^2}{\sigma_1(u)^2 + \int_{\mathbb{R}} \sigma_2(u, y)^2 K(u; dy)} dA_u \ . \end{aligned}$$

Therefore, under condition (2), the mean-variance tradeoff K_t is a bounded process. However, the minimal martingale measure $\widehat{Q}$ might not be equivalent to P and $\widehat{Q} \notin \mathcal{M}_p^e$. We have the following lemma.

Lemma 2.2. *Let $\lambda_t = \frac{\alpha_t}{\sigma_1(t)}$, and define $\frac{dQ^0}{dP}\big|_{\mathcal{F}} := Z_T^0$, where $Z_t^0 = \mathcal{E}(-\int \lambda_u dM_u)_t$, then $Q^0 \in \mathcal{M}_p^e$ and Q^0 satisfies the reverse Hölder inequality $R_p(P)$, i.e. there exists a constant C such that*

$$E\left((Z_T^0)^p|\mathcal{F}_\tau\right) \leq C(Z_\tau^0)^p$$

for any stopping time τ.

Proof. It is easy to see that $\int_0^T \lambda_u^2 dA_u \leq \frac{1}{k^2}\int_0^T \alpha_u^2 dA_u \leq \frac{K}{k^2} < \infty$, thus $Q^0 \in \mathcal{M}^e$. By Itô's formula, we have

$$(Z_t^0)^p = \mathcal{E}\big(-p\int \lambda_u dM_u\big)_t \exp\big(\frac{p(p-1)}{2}\int_0^t \lambda_u^2 dA_u\big),$$

thus there exists a constant C such that

$$E\{(Z_t^0)^p\} \leq CE\mathcal{E}\big(-p\int \lambda_u dM_u\big)_T = C < \infty$$

and

$$E\left(\frac{(Z_T^0)^p}{(Z_\tau^0)^p}|\mathcal{F}_\tau\right) \leq CE\mathcal{E}(-p\int \lambda_u I_{\{u\geq\tau\}}dM_u)_T = C < \infty. \qquad \square$$

From Lemma 2.2, we see that $\mathcal{M}_p^e \neq \emptyset$, so that we may now consider the p-optimal martingale measure. It is more convenient for us to consider this problem under Q^0. Let $\widetilde{M}_t := M_t + \int_0^t \lambda_u dA_u$, it is easy to see that $\widetilde{M}$ is a continuous local Q^0-martingale. Under Q^0, the discounted price process S can be rewritten as

$$S_t = S_0 + \int_0^t \sigma_1(u)d\widetilde{M}_u + \int_0^t\int_{\mathbb{R}} \sigma_2(u,y)\widetilde{N}(du,dy).$$

For all $Q \in \mathcal{M}^e$, let $Z_t^{Q,Q^0} = E_{Q^0}\big(\frac{dQ}{dQ^0}|\mathcal{F}_t\big)$, there exists a predictable process $l_1(t)$ and a $\widetilde{\mathcal{P}}$-measurable function $l_2(t,y) > -1$ such that

$$Z_t^{Q,Q^0} = \mathcal{E}\{\int_0^\cdot l_1(u)d\widetilde{M}_u + \int_0^\cdot\int_{\mathbb{R}} l_2(u,y)\widetilde{N}(du,dy)\}_t\ .$$

Since SZ^{Q,Q^0} is a local Q^0-martingale, $[S, Z^{Q,Q^0}]$ is a local Q^0-martingale. On the other hand,

$$[S, Z^{Q,Q^0}]_t = S_0 + \int_0^t Z^{Q,Q^0}_{u-}\sigma_1(u)l_1(u)dA_u + \int_0^t \int_{\mathbb{R}} Z^{Q,Q^0}_{u-}\sigma_2(u,y)l_2(u,y)\mu(du,dy) = S_0 + \int_0^t Z^{Q,Q^0}_{u-}\{\sigma_1(u)l_1(u) + \int_{\mathbb{R}} \sigma_2(u,y)l_2(u,y)K(\omega,u;dy)\}dA_u + \int_0^t \int_{\mathbb{R}} Z^{Q,Q^0}_{u-}\sigma_2(u,y)l_2(u,y)\widetilde{N}(du,dy),$$

which implies that

$$l_1(u) = -\frac{1}{\sigma_1(u)}\int_{\mathbb{R}} \sigma_2(u,y)l_2(u,y)K(\omega,u;dy) \qquad dA_u \times dP\text{-a.s.}$$

For $l(t,y) \in G^1_{\mathrm{loc}}(\mu)$, we introduce

$$\widetilde{l}(u) := \int_{\mathbb{R}} \frac{\sigma_2(u,y)}{\sigma_1(u)} l(u,y)K(\omega,u;dy),$$

and

$$Z_t(l) := \mathcal{E}\{-\int_0^{\cdot} \widetilde{l}(u)d\widetilde{M}_u + \int_0^{\cdot}\int_{\mathbb{R}} l(u,y)\widetilde{N}(du,dy)\}_t, \tag{3}$$

and let

$$\mathcal{L}_p(Q^0) = \left\{ \begin{array}{ll} l(t,y); & l \text{ is } \widetilde{\mathcal{P}}\text{-measurable function such that } Z(l) \\ & \text{ is a uniformly } Q^0\text{-martingale with} \\ & E_{Q^0}\left\{(Z^0_T)^{p-1}Z_T(l)^p\right\} < \infty \end{array} \right\},$$

then $\mathcal{M}^e_p$ can be rewritten as

$$\mathcal{M}^e_p = \left\{ Q \sim Q^0 : \frac{dQ}{dQ^0}\Big|_{\mathcal{F}_T} = Z_T(l), l \in \mathcal{L}_p(Q^0) \right\}.$$

Thus the problem (1) can be represented as the following

$$\min_{l \in \mathcal{L}_p(Q^0)} E_{Q^0}\left\{ \left(Z^0_T\right)^{p-1} Z_T(l)^p \right\}, \quad p \geq 1. \tag{4}$$

3. Backward Semimartingale Equation

To make our notation more simple, in this section we let $\xi = \left(Z_T^0\right)^{p-1}$. For $l \in \mathcal{L}_p(Q^0)$ and for a fixed $t \in [0,T]$, let

$$Z_{t,s}(l) := \mathcal{E}\Big\{ -\int_0^{\cdot} \widetilde{l}(u)1_{u>t}d\widetilde{M}_u + \int_0^{\cdot}\int_{\mathbb{R}} l(u,y)1_{u>t}\widetilde{N}(du,dy)\Big\}_s ,$$

then the value process of the problem (4) is given by

$$V_t(p) := \operatorname*{ess\,inf}_{l\in\mathcal{L}_p(Q^0)} E_{Q^0}\left\{\xi Z_{t,T}(l)^p \middle| \mathcal{F}_t\right\}.$$

We now consider the backward semimartingale equation of V. Similar to El Karoui and Quenez [15], we have the following lemma.

Lemma 3.1. (Optimality Principle) *1) There exists an RCLL semimartingale, still denoted by $V_t(p)$, such that for each $t \in [0,T]$*

$$V_t(p) = \operatorname*{ess\,inf}_{l\in\mathcal{L}_p(Q^0)} E_{Q^0}\left\{\xi Z_{t,T}(l)^p \middle| \mathcal{F}_t\right\}, \quad Q^0\text{-a.s.}$$

$V_t(p)$ is the largest RCLL process equal to $(Z_T^0)^{p-1}$ at time T such that $Z_t(l)^pV_t(p)$ is a Q^0-submartingale for every $l \in \mathcal{L}_p(Q^0)$.

2) The following two properties are equivalent:

(i) there exists a $l^* \in \mathcal{L}_p(Q^0)$ such that $Z_t(l^*)^p\, V_t(p)$ is a Q^0-martingale;

(ii) there exists a $l^* \in \mathcal{L}_p(Q^0)$ such that the measure Q^* defined by

$$\left.\frac{dQ^*}{dQ^0}\right|_{\mathcal{F}_T} := Z_T(l^*),$$

is p-optimal martingale measure.

Proof. For a fixed t, let

$$\Gamma_t := \left\{E_{Q^0}\big[\xi Z_{t,T}(l)^p\big|\mathcal{F}_t\big]\,; \quad l \in \mathcal{L}_p(Q^0)\right\},$$

then we see that Γ_t is stable under the $\wedge$-operation. Indeed, for any $l_1, l_2 \in \mathcal{L}_p(Q^0)$, let

$$A = \big\{\omega; E_{Q^0}\big[\xi Z_{t,T}(l_1)^p\big|\mathcal{F}_t\big] \le E_{Q^0}\big[\xi Z_{t,T}(l_2)^p\big|\mathcal{F}_t\big]\big\} ,$$

it is easy to see that A is $\mathcal{F}_t$-measurable. Thus

$$
\begin{aligned}
&E_{Q^0}\big[\xi Z_{t,T}(l_1)^p \big| \mathcal{F}_t\big] \wedge E_{Q^0}\big[\xi Z_{t,T}(l_2)^p \big| \mathcal{F}_t\big] \\
&\qquad = E_{Q^0}\big[\xi\big\{Z_{t,T}(l_1)^p I_A + Z_{t,T}(l_2)^p I_{A^c}\big\}\big|\mathcal{F}_t\big] \\
&\qquad = E_{Q^0}\big[\xi\big\{Z_{t,T}(l_1) I_A + Z_{t,T}(l_2) I_{A^c}\big\}^p\big|\mathcal{F}_t\big] \\
&\qquad = E_{Q^0}\big[\xi Z_{t,T}(l^*)^p \big| \mathcal{F}_t\big] ,
\end{aligned}
$$

where

$$
\begin{aligned}
l^*(s,y) &= \frac{Z_{t,s-}(l_1) l_1(s,y) I_A + Z_{t,s-}(l_2) l_2(s,y) I_{A^c}}{Z_{t,s-}(l_1) I_A + Z_{t,s-}(l_2) I_{A^c}} I_{s>t} \\
&= \{l_1(s,y) I_A + l_2(s,y) I_{A^c}\} I_{s>t} \in \mathcal{L}_p(Q^0).
\end{aligned}
$$

Thus $E_{Q^0}\big[\xi Z_{t,T}(l_1)^p \big| \mathcal{F}_t\big] \wedge E_{Q^0}\big[\xi Z_{t,T}(l_2)^p \big| \mathcal{F}_t\big] \in \Gamma_t$. The rest of the proof is the same as El Karoui and Quenez [15] or Laurent and Pham [20]. □

We now consider the following backward martingale equation

$$
\begin{cases}
a_t = a_0 + \displaystyle\int_0^t a_{u-}\left[\sigma_1(u)\theta_1(u) + (p-1)\widetilde{f(\theta_2)}_u\right] d\widetilde{M}_u \\
\quad + \displaystyle\int_0^t \int_{\mathbb{R}} a_{u-}\Big\{[1+\sigma_2(u,y)\theta_1(u)][1+\theta_2(u,y)] - 1\Big\}\widetilde{N}(du,dy) \\
\quad + \dfrac{p(p-1)}{2}\displaystyle\int_0^t a_{u-}\widetilde{f(\theta_2)}_u^2 dA_u \\
\quad + \displaystyle\int_0^t \int_{\mathbb{R}} a_{u-}[1+\sigma_2(u,y)\theta_1(u)]\{\theta_2(u,y) \\
\quad + (p-1) f(\theta_2)(u,y)\} K(u;dy) dA_u \\
a_T = (Z_T^0)^{p-1} ,
\end{cases}
\tag{5}
$$

where

$$
f(\theta_2)(u,y) := (1+\theta_2(u,y))^{-\frac{1}{p-1}} - 1
$$

and

$$
\widetilde{f(\theta_2)}_u := \int_{\mathbb{R}} \frac{\sigma_2(u,y)}{\sigma_1(u)}[(1+\theta_2(u,y))^{-\frac{1}{p-1}} - 1] K(u;dy).
$$

Definition 3.2. A *solution* of (5) is a pair (a, θ_1, θ_2) satisfying (5) such that

(i) θ_1 is a predictable process with $\theta_1(u)\sigma_2(u,y) > -1$ for all (u,y), θ_2 is a $\widetilde{\mathcal{P}}$-measurable $\widetilde{N}$-integrable function with $\theta_2 > -1$ and a is a strictly positive RCLL Q^0-semimartingale;

(ii) for all $l \in \mathcal{L}_p(Q^0)$, $a_t Z_t(l)^p$ belongs to the class (D), i.e., the family of random variables $a_\tau Z_\tau(l)^p I_{\tau \le T}$ for all stopping times τ is uniformly integrable;

(iii) $\mathcal{E}\big\{-\int_0^{\cdot}\widetilde{f(\theta_2)}_u d\widetilde{M}_u+\int_0^{\cdot}\int_{\mathbb{R}} f(\theta_2)(u,y)d\widetilde{N}(du,dy)\big\}_t$ is a uniformly integrable Q^0-martingale.

Remark. Note that a similar BME was derived in Bobrovnytska and Schweizer [3] for the mean variance case. The situation here, however, is much more general.

Theorem 3.3. *If (5) has a solution which is denoted by (a,θ_1,θ_2), then $a_t=V_t(p)$ a.s. for all $t\in[0,T]$ and the p-optimal martingale measure Q^* is equivalent to P which is given by*

$$\frac{dQ^*}{dQ^0}\Big|_{\mathcal{F}_T}=\mathcal{E}\big\{-\int_0^{\cdot}\widetilde{f(\theta_2)}_u d\widetilde{M}_u+\int_0^{\cdot}\int_{\mathbb{R}} f(\theta_2)(u,y)d\widetilde{N}(du,dy)\big\}_T\,. \tag{6}$$

To prove the above theorem, we need the following lemma:

Lemma 3.4. *Let $\Psi(x_1,x_2):=((1+x_1)^p-1)(1+x_2)-px_1,\ x_1>-1,x_2>0$. For a fixed $x_2>-1$, let*

$$f(x_2)=(1+x_2)^{-\frac{1}{p-1}}-1,$$

then for all $x_1>-1$,

$$\Psi(x_1,x_2)\ge\Psi(f(x_2),x_2).$$

And $\Psi(f(x_2),x_2)=p-(1+x_2)-(p-1)(1+x_2)^{-\frac{1}{p-1}}=-x_2-(p-1)f(x_2)$.

Proof of Theorem 3.3. By Itô's formula, we see

$$\begin{aligned}Z_t(l)^p&=1-\int_0^t pZ_{u-}(l)^p\,\widetilde{l}(u)d\widetilde{M}_u\\&+\int_0^t\int_{\mathbb{R}} Z_{u-}(l)^p\left\{(1+l(u,y))^p-1\right\}\widetilde{N}(du,dy)+\int_0^t Z_{u-}(l)^p\left\{\frac{1}{2}p(p-1)\widetilde{l}(u)^2\right.\\&\left.+\int_{\mathbb{R}}\left\{(1+l(u,y))^p-1-p\,l(u,y)\right\}K(u;dy)\right\}dA_u\,.\end{aligned}$$

And once again by applying Itô's formula to $a_tZ_t(l)$,

$$\begin{aligned}a_tZ_t(l)^p&=a_0+\int_0^t a_{u-}Z_{u-}(l)^p\left[\sigma_1(u)\theta_1(u)+(p-1)\widetilde{f(\theta_2)}_u\right]d\widetilde{M}_u\\&+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]-1\Big\}\widetilde{N}(du,dy)\\&+\frac{p(p-1)}{2}\int_0^t a_{u-}Z_{u-}(l)^p\widetilde{f(\theta_2)}_u^2 dA_u\end{aligned}$$

$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p[1+\sigma_2(u,y)\theta_1(u)]\{\theta_2(u,y)+(p-1)f(\theta_2)(u,y)\}K(u;dy)dA_u$$
$$-\int_0^t pa_{u-}Z_{u-}(l)^p\,\widetilde{l}(u)d\widetilde{M}_u+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\left\{(1+l(u,y))^p-1\right\}\widetilde{N}(du,dy)$$
$$+\int_0^t a_{u-}Z_{u-}(l)^p\left\{\frac{p(p-1)}{2}\widetilde{l}(u)^2+\int_{\mathbb{R}}\left\{(1+l(u,y))^p-1-p\,l(u,y)\right\}K(u;dy)\right\}dA_u$$
$$-\int_0^t pa_{u-}Z_{u-}(l)^p\left[\sigma_1(u)\theta_1(u)+(p-1)\widetilde{f(\theta_2)}_u\right]\widetilde{l}(u)dA_u$$
$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]-1\Big\}\left\{(1+l(u,y))^p-1\right\}\widetilde{N}(du,dy)$$
$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]-1\Big\}\times\left\{(1+l(u,y))^p-1\right\}K(u;dy)dA_u$$

$$a_tZ_t(l)^p=a_0+\int_0^t a_{u-}Z_{u-}(l)^p\left[\sigma_1(u)\theta_1(u)+(p-1)\widetilde{f(\theta_2)}_u-p\,\widetilde{l}(u)\right]d\widetilde{M}_u$$
$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]\,[1+l(u,y)]^p-1\Big\}\widetilde{N}(du,dy)$$
$$+\int_0^t a_{u-}Z_{u-}(l)^p\Big\{\frac{p(p-1)}{2}[\widetilde{f(\theta_2)}_u-\widetilde{l}(u)]^2$$
$$+(p-1)\sigma_1(u)\theta_1(u)\widetilde{f(\theta_2)}_u-p\sigma_1(u)\theta_1(u)\,\widetilde{l}(u)\Big\}dA_u$$
$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{\theta_2(u,y)[1+\sigma_2(u,y)\theta_1(u)]-(1-p)f(\theta_2)(u,y)$$
$$+[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]\left\{(1+l(u,y))^p-1\right\}-p\,l(u,y)\Big\}K(u;dy)dA_u$$
$$=a_0+\int_0^t a_{u-}Z_{u-}(l)^p\left[\sigma_1(u)\theta_1(u)+(p-1)\widetilde{f(\theta_2)}_u-p\,\widetilde{l}(u)\right]d\widetilde{M}_u$$
$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]\,[1+l(u,y)]^p-1\Big\}\widetilde{N}(du,dy)$$
$$+\frac{p(p-1)}{2}\int_0^t a_{u-}Z_{u-}(l)^p[\widetilde{f(\theta_2)}_u-\widetilde{l}(u)]^2dA_u$$
$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{(p-1)\sigma_2(u,y)\theta_1(u)f(\theta_2)(u,y)-p\sigma_2(u,y)\theta_1(u)\,l(u,y)\Big\}K(u;dy)dA_u$$
$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{\theta_2(u,y)[1+\sigma_2(u,y)\theta_1(u)]-(1-p)f(\theta_2)(u,y)$$
$$+[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]\left\{(1+l(u,y))^p-1\right\}-p\,l(u,y)\Big\}K(u;dy)dA_u$$
$$=a_0+\int_0^t a_{u-}Z_{u-}(l)^p\left[\sigma_1(u)\theta_1(u)+(p-1)\widetilde{f(\theta_2)}_u-p\,\widetilde{l}(u)\right]d\widetilde{M}_u$$
$$+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]\,[1+l(u,y)]^p-1\Big\}\widetilde{N}(du,dy)$$

$$
\begin{aligned}
&+\frac{p(p-1)}{2}\int_0^t a_{u-}Z_{u-}(l)^p[\widetilde{f(\theta_2)}_u-\widetilde{l}(u)]^2dA_u \\
&+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p[1+\sigma_2(u,y)\theta_1(u)]\Big\{\theta_2(u,y)+(p-1)f(\theta_2)(u,y) \\
&\quad+\{(1+l(u,y))^p-1\}-p\,l(u,y)\Big\}K(u;dy)dA_u \\
&=a_0+\int_0^t a_{u-}Z_{u-}(l)^p\left[\sigma_1(u)\theta_1(u)+(p-1)\widetilde{f(\theta_2)}_u-p\,\widetilde{l}(u)\right]d\widetilde{M}_u \\
&+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p\Big\{[1+\theta_2(u,y)][1+\sigma_2(u,y)\theta_1(u)]\,[1+l(u,y)]^p-1\Big\}\widetilde{N}(du,dy) \\
&+\frac{p(p-1)}{2}\int_0^t a_{u-}Z_{u-}(l)^p[\widetilde{f(\theta_2)}_u-\widetilde{l}(u)]^2dA_u \\
&+\int_0^t\int_{\mathbb{R}} a_{u-}Z_{u-}(l)^p[1+\sigma_2(u,y)\theta_1(u)]\{-\Psi(f(\theta_2)(u,y),\theta_2(u,y)) \\
&\quad+\Psi(l(u,y),\theta_2(u,y))\}K(u;dy)dA_u\ .
\end{aligned}
$$

By Lemma 3.4, for any $l\in\mathcal{L}_p(Q^0)$,

$$\Psi(l(u,y),\theta_2(u,y))\geq\Psi(f(\theta_2)(u,x),\theta_2(u,x)),$$

thus $a_tZ_t(l)^p$ is a local submartingale for all $l\in\mathcal{L}_p(Q^0)$. By Definition 3.2 (ii), $a_tZ_t(l)^p$ belongs to the class (D). Therefore, $a_tZ_t(l)^p$ is a true submartingale for all $l\in\mathcal{L}_p(Q^0)$. As $a_T=(Z_T^0)^{p-1}$, by Lemma 3.1, we have

$$a_t\leq V_t(p)\ .$$

On the other hand, let $\widehat{l}^*(u,y):=f(\theta_2)(u,y)$, by Definition 3.2 (iii), we see that $Z_t(\widehat{l}^*)$ is a uniformly integrable martingale under Q^0, thus we can define a new measure $\widehat{Q}^*$ by

$$\frac{d\widehat{Q}^*}{dQ^0}\Big|_{\mathcal{F}}=Z_T(\widehat{l}^*),$$

which is equivalent to Q^0, since $\widehat{l}^*>-1$. It is easy to see that $a_tZ_t(\widehat{l}^*)^p$ is a positive local Q^0-martingale, thus a Q^0-supermartingale. Therefore,

$$E\{(\frac{d\widehat{Q}^*}{dP})^p\}=E_{Q^0}\{a_TZ_T(\widehat{l}^*)^p\}\leq a_0<\infty,$$

from which we know that $\widehat{Q}^*\in\mathcal{M}_p^e$ and $\widehat{l}^*\in\mathcal{L}_p(Q^0)$. And once again by Defintion 3.2 (ii), $a_tZ_t(\widehat{l}^*)^p$ belongs to the class (D), thus $a_tZ_t(\widehat{l}^*)^p$ is a uniformly Q^0-martingale. Thus,

$$a_t=E_{Q^0}(Z_{t,T}(a_T\widehat{l}^*)^p|\mathcal{F}_t)\geq \operatorname*{ess\,inf}_{l\in\mathcal{L}_p(Q^0)}E_{Q^0}\left\{a_TZ_{tT}(l)^p\middle|\mathcal{F}_t\right\}=V_t(p).$$

Thus $a_t = V_t$ a.s. for all $t \in [0,T]$. From this we see that $\widehat{Q}^*$ is the p-optimal martingale measure. $\square$

When $\sigma_2(u,y) = 0$, i.e., S is a continuous process, then for all $l \in \mathcal{L}_p(Q^0)$, $\widetilde{l}(u) = 0$. Thus in this case, (5) can also be rewritten as the following form which is a more simple backward martingale equation (BME):

$$\begin{cases} a_t = a_0 + \int_0^t \int_{\mathbb{R}} a_{u-}[(p-1)f(\theta_2)(u,y) + \theta_2(u,y)]K(u;dy)dA_u \\ \qquad\qquad + \int_0^t a_{u-}\theta_1(u)d\widetilde{M}_u + \int_0^t \int_{\mathbb{R}} a_{u-}\theta_2(u,y)d\widetilde{N}(du,dy) \ , \\ a_T = (Z_T^0)^{p-1}, \end{cases} \tag{7}$$

where $f(\theta_2)(u,y) := (1+\theta_2(u,y))^{-\frac{1}{p-1}} - 1$.

Similarly, a *solution* of (7) is a pair (a, θ_1, θ_2) satisfying (7) such that:

(i) θ_1 is a predictable $\widetilde{M}$-integrable process, θ_2 is a $\widetilde{\mathcal{P}}$-measurable $\widetilde{N}$-integrable function and a is a strictly positive RCLL Q^0-semimartingale;

(ii) for all $l \in \mathcal{L}_p(Q^0)$, $Z_t(l)^p a_t$ belongs to the class (D), i.e., the family of random variables $Z_\tau(l)^p a_\tau I_{\tau \le T}$ for all stopping times τ is uniformly integrable;

(iii) $\mathcal{E}\{\int_0^\cdot \int_{\mathbb{R}} f(\theta_2)(u,y)d\widetilde{N}(du,dy)\}_t$ is a uniformly integrable Q^0-martingale.

Remark. Since there exists inaccessable jumps in our model, the conditions of Proposition 1 of Mania, Santacroce and Tevzadze [21] are not satified, which implies that p-optimal martingale measure Q^* might not be equivalent to P.

Corollary 3.5. *If $\sigma_2(u,y) = 0$, assume that the p-optimal martingale measure Q^* is equivalent to P, i.e., $Q^* \sim P$, then for the value process $V(p)$, there exists a predicatble $\widetilde{M}$-integrable process θ_1^* and a $\widetilde{\mathcal{P}}$-measurable function $\theta_2^* \in \mathcal{L}_p(Q^0)$ such that $(V(p), \theta_1^*, \theta_2^*)$ is a unique solution of (7).*

Reversely, if (a, θ_1, θ_2) is a solution of (7), then $a_t = V_t(P)$ a.s. for all $t \in [0,T]$ and the p-optimal martingale measure Q^* is equivalent to P which is given by

$$\left.\frac{dQ^*}{dQ^0}\right|_{\mathcal{F}_T} = \mathcal{E}\{-\int_0^\cdot \int_{\mathbb{R}} f(\theta_2)(u,y)\widetilde{N}(du,dy)\}_T \ . \tag{8}$$

In other word, the following two properties are equivalent:

(1) BME (7) has a unique solution;

(2) the p-optimal martingale measure Q^* is equivalent to P.

Proof. According to Lemma 3.1, $V_t(p)$ is a positive RCLL semimartingale under Q^0, according to Assumption 2.1, we see that it can be represented as the following form

$$V_t(p) = V_0(p) + \int_0^t V_{u-}(p)\ \theta(u) dA_u + \int_0^t V_{u-}(p)\theta_1(u) d\widetilde{M}_u + \int_0^t \int_{\mathbb{R}} V_{u-}(p)\theta_2(u,y)\widetilde{N}(du,dy),$$

where θ and θ_1 are predictable processes and θ_2 is a $\widetilde{\mathcal{P}}$-measurable function with $\theta_2 > -1$. Applying the Itô formula to $Z_t(l)^p$, we obtain

$$\begin{aligned} Z_t(l)^p &= 1 + \int_0^t \int_{\mathbb{R}} Z_{u-}(l)^p \left\{(1 + l(u,y))^p - 1\right\} \widetilde{N}(du,dy) \\ &\quad + \int_0^t \int_{\mathbb{R}} Z_{u-}(l)^p \left\{(1 + l(u,y))^p - 1 - p\, l(u,y)\right\} K(u;dy) dA_u\ . \end{aligned}$$

Again, making use of Itô's formula to $Z_t(l)^p\, V_t(p)$, we have

$$\begin{aligned} Z_t(l)^p\, V_t(p) &= V_0(p) + \int_0^t Z_{u-}(l)^p\, V_{u-}(p)\theta_1(u) d\widetilde{M}_u \\ &\quad + \int_0^t \int_{\mathbb{R}} Z_{u-}(l)^p\, V_{u-}(p) \left\{[1 + \theta_2(u,y)]\big(1 + l(u,y)\big)^p - 1\right\} \widetilde{N}(du,dy) \\ &\quad + \int_0^t Z_{u-}(l)^p\, V_{u-}(p) \bigg\{\theta(u) + \int_{\mathbb{R}} \{[(1 + l(u,y))^p - 1] \\ &\qquad [1 + \theta_2(u,y)] - p\, l(u,y)\} K(u;dy)\bigg\} dA_u\ . \end{aligned}$$

According to Lemma 3.4, $Z_t(l)^p\, V_t(p)$ can be rewritten as

$$\begin{aligned} Z_t(l)^p\, V_t(p) &= V_0(p) + \int_0^t Z_{u-}(l)^p\, V_{u-}(p)\theta_1(u) d\widetilde{M}_u + \int_0^t \int_{\mathbb{R}} Z_{u-}(l)^p\, V_{u-}(p) \\ &\quad \times \left\{\theta_2(u,y) + [1 + \theta_2(u,y)]\big[\,(1 + l(u,y))^p - 1\big]\right\} \widetilde{N}(du,dy) \\ &\quad + \int_0^t Z_{u-}(l)^p\, V_{u-}(p) \bigg\{\theta(u) + \int_{\mathbb{R}} \Psi(f(\theta_2)(u,y), \theta_2(u,y)) K(u;dy) \\ &\quad + \int_{\mathbb{R}} \{\Psi(l(u,y), \theta_2(u,y)) - \Psi(f(\theta_2)(u,y), \theta_2(u,y))\} K(u;dy)\bigg\} dA_u\ . \end{aligned}$$

From Lemma 3.1, $Z_t(l)^p\, V_t(p)$ is a Q^0-submartingale for every $l \in \mathcal{L}_p(Q^0)$. If the p-optimal martingale measure Q^* is equivalent to P, then there exists a

$l^* \in \mathcal{L}_p(Q^0)$ such that $Z_t(l^*)^p V_t(p)$ is a uniformly Q^0-martingale. Thus

$$\begin{aligned} l^*(u,y) &= f(\theta_2(u,y)); \\ \theta(u) &= -\int_{\mathbb{R}} \Psi(f(\theta_2)(u,y), \theta_2(u,y))K(u;dy) \\ &= \int_{\mathbb{R}} [\theta_2(u,y) + (p-1)f(\theta_2)(u,y)]K(u;dy) \ . \end{aligned}$$

Therefore,

$$\begin{aligned} V_t(p) &= V_0(p) + \int_0^t \int_{\mathbb{R}} [\theta_2(u,y) + (p-1)f(\theta_2)(u,y)]K(u;dy)dA_u \\ &+ \int_0^t V_{u-}\theta_1(u)d\widetilde{M}_u + \int_0^t \int_{\mathbb{R}} V_{u-}(p)\theta_2(u,y)\widetilde{N}(du,dy) \ . \end{aligned}$$

In other word, $(V(p), \theta_1, \theta_2)$ is the solution of (7). □

The following theorem is similar to Corollary 2 of Mania, Santacroce and Tevzadze [21].

Theorem 3.6. *The BME (5) has a solution if and only if there exists a $\widetilde{\mathcal{P}}$-measurable function $l^* \in \mathcal{L}_p(Q^0)$ such that*

$$(Z_T^0)^{p-1} Z_T(l^*)^{p-1} = c + \int_0^T h_u dS_u,$$

for a constant c and an X-integrable predictable process h such that $(\int_0^t h_u\, dS_u)_{t\in[0,T]}$ is a Q-martingale for every $Q \in \mathcal{M}_p^e$.

Proof. "$\Longrightarrow$": If BME (5) has a solution which is denoted by (a, θ_1, θ_2), let

$$l^*(u,y) = f(\theta_2)(u,y),$$

from the proof of Theorem 3.3, we see that $E_{Q^0}[a_T Z_T(l^*)^p] = E_{Q^0}[(Z_T^0)^{p-1} Z_T(l^*)^p] < \infty$, thus $l^* \in \mathcal{L}_p(Q^0)$. We first apply Itô's formula to $Z_t(l^*)^{p-1}$, then

$$\begin{aligned} Z_t(l^*)^{p-1} &= 1 - \int_0^t (p-1) Z_{u-}(l^*)^{p-1}\, \widetilde{f(\theta_2)}_u d\widetilde{M}_u \\ &+ \int_0^t \int_{\mathbb{R}} Z_{u-}(l^*)^{p-1} \left\{ (1 + f(\theta_2)(u,y))^{p-1} - 1 \right\} \widetilde{N}(du,dy) \\ &+ \int_0^t Z_{u-}(l^*)^{p-1} \frac{(p-1)(p-2)}{2} \widetilde{f(\theta_2)}_u^2 dA_u \\ &+ \int_0^t \int_{\mathbb{R}} Z_{u-}(l^*)^{p-1} \left\{ (1 + f(\theta_2)(u,y))^{p-1} - 1 - (p-1)\, f(\theta_2)(u,y) \right\} \end{aligned}$$

$$\times K(u;dy)dA_u \,.$$

Applying Itô's formula to $a_t Z_t(l^*)^{p-1}$, then

$$\begin{aligned}
a_t Z_t(l^*)^{p-1} &= a_0 + \int_0^t a_{u-} Z_{u-}(l^*)^{p-1}\left[\sigma_1(u)\theta_1(u) + (p-1)\widetilde{f(\theta_2)}_u\right] d\widetilde{M}_u \\
&\quad + \int_0^t \int_{\mathbb{R}} a_{u-} Z_{u-}(l^*)^{p-1}\Big\{[1+\sigma_2(u,y)\theta_1(u)][1+\theta_2(u,y)] - 1\Big\}\widetilde{N}(du,dy) \\
&\quad + \frac{p(p-1)}{2}\int_0^t a_{u-} Z_{u-}(l^*)^{p-1}\widetilde{f(\theta_2)}_u^2 dA_u \\
&\quad + \int_0^t \int_{\mathbb{R}} a_{u-} Z_{u-}(l^*)^{p-1}[1+\sigma_2(u,y)\theta_1(u)]\{\theta_2(u,y) + (p-1)f(\theta_2)(u,y)\}K(u;dy)dA_u \\
&\quad - \int_0^t (p-1) a_{u-} Z_{u-}(l^*)^{p-1}\,\widetilde{f(\theta_2)}_u d\widetilde{M}_u \\
&\quad + \int_0^t \int_{\mathbb{R}} a_{u-} Z_{u-}(l^*)^{p-1}\left\{(1+f(\theta_2)(u,y))^{p-1} - 1\right\}\widetilde{N}(du,dy) \\
&\quad + \int_0^t a_{u-} Z_{u-}(l^*)^{p-1}\frac{(p-1)(p-2)}{2}\widetilde{f(\theta_2)}_u^2 dA_u \\
&\quad + \int_0^t \int_{\mathbb{R}} a_{u-} Z_{u-}(l^*)^{p-1}\left\{(1+f(\theta_2)(u,y))^{p-1} - 1 - (p-1)\,f(\theta_2)(u,y)\right\}K(u;dy)dA_u \\
&- \int_0^t a_{u-} Z_{u-}(l^*)^{p-1}[\sigma_1(u)\theta_1(u) + (p-1)\widetilde{f(\theta_2)}_u](p-1)\widetilde{f(\theta_2)}_u dA_u \\
&+ \int_0^t \int_{\mathbb{R}} a_{u-} Z_{u-}(l^*)^{p-1}\big\{[1+\sigma_2(u,y)\theta_1(u)][1+\theta_2(u,y)] - 1\big\} \\
&\big\{(1+f(\theta_2)(u,y))^{p-1} - 1\big\}\widetilde{N}(du,dy) \\
&+ \int_0^t \int_{\mathbb{R}} a_{u-} Z_{u-}(l^*)^{p-1}\big\{[1+\sigma_2(u,y)\theta_1(u)][1+\theta_2(u,y)] - 1\big\} \\
&\big\{(1+f(\theta_2)(u,y))^{p-1} - 1\big\}K(u;dy)dA_u \\
&= a_0 + \int_0^t a_{u-} Z_{u-}(l^*)^{p-1}\sigma_1(u)\theta_1(u)d\widetilde{M}_u \\
&\quad + \int_0^t \int_{\mathbb{R}} a_{u-} Z_{u-}(l^*)^{p-1}\,\{[1+\sigma_2(u,y)\theta_1(u)][1+\theta_2(u,y)] \\
&(1+f(\theta_2)(u,y))^{p-1} - 1\}\,\widetilde{N}(du,dy) \\
&\quad + \int_0^t \int_{\mathbb{R}} a_{u-} Z_{u-}(l^*)^{p-1}[1+\sigma_2(u,y)\theta_1(u)]\big\{\theta_2(u,y) \\
&\qquad + [1+\theta_2(u,y)]\big[(1+f(\theta_2)(u,y))^{p-1} - 1\big]\big\}K(u;dy)dA_u \\
&= a_0 + \int_0^t a_{u-} Z_{u-}(l^*)^{p-1}\theta_1(u)dS_u \,,
\end{aligned}$$

therefore, $(Z_T^0)^{p-1} Z_T(l^*)^{p-1} = a_0 + \int_0^T a_{u-} Z_{u-}(l^*)^{p-1}\theta_1(u)dS_u$. The rest is the same as the proof of Corollary 2 of Mania, Santacroce and Tevzadze [21].

"$\Longleftarrow$": If there exists a $\widetilde{\mathcal{P}}$-measurable function $l^* \in \mathcal{L}_p(Q^0)$ such that

$$(Z_T^0)^{p-1} Z_T(l^*)^{p-1} = c + \int_0^T h_u dS_u,$$

for a constant c and an X-integrable predictable process h such that $(\int_0^t h_u$

$dS_u)_{t\in[0,T]}$ is a Q-martingale for every $Q \in \mathcal{M}_p^e$, let

$$Y_t := E_{Q^0}((Z_T^0)^{p-1} Z_T(l^*)^{p-1}|\mathcal{F}_t) = c + \int_0^t h_u dS_u,$$

it is easy to see that Y_t is a positive uniformly integrable Q^0-martingale with $Y_T = (Z_T^0)^{p-1} Z_T(l^*)^{p-1}$, thus it can be rewritten as the following

$$Y_t = c + \int_0^t Y_{u-}\hat{h}_u\sigma_1(u)d\widetilde{M}_u + \int_0^t \int_{\mathbb{R}} Y_{u-}\hat{h}_u\sigma_2(u,y)\widetilde{N}(du,dy),$$

since Y is positive, $\hat{h}_u\sigma_2(u,y) > -1$. On the other hand, applying Itô's formula to $Z_t(l^*)^{1-p}$, we have

$$\begin{aligned}
&Z_t(l^*)^{1-p} = 1 - \int_0^t (1-p)Z_{u-}(l^*)^{1-p}\widetilde{l}^*(u)d\widetilde{M}_u \\
&+ \int_0^t \int_{\mathbb{R}} Z_{u-}(l^*)^{1-p}\{(1+l^*(u,y))^{1-p} - 1\}\widetilde{N}(du,dy) \\
&+ \frac{p(p-1)}{2}\int_0^t Z_{u-}(l^*)^{1-p}\widetilde{l}^*(u)^2 dA_u \\
&+ \int_0^t \int_{\mathbb{R}} Z_{u-}(l^*)^{1-p}\{(1+l^*(u,y))^{1-p} - 1 - (1-p)l^*(u,y)\}K(u;dy)dA_u
\end{aligned}$$

Let $a_t := Y_t Z_t(l^*)^{1-p}$, it is easy to see that $a_T = (Z_T^0)^{p-1}$. And once again by applying Itô's formula to $Y_t Z_t(l^*)^{1-p}$, we have

$$\begin{aligned}
a_t &= c + \int_0^t a_{u-}[\sigma_1(u)\hat{h}_u + (p-1)\widetilde{l}^*(u)]d\widetilde{M}_u \\
&+ \int_0^t \int_{\mathbb{R}} a_{u-}\left\{\hat{h}_u\sigma_2(u,y) + \{(1+l^*(u,y))^{1-p} - 1\}[1+\sigma_2(u,y)\hat{h}_u]\right\}\widetilde{N}(du,dy) \\
&+ \int_0^t a_{u-}\left\{\frac{p(p-1)}{2}\widetilde{l}^*(u)^2 + (p-1)\sigma_1(u)\hat{h}_u\widetilde{l}^*(u)\right\}dA_u + \\
&\int_0^t \int_{\mathbb{R}} a_{u-}\{[(1+l^*(u,y))^{1-p} - 1][1+\sigma_2(u,y)\hat{h}_u] - (1-p)l^*(u,y)\}K(u;dy)dA_u \\
&= c + \int_0^t a_{u-}[\sigma_1(u)\hat{h}_u + (p-1)\widetilde{l}^*(u)]d\widetilde{M}_u \\
&+ \int_0^t \int_{\mathbb{R}} a_{u-}\left\{(1+l^*(u,y))^{1-p}[1+\sigma_2(u,y)\hat{h}_u] - 1\right\}\widetilde{N}(du,dy) \\
&+ \frac{p(p-1)}{2}\int_0^t a_{u-}\widetilde{l}^*(u)^2 dA_u
\end{aligned}$$

$$+\int_0^t \int_{\mathbb{R}} a_{u-}[1+\sigma_2(u,y)\hat{h}_u]\{[(1+l^*(u,y))^{1-p}-1]+(p-1)l^*(u,y)\}K(u;dy)dA_u.$$

Let

$$\begin{array}{ll} \theta_1(u) & := \hat{h}_u\,, \\ \theta_2(u,y) & := (1+l^*(u,y))^{1-p}-1, \end{array}$$

by simple computation, we see that $f(\theta_2)(u,y) = l^*(u,y)$, therefore, (a, θ_1, θ_2) is a pair satisfying (5). Since $l^*(u,y) \in \mathcal{L}_p(Q^0)$, we see that

$$Z_t(l^*) := \mathcal{E}\{-\int_0^\cdot \widetilde{f(\theta_2)}_u d\widetilde{M}_u + \int_0^\cdot \int_{\mathbb{R}} f(\theta_2)(u,y)\widetilde{N}(du,dy)\}_t$$

is a positive uniformly Q^0-martingale, thus we can define a new measure as the following

$$\left.\frac{dQ^*}{dQ^0}\right|_{\mathcal{F}} = Z_T(l^*),$$

we see that for every $Q \in \mathcal{M}_p^e$, $E_{Q^*}[(Z_T^0)^{p-1}(Z_T(l^*))^{p-1}] = E_Q[(Z_T^0)^{p-1}\, Z_T(l^*)^{p-1}] = c$. Thus for every $l \in \mathcal{L}_p(Q^0)$, we have

$$E_{Q^0}[(Z_T^0)^{p-1}(Z_T(l^*))^{p-1}(Z_T(l^*) - Z_T(l))] = 0,$$

by Proposition A of Appendix of Mania, Santacroce and Tevzadze [21], we find that Q^* is p-optimal martingale measure, thus

$$a_t Z_t(l^*)^p = Y_t Z_t(l^*) = E_{Q^0}\{(Z_T^0)^{p-1} Z_T(l^*)^p | \mathcal{F}_t\},$$

from which we see that

$$a_t = E_{Q^0}\{(Z_T^0)^{p-1} Z_{t,T}(l^*)^p|\mathcal{F}_t\} = \operatorname*{ess\,inf}_{l\in\mathcal{L}_p(Q^0)} E_{Q^0}\{(Z_T^0)^{p-1} Z_{t,T}(l)^p|\mathcal{F}_t\}$$

and for all $l \in \mathcal{L}_p(Q^0)$ and all stopping time τ,

$$a_\tau Z_\tau(l)^p \le E_{Q^0}\big((Z_T^0)^{p-1} Z_T(l)^p|\mathcal{F}_\tau\big).$$

Thus $aZ(l)^p$ is in the class (D) under measure Q^0. Thus (a, θ_1, θ_2) is a solution of BME (5) in the meaning of Definition 3.2. □

Remark. Note that for $q = 2$ and deterministic mean-variance tradeoff $K_T$we have: If the minimal martingale measure $\widehat{Q}$ belongs to $\mathcal{M}_p^e$, according to Theorem 3.6 and Schweizer [24], then (5) has a solution.

Corollary 3.7. *Assume that (5) has a solution, let* $\widetilde{a}_t := a_t^{-\frac{1}{p-1}}$*, then* $\widetilde{a}_t$ *is a solution of the following BME*

$$\begin{cases} (a_t)^{-\frac{1}{p-1}} = \widetilde{a}_0 - \dfrac{1}{p-1}\displaystyle\int_0^t \widetilde{a}_{u-}\left[\sigma_1(u)\theta_1(u) + (p-1)\widetilde{f(\theta_2)}_u\right] d\widetilde{M}_u \\ + \displaystyle\int_0^t\int_{\mathbb{R}} \widetilde{a}_{u-}\left\{[1+\sigma_2(u,y)\theta_1(u)]^{-\frac{1}{p-1}}[1+\theta_2(u,y)]^{-\frac{1}{p-1}} - 1\right\}\widetilde{N}(du,dy) \\ + \dfrac{1}{2}\dfrac{p}{(p-1)^2}\displaystyle\int_0^t \widetilde{a}_{u-}\sigma_1(u)^2\theta_1(u)^2 dA_u \\ + \dfrac{1}{p-1}\displaystyle\int_0^t\int_{\mathbb{R}} \widetilde{a}_{u-}[1+\theta_2(u,y)]^{-\frac{1}{p-1}} \\ \times\left\{\sigma_2(u,y)\theta_1(u) + (p-1)f(\sigma_2(u,y)\theta_1(u))\right\}K(u;dy)dA_u \\ \widetilde{a}_T = (Z_T^0)^{-1}. \end{cases} \tag{9}$$

Especially, when $\sigma_2(u,y) = 0$*, i.e.,* S *is a continuous process,* $\widetilde{a}_t$ *is a solution of the following stochastic Riccati equation (SRE)*

$$\begin{cases} \widetilde{a}_t = \widetilde{a}_0 + \dfrac{1}{2}\dfrac{p}{(p-1)^2}\displaystyle\int_0^t \widetilde{a}_{u-}\theta_1(u)^2 dA_u - \dfrac{1}{p-1}\displaystyle\int_0^t \widetilde{a}_{u-}\theta_1(u)d\widetilde{M}_u \\ \qquad + \displaystyle\int_0^t\int_{\mathbb{R}} \widetilde{a}_{u-}f(\theta_2)(u,y)\widetilde{N}(du,dy)\ , \\ \widetilde{a}_T = (Z_T^0)^{-1}. \end{cases} \tag{10}$$

4. Example

In this section, we will consider a somewhat simpler model so that we could compare our results with Jeanblanc, Kloeppel and Miyahara [14]. We assume that W is a Brownian motion and $N(dt,dy)$ is a Poisson random measure with the compensator $dt\nu(dy)$, i.e., $\widetilde{N}(du,dy) := N(du,dy) - \nu(dy)du$ is a martingale random measure (see Jacod [13]). In this section, we assume the discounted price S is given by

$$S_t = S_0 + bt + \sigma W_t + \int_0^t\int_{\mathbb{R}} yI_{\{|y|<R\}}\widetilde{N}(du,dy), \tag{11}$$

where b, σ and R are constants with $\sigma > 0$ and $R > 0$. In this simple case, we will try to solve the BME (5).

In this case, Q^0 is given by

$$\frac{dQ^0}{dP}|_{\mathcal{F}} = Z_T^0,$$

where $Z_t^0 := \exp\big\{ -\frac{b}{\sigma}W_t - \frac{b^2}{2\sigma^2}t\big\}$ is a uniformly integrable martingale. It is easy to see that $Q^0 \in \mathcal{M}_p^e$ and

$$\widetilde{W}_t := W_t + \frac{b}{\sigma}t$$

is a Q^0-Brownian motion. For a $\widetilde{\mathcal{P}}$-measurable function $l(u,y)$ such that $\int_0^T \int_{\mathbb{R}} |l(u,y)|\nu(dy)du < \infty$, let

$$\widetilde{l}(u) := \frac{1}{\sigma}\int_{\mathbb{R}} l(u,y)yI_{|y|\le R}\nu(dy),$$

and

$$Z_t(l) := \mathcal{E}\{-\int_0^{\cdot} \widetilde{l}(u)d\widetilde{W}_u + \int_0^{\cdot}\int_{\mathbb{R}} l(u,y)d\widetilde{N}(du,dy)\}_t.$$

Especially, when $l(u,y) \equiv g(y)$ for a deterministic function $g(y)$,

$$\widetilde{g} = \frac{1}{\sigma}\int_{\mathbb{R}} yg(y)I_{|y|\le R}\nu(dy)$$

is a deterministic constant and $Z_t(g)$ is then given by

$$Z_t(g) := \mathcal{E}\{-\widetilde{g}\,\widetilde{W}_{\cdot} + \int_0^{\cdot}\int_{\mathbb{R}} g(y)d\widetilde{N}(du,dy)\}_t.$$

It is easy to see from Itô's formula that

$$\begin{aligned}(Z_t^0 Z_t(g))^p \quad &= 1 - p\int_0^t (Z_{u-}^0 Z_{u-}(g))^p[\widetilde{g} + \frac{b}{\sigma}]\,dW_u \\ &+ \int_0^t\int_{\mathbb{R}} (Z_{u-}^0 Z_{u-}(g))^p\big\{(1+g(y))^p - 1\big\}\widetilde{N}(du,dy) \\ &+ \frac{p(p-1)}{2}\int_0^t (Z_{u-}^0 Z_{u-}(g))^p[\widetilde{g} + \frac{b}{\sigma}]^2 du \\ &+ \int_0^t\int_{\mathbb{R}} (Z_{u-}^0 Z_{u-}(g))^p\big\{(1+g(y))^p - 1 - p\,g(y)\big\}\nu(dy)du\ .\end{aligned}$$

Thus

$$\begin{aligned}E(Z_t^0 Z_t(g))^p &= 1 + \frac{p(p-1)}{2}\int_0^t E(Z_{u-}^0 Z_{u-}(g))^p[\widetilde{g} + \frac{b}{\sigma}]^2 du \\ &+ \int_0^t\int_{\mathbb{R}} E(Z_{u-}^0 Z_{u-}(g))^p\big\{(1+g(y))^p - 1 - p\,g(y)\big\}\nu(dy)du\ .\end{aligned}$$

Similar to Jeanblanc, Kloeppel and Miyahara [14], we have the following lemma.

Lemma 4.1. *Assume that there exists a constant λ^* such that $1 + \frac{\lambda^*}{p} y I_{|y|\leq R} > 0$ and that*

$$\frac{\lambda^*}{p(p-1)}\sigma^2 + \int_{\mathbb{R}} y\left\{\left(1+\frac{\lambda^*}{p}y\right)^{\frac{1}{p-1}} - 1\right\} I_{|y|\leq R}\nu(dy) + b = 0. \tag{12}$$

Let

$$g^*(y) = \left(1+\frac{\lambda^*}{p} y I_{|y|\leq R}\right)^{\frac{1}{p-1}} - 1,$$

if $g^*(y) \in \mathcal{L}_p(Q^0)$, then we can define a new measure Q^* by

$$\frac{dQ^*}{dQ^0}\Big|_{\mathcal{F}} = Z_T(g^*) = \mathcal{E}\{-\widetilde{g}^* \,\widetilde{W}_\cdot + \int_0^\cdot \int_{\mathbb{R}} g^*(y)\widetilde{N}(du,dy)\}_T,$$

and then Q^* is p-optimal martingale measure.

According to Lemma 4.1, we can give the solution of BME (5) as in the following theorem.

Theorem 4.2. *Under the conditions of Lemma 4.1, assume that*

$$k := \int_{\mathbb{R}} \left\{(1+g^*(y))^{p-1} - 1 - (p-1)\, g^*(y)\right\}\nu(dy) < \infty\ ,$$

then the solution of BME (5) is given by

$$\begin{cases} \theta_1(u) &= \dfrac{\lambda^*}{p}, \\ \theta_2(u,y) &= \dfrac{1}{1+\frac{\lambda^*}{p} y I_{|y|\leq R}} - 1, \\ a_t &= K\mathcal{E}\{\dfrac{\lambda^*}{p} S\}_t Z_t(g^*)^{1-p}\ , \end{cases}$$

where $K = \exp\left\{\left(\frac{\lambda^{*2}\sigma^2(p-2)}{2p^2(p-1)} + k - \frac{\lambda^*}{p}b\right)\times T\right\}$.

Proof. By applying Itô's formula to $(Z_t^0 Z_t(g^*))^{p-1}$, we have

$$\begin{aligned}(Z_t^0 Z_t(g^*))^{p-1} &= 1 - (p-1)\int_0^t (Z_{u-}^0 Z_{u-}(g^*))^{p-1}[\widetilde{g}^* + \frac{b}{\sigma}]\, dW_u \\ &\quad + \int_0^t \int_{\mathbb{R}} (Z_{u-}^0 Z_{u-}(g^*))^{p-1}\left\{(1+g^*(y))^{p-1} - 1\right\}\widetilde{N}(du,dy) \\ &\quad + \frac{(p-1)(p-2)}{2}\int_0^t (Z_{u-}^0 Z_{u-}(g^*))^{p-1}[\widetilde{g}^* + \frac{b}{\sigma}]^2 du\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}}(Z^0_{u-}Z_{u-}(g^*))^{p-1}\big\{(1+g^*(y))^{p-1}-1-(p-1)\,g^*(y)\big\}\nu(dy)du \\
&\qquad = 1+\frac{\lambda^*}{p}\int_0^t(Z^0_{u-}Z_{u-}(g^*))^{p-1}\sigma\, dW_u \\
&\qquad\quad +\frac{\lambda^*}{p}\int_0^t\int_{\mathbb{R}}(Z^0_{u-}Z_{u-}(g^*))^{p-1}yI_{|y|\le R}\widetilde{N}(du,dy) \\
&\qquad\quad +\frac{(p-2)}{2(p-1)}\frac{\lambda^{*2}\sigma^2}{p^2}\int_0^t(Z^0_{u-}Z_{u-}(g^*))^{p-1}du \\
&+\int_0^t\int_{\mathbb{R}}(Z^0_{u-}Z_{u-}(g^*))^{p-1}\big\{(1+g^*(y))^{p-1}-1-(p-1)\,g^*(y)\big\}\nu(dy)du \\
&= 1+\frac{\lambda^*}{p}\int_0^t(Z^0_{u-}Z_{u-}(g^*))^{p-1}bdu+\frac{\lambda^*}{p}\int_0^t(Z^0_{u-}Z_{u-}(g^*))^{p-1}\sigma\, dW_u \\
&\qquad\quad +\frac{\lambda^*}{p}\int_0^t\int_{\mathbb{R}}(Z^0_{u-}Z_{u-}(g^*))^{p-1}yI_{|y|\le R}\widetilde{N}(du,dy) \\
&\qquad +\int_0^t(Z^0_{u-}Z_{u-}(g^*))^{p-1}\left\{\frac{(p-2)}{2(p-1)}\frac{\lambda^{*2}\sigma^2}{p^2}+k-\frac{\lambda^*}{p}b\right\}du \\
&\qquad\qquad = \mathcal{E}\{\frac{\lambda^*}{p}S.\}_t\exp\left\{\Big(\frac{(p-2)}{2(p-1)}\frac{\lambda^{*2}\sigma^2}{p^2}+k-\frac{\lambda^*}{p}b\Big)\times t\right\}\ .
\end{aligned}
$$

Therefore,

$$
(Z^0_T Z_T(g^*))^{p-1}=K\mathcal{E}\{\frac{\lambda^*}{p}S\}_T\ ,
$$

and Theorem 4.2 follows from the proof of Theorem 3.6. □

Remark. A problem very similar to the one here is considered in the also technically interesting article cited several times before by Jeanblanc, Kloeppel and Miyahara [14] and these authors also treat the convergence of the q-optimal martingale measures to the entropy minimal martingale measure. The situation here however is more general than in the pure Lévy-case considered there and the methods are completely different. In Kohlmann and Xiong [18] the minimal entropy measure in the general jump case is described using the new techniques of this paper and a similar convergence result is derived.

5. The Problem of Optimal Utility

We now return to the general case and assume that BME (5) has a solution. We consider the problem of optimal utility of an investor with the following utility function

$$U(x) = -\left|1 - \frac{x}{k_0}\right|^q,$$

where k_0 is a positive constant and q is given by $\frac{1}{p} + \frac{1}{q} = 1$. This kind of utility function was first considered in Kohlmann and Niethammer [16]. We now try to give an explicit solution by the solution of BME (5).

Definition 5.1. Given a initial wealth $x > 0$, a predictable process $\pi = (\pi_t)_{0\le t\le T}$ is called *admissible self-financing strategy* if the stochastic integral $(\pi \cdot S)_t = \int_0^t \pi_u dS_u$ is well defined for all $t \in [0, T]$, and the wealth process defined by $X_t^\pi = x + \int_0^t \pi_u dS_u$ is a uniformly integrable Q^*-martingale. The set of all admissible self-financing strategy is denoted by $\mathrm{Adm}(x)$.

Remark. If π is a predictable S-integrable process with

$$E\left[\left(\sup_{0\le t\le T}\left|\int_0^t \pi_u dS_u\right|\right)^q\right] < \infty,$$

then $\pi \in \mathrm{Adm}(x)$, which follows from

$$E_{Q^*}\left[\sup_{0\le t\le T}\left|\int_0^t \pi_u dS_u\right|\right] \le \left(E\left[\left(Z_T^0 Z_T(l^*)\right)^p\right]\right)^{\frac{1}{p}} \left(E\left[\left(\sup_{0\le t\le T}\left|\int_0^t \pi_u dS_u\right|\right)^q\right]\right)^{\frac{1}{q}} < \infty\,.$$

We now consider the following optimization problem

$$\sup_{\pi\in Adm(x)} EU(X_T^\pi). \tag{13}$$

It is easy to see that for a $q > 1$,

$$U'(x) = \begin{cases} \frac{q}{k_0}(1 - \frac{x}{k_0})^{q-1}, & \text{if } x \le k_0; \\ -\frac{q}{k_0}(\frac{x}{k_0} - 1)^{q-1}, & \text{if } x > k_0. \end{cases}$$

Let $I(y) = (U')^{-1}(y)$, thus $I(y)$ is given by

$$I(y) = \begin{cases} k_0 + k_0(\frac{k_0}{q})^{\frac{1}{q-1}}(-y)^{\frac{1}{q-1}}, & \text{if } y \le 0; \\ k_0 - k_0(\frac{k_0}{q})^{\frac{1}{q-1}} y^{\frac{1}{q-1}}, & \text{if } y > 0. \end{cases}$$

Let us recall here that $a_0 = E[(Z_T^0)^p Z_T(l^*)^p]$. As to the optimal strategy, we have the following theorem.

Theorem 5.2. *Assume that the BME (5) has a solution denoted by (a, θ_1, θ_2). Let $l^* = f(\theta_2)(u, y)$ and*

$$\lambda^* = \frac{q|k_0 - x|^{q-1}}{(k_0)^q (a_0)^{q-1}}.$$

(1) If $x < k_0$, then the optimal strategy π^* is given by

$$\pi_u^* = -k_0(\frac{k_0}{q})^{\frac{1}{q-1}}(\lambda^*)^{\frac{1}{q-1}} a_{u-} Z_{u-}(l^*)^{p-1}\theta_1(u).$$

(2) If $x \geq k_0$, then the optimal strategy π^* is given by

$$\pi_u^* = k_0(\frac{k_0}{q})^{\frac{1}{q-1}}(\lambda^*)^{\frac{1}{q-1}} a_{u-} Z_{u-}(l^*)^{p-1}\theta_1(u).$$

Proof. We only prove (i), the proof of (ii) is the same. From the proof of Theorem 3.6, it is easy to see that

$$(Z_T^0)^{p-1} Z_T(l^*)^{p-1} = a_0 + \int_0^T a_{u-} Z_{u-}(l^*)^{p-1}\theta_1(u) dS_u,$$

therefore, $I(\lambda^* Z_T^0 Z_T(l^*))$ can be written as

$$\begin{aligned} I(\lambda^* Z_T^0 Z_T(l^*)) &= k_0 - k_0(\frac{k_0}{q})^{\frac{1}{q-1}}(\lambda^*)^{\frac{1}{q-1}}(Z_T^0)^{\frac{1}{q-1}}(Z_T(l^*))^{\frac{1}{q-1}} \\ &= k_0 - k_0(\frac{k_0}{q})^{\frac{1}{q-1}}(\lambda^*)^{\frac{1}{q-1}}(Z_T^0)^{p-1}(Z_T(l^*))^{p-1} \\ &= k_0 - k_0(\frac{k_0}{q})^{\frac{1}{q-1}}(\lambda^*)^{\frac{1}{q-1}} a_0 - \int_0^T k_0(\frac{k_0}{q})^{\frac{1}{q-1}}(\lambda^*)^{\frac{1}{q-1}} a_{u-} Z_{u-}(l^*)^{p-1}\theta_1(u) dS_u \\ &= k_0 - |k_0 - x| + \int_0^T \pi_u^* dS_u = x + \int_0^T \pi_u^* dS_u\ , \end{aligned}$$

and from Theorem 3.6, we see that $\int_0^t \pi_u^* dS_u$ is a uniformly integrable Q^*-martingale, thus $\pi^* \in Adm(x)$. Furthermore, for any given $\pi \in Adm(x)$,

$$\begin{aligned} &E(U(I(\lambda^* Z_T^0 Z_T(l^*)))) - E(U(X_T^\pi)) \\ &\quad \geq E\left\{U'\big(I(\lambda^* Z_T^0 Z_T(l^*))\big)\big[I(\lambda^* Z_T^0 Z_T(l^*)) - X_T^\pi\big]\right\} \end{aligned}$$

$$= E\left\{\lambda^* Z_T^0 Z_T(l^*)\left[I(\lambda^* Z_T^0 Z_T(l^*)) - X_T^\pi\right]\right\}$$
$$= \lambda^* E_{Q^*}\left\{I(\lambda^* Z_T^0 Z_T(l^*)) - X_T^\pi\right\} = \lambda^*(x - x) = 0\,. \quad \square$$

Remark. Here the reader might expect the derivation of the convergence results on the control problems corresponding to the convergence of the measures mentioned in above remark along the ideas in Kohlmann and Niethammer [16]. After establishing the martingale convergence results and the BME-characterization of the minimal entropy martingale measure in Kohlmann and Xiong [18] this convergence on, say, the other side of the duality will be derived in a paper we are finishing at the moment.

Conclusion. The p-optimal martingale measures in an incomplete financial market model with inaccessible jumps described by a random jump measure are derived by making use of the dynamic programming approach to obtain a backward martingale equation (BME) with the property that if the BME has a solution, then the p-optimal martingale measure is equivalent to the original measure. Furthermore we give a description of the p-optimal martingale measure by the solution of the BME.

In the simpler Lévy case an explicit solution of the BME is derived. As an application, we consider the optimal utility of an investor with utility function $U(x) = -|1 - \frac{x}{k_0}|^q$, and explicitly describe the optimal strategy by the solution of the BME.

Martingale measures, i.e. probability measures under which certain given sets of stochastic processes on a basic space $(\Omega, \mathbb{F}, P)$ are (local) martingales, have been playing a manifold role in theoretical stochastics, in control theory and in applied mathematical finance for many years and still are subject of actual research today. They appear in a broad variety of applications e.g. in the well known Stroock-Varadhan martingale problems, in martingale related formulations of optimality principles in control theory and they turned out to be the basic tools in modeling financial markets. In the beginning discrete models were considered, then models basing on stochastic differential equations, on continuous semimartingales. During the last years the need of a highly realistic model then turned the researchers' interest towards Lévy-process and most recently to more and more general jump process models. In the seminal papers on these models the basic notions of no-arbitrage, completeness, etc. are characterized by existence and uniqueness results for (possibly signed) martingale measures which are equivalent or absolutely continuous with respect to the basic probability with a highlight in the different forms of Fundamental Asset Pricing Theorems (see the seminal works by Harrison, Kreps, Pliska, Yan, Delbaen, Schachermayer and many others). They play a dominant role in measuring risk, and in capital asset pricing and portfolio optimization problems these measures appear in the dual formulations of the underlying control problems in the sense of functional duality as well as in the sense of adjoint duality (see the works by Bismut, Rockafellar, Ekeland, etc.). The latter then is related to the notion of backward stochastic differential equations first described by Pardoux and Peng. It goes beyond the scope of this paper to explain the use of martingale measures in more detail or even to relate certain seminal results to the names of the authors of the meanwhile extremely broad spectrum of articles in the recent literature. Instead, we here presented some of the techniques to derive some new results on a recently very actively pursued question in incomplete markets related to measuring the distance of a martingale measure from the basic measure and relating the results to the corresponding portfolio/hedging problem.

Acknowledgments

This research is supported by National Natural Science Foundation of China under Grant No. 70671069. The second author thanks the University of Konstanz for its support and hospitality.

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