

A SURVEY OF HUEBSCHMANN AND STASHEFF'S PAPER:
FORMAL SOLUTION OF THE MASTER EQUATION
VIA HPT AND DEFORMATION THEORY

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Abstract: The construction of universal twisting cocycle solving the most general form of Maurer-Cartan equation [8] is explained in detail, together with the relation with the *almost contraction* of [3]. In this way the applications to derived brackets and deformation quantization are emphasized.

The relation between the Huebschmann-Stasheff construction and the Kuranishi functor of deformation theory is mentioned.

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1. Introduction

These notes, based on the paper [8] by Huebschmann and Stasheff, were prepared for a series of talks at Illinois State University with the intention of applying Homological Perturbation Theory (HPT) to the construction of derived brackets [11, 16], and eventually writing Part II of the paper [1].

Derived brackets are obtained by deforming the initial bracket via a derivation of the bracket. In [3] it was demonstrated that such deformations correspond to solutions of the Maurer-Cartan equation, and the role of an “almost contraction” was noted. This technique (see also [9]) is very similar to the itera-

tive procedure of [8] for finding the most general solution of the Maurer-Cartan equation, i.e. the deformation of a given structure in a prescribed direction.

The present article, besides providing additional details of the condensed article [8], forms a theoretical background for understanding and generalizing the current techniques that give rise to derived brackets. The generalization, which will be the subject matter of [2], will be achieved by using Stasheff and Huebschmann's universal solution. A second application of the universal solution will be in deformation quantization and will help us find the coefficients of star products in a combinatorial manner, rather than as a byproduct of string theory which underlies the original solution given by Kontsevich [10].

HPT is often used to replace given chain complexes by homotopic, smaller, and more readily computable chain complexes (to explore “small” or “minimal” models). This method may prove to be more efficient than “spectral sequences” in computing (co)homology. One useful tool in HPT is

Lemma 1. (Basic Perturbation Lemma (BPL)) *Given a contraction of N onto M and a perturbation ∂ of d_N , under suitable conditions there exists a perturbation d_∂ of d_M such that $H(M, d_M + d_\partial) = H(N, d_N + \partial)$.*

The main question is: under what conditions does the BPL allow the preservation of the data structures (DGA's, DG coalgebras, DGLA's, etc.)? (We will use the self-explanatory abbreviations such as DG for “differential graded”, DGA for “differential graded (not necessarily associative) algebra”, and DGLA for “differential graded Lie algebra”.)

Another prominent idea is that of a “(universal) twisting cochain” as a solution of the “master equation”:

Proposition 1. *Given a contraction of N onto M and a twisting cochain $N \rightarrow A$ (A some DGA), there exists a unique twisting cochain $M \rightarrow A$ that factors through the given one and which can be constructed inductively.*

The explicit formulas are reminiscent of the Kuranishi map [13] (p. 17), and the relationship will be investigated elsewhere.

Note. We will assume that the ground ring is a field F of characteristic zero.

2. Perturbations of (Co)Differentials

2.1. Derivations of the Tensor Algebra

For any vector space V over F we have the isomorphism $\text{Der}(TV) \cong \text{Hom}(V, TV)$, where TV denotes the (augmented) tensor algebra on V . Namely, every linear map f from V into TV extends uniquely into a derivation of the algebra TV via the formula

$$\hat{f}(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes f(v_i) \otimes \cdots \otimes v_n.$$

Equivalently, every derivation of TV is determined by its restriction to V .

2.2. Coderivations of the Tensor Coalgebra

Similarly, we have the isomorphism $\text{Coder}(T^cV) \cong \text{Hom}(T^cV, V)$, where T^cV is the (coaugmented) coassociative tensor coalgebra of V , with counit $\eta : T^cV \rightarrow F$ (projection onto F), and comultiplication

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^n (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

Every linear map $f = f_1 + f_2 + \cdots + f_n + \cdots : T^cV \rightarrow V$ (where $f_i : V^{\otimes i} \rightarrow V$) factors uniquely through a coderivation \hat{f} of T^cV defined via the formula

$$\begin{aligned} \hat{f}(v_1 \otimes \cdots \otimes v_n) &= \sum_{i=1}^n v_1 \otimes \cdots \otimes f_1(v_i) \otimes \cdots \otimes v_n \\ &\quad + \sum_{i=1}^{n-1} v_1 \otimes \cdots \otimes f_2(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n \\ &\quad \vdots \\ &\quad + f_n(v_1 \otimes \cdots \otimes v_n). \end{aligned}$$

That is, each coderivation on T^cV is determined by itself followed by the projection onto V . Recall that the condition for \hat{f} to be a coderivation can be written as $\Delta\hat{f} = (1 \otimes \hat{f} + \hat{f} \otimes 1)\Delta$.

2.3. Coderivations of the Symmetric Coalgebra

Let us consider the cofree cocommutative counital coassociative algebra ST^cV on the vector space V as a subspace of T^cV . The symmetric group Σ_n acts on the left on $V^{\otimes n}$ via $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. Then

$$ST^cV = \bigoplus_{n \geq 0} (V^{\otimes n})^{\Sigma_n}$$

is the space of invariants of this action. The action is compatible with the coproduct on ST^cV , so ST^cV is a subcoalgebra of T^cV which is cocommutative. Note that $ST(V)$ is not a subalgebra with respect to the tensor multiplication in $T(V)$; the product has to be symmetrized so that it projects back onto this subspace (reminiscent of what T. Voronov does with derived brackets). The projection (symmetrization) map $P : T^cV \rightarrow ST^cV$ is given by

$$P(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma(v_1 \otimes \cdots \otimes v_n).$$

This is not a coalgebra map, but is a retraction of the canonical inclusion $ST^cV \hookrightarrow T^cV$. Now a coderivation $D : T^cV \rightarrow T^cV$ induces a coderivation $D_S : ST^cV \rightarrow ST^cV$ by the composition $ST^cV \hookrightarrow T^cV \xrightarrow{f} T^cV \xrightarrow{P} ST^cV$. In particular, a coderivation of T^cV induces one of $\Lambda^cV = ST^c[sV]$, where V is thought of as living in degree zero: we introduce the *graded* symmetric coalgebra below. Once again, coderivations of ST^cV are determined by their projections onto ST_1^cV ; a map $f = f_1 + f_2 + \cdots : ST^c(V) \rightarrow V$ determines a coderivation \hat{f} as in the tensor coalgebra case.

In the remaining part of this survey, we choose to identify ST^cV with the abstract symmetric coalgebra S^cV under the isomorphism

$$v_1 \cdots v_n \mapsto P(v_1 \otimes \cdots \otimes v_n).$$

The coproduct in S^cV is given by

$$\Delta(v_1 \cdots v_n) = \sum_{i=0}^n \sum_{\sigma \in \Sigma_{i, n-i}} v_{\sigma(1)} \cdots v_{\sigma(i)} \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)}.$$

2.4. DGLA's and Perturbations of the Codifferential

Definition 1. For any chain complex (X, d) , and odd ∂ , with $(d + \partial)^2 = 0$, we say that ∂ is a *perturbation* of the differential d . We call $d + \partial$ the

perturbed differential. Equivalently, we have $[d, \partial] + \partial\partial = 0$ in $\text{End}(X)$. If ∂ is also compatible with an existing coalgebra structure on X , we say that it is a *coalgebra perturbation*.

Let (g, d) be a graded chain complex (d lowers degrees) with a bracket $[\ , \]$ that is skew-symmetric (not necessarily Leibniz or a chain map). Consider the differential graded symmetric coalgebra $S^c[sg]$, the differential d being induced by that on g . Also let ∂ be the coderivation on $S^c[sg]$ of degree -1 induced by the bracket.

Proposition 2. *The bracket $[\ , \]$ turns (g, d) into a DGLA if and only if ∂ is a coalgebra perturbation of d . Also, any DGLA structure on g is determined by the coalgebra perturbation induced from the bracket.*

When g is an ordinary (degree-zero) Lie algebra over a field, $S^c[sg] = \Lambda^c g$ with differential ∂ corresponding to the bracket is the ordinary Koszul or Chevalley-Eilenberg complex computing the homology of g with coefficients in the field.

2.5. Strongly Homotopy Lie Algebras

Definition 2. Let (g, d) be a chain complex and let d also denote the codifferential in $S^c[sg]$ induced by d . A *strongly homotopy Lie* (sh-Lie, or L_∞) structure on g is a perturbation $\partial = \partial_2 + \dots + \partial_n + \dots$ of d , i.e. an odd coderivation satisfying $[d, \partial] + \partial\partial = 0$ and $\partial\eta = 0$ (recall that η is the counit) so that the sum $d + \partial$ endows $S^c[sg]$ with a new coaugmented DG coalgebra structure.

The corresponding mega-map $\ell_2 + \dots + \ell_n + \dots$ from $S^c[sg]$ to g extends the differential $\ell_1 = d : sg \rightarrow g$, and the lower identities satisfied by

$$\ell = \ell_1 + \ell_2 + \dots + \ell_n + \dots$$

read as follows:

$$\begin{aligned} \ell_1^2 &= 0, \\ \ell_1(\ell_2(a, b)) \pm \ell_2(\ell_1(a), b) \pm \ell_2(\ell_1(b), a) &= 0, \\ \ell_1(\ell_3(a, b, c)) \pm \ell_3(\ell_1(a), b, c) \pm \ell_3(\ell_1(b), a, c) \pm \ell_3(\ell_1(c), a, b) \\ &\pm \ell_2(\ell_2(a, b), c) \pm \ell_2(\ell_2(a, c), b) \pm \ell_2(\ell_2(b, c), a) &= 0. \end{aligned}$$

An sh-Lie morphism between two sh-Lie (or DGL) algebras $(g, d + \dots)$ and $(g', d' + \dots)$ is a collection of chain maps $F_n : S_n^c[sg] \rightarrow S_n^c[sg']$, satisfying

$\Delta'F(u) = (F \otimes F)(\Delta u)$. Then F is uniquely determined by its projection onto sg' , that is, we may assume $F_n : S_n^c[sg] \rightarrow sg'$.

Definition 3. A *quasi-isomorphism* F between sh-Lie algebras g, g' is an sh-Lie morphism such that $F_1 : sg \rightarrow sg'$ induces an isomorphism between $H(g, d)$ and $H(g', d')$.

Remark 1. Quasi-isomorphisms between DGLA's are especially important in deformation theory. Such a map gives a one-to-one correspondence between moduli spaces of solutions to MC equations in $\hbar g[[\hbar]]$ and $\hbar g'[[\hbar]]$ (see [5]): given a quasi-isomorphism $F : S^c[sg] \rightarrow sg'$, we define $\tilde{F} : \hbar g[[\hbar]] \rightarrow \hbar g'[[\hbar]]$ by

$$\tilde{F}(r) = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(r, \dots, r)$$

(also see [6]).

2.6. The Hochschild Chain Complex and DGA's

Let (A, μ) be a unital associative algebra (possibly graded), and $T^c[sA]$ denote the tensor coalgebra on the suspension of A . We recall that

$$\text{Coder}(T^c[sA]) \cong \text{Hom}(T^c[sA], A).$$

In particular, the associative multiplication $\mu \in \text{Hom}(T^c[sA], A)$ corresponds to a square-zero coderivation $\partial : T^c[sA] \rightarrow T^c[sA]$ defined by

$$\begin{aligned} & \partial(a_1 \otimes \dots \otimes a_n) \\ &= \sum_{i=1}^{n-1} (-1)^{i+1} (a_1 \otimes \dots \otimes \mu(a_i \otimes a_{i+1}) \otimes \dots \otimes a_n) \\ & \quad + (-1)^{n+1} (\mu(a_n \otimes a_1) \otimes a_2 \otimes \dots \otimes a_{n-1}). \end{aligned}$$

The condition that ∂ is a codifferential is equivalent to the associativity condition $m \circ m = 0$, where \circ is the Gerstenhaber composition on multilinear maps (a right pre-Lie map). The complex $(\text{Hom}(T^c[sA], A), \partial)$ is known as the *Hochschild chain complex*.

Now let (A, μ, d) be a DGA. Then $d + \mu \in \text{Hom}(T^c[sA], A)$ corresponds to a perturbed codifferential $d + \partial$ satisfying $(d + \partial)^2 = 0$, which is equivalent to the identities $d^2 = 0$ and $[d, \partial] + \partial\partial = 0$. The latter can also be split into $[d, \partial] = 0$ and $\partial\partial = 0$.

Proposition 3. *The multiplication μ turns (A, d) into a DGA if and only if ∂ is a coalgebra perturbation of d . Also, any DGA structure on A is determined by the coalgebra perturbation induced from μ .*

2.7. Strongly Homotopy Associative Algebras

Definition 4. Let (A, d) be a chain complex and let d also denote the codifferential in $T^c[sA]$ induced by d . A *strongly homotopy associative* (or A_∞) structure on A is a perturbation $\partial = \partial_2 + \dots + \partial_n + \dots$ of d , i.e. an odd coderivation satisfying $[d, \partial] + \partial\partial = 0$ and $\partial\eta = 0$ so that the sum $d + \partial$ endows $T^c[sA]$ with a new coaugmented DG coalgebra structure.

The corresponding mega-map $m_2 + \dots + m_n + \dots$ from $T^c[sA]$ to A extends the differential $m_1 = d : sA \rightarrow A$, and the lower identities satisfied by

$$m = m_1 + m_2 + \dots + m_n + \dots$$

read as follows:

$$\begin{aligned} m_1^2 &= 0 \\ m_1(m_2(a, b)) \pm m_2(m_1(a), b) \pm m_2(m_1(b), a) &= 0 \\ m_1(m_3(a, b, c)) \pm m_3(m_1(a), b, c) \pm m_3(a, m_1(b), c) \\ &\quad \pm m_3(a, b, m_1(c)) = 0. \end{aligned}$$

The mega-identity is $m \circ m = 0$, sometimes written in the braces notation $\{m\}\{m\} = 0$.

3. Master Equation

If (A, d) is a differential graded associative algebra (DGA), then the equation

$$d\tau = \tau\tau \tag{1}$$

is called the *Master Equation (ME)* (or *Maurer-Cartan equation (MCE)*, etc.). Similarly if (g, d) is a DGLA, then the equation

$$d\tau = \frac{1}{2}[\tau, \tau] \tag{2}$$

is also called the Master Equation. Sometimes the sign convention is

$$d\tau + \tau\tau = 0, \tag{3}$$

or

$$d\tau + \frac{1}{2}[\tau, \tau] = 0. \quad (4)$$

Clearly any solution of such an equation must be an odd element of the algebra. Moreover, in case A is the graded universal enveloping algebra of g , or g is the Lie algebra obtained from A by the usual bracket, then solutions of the DGLA master equation are also solutions of the DGA master equation.

Remark 2. If τ is a solution of equation (2) or equation (4) in a DGLA g , then the odd derivation $d_\tau = d - \text{ad}_\tau$ or $d_\tau = d + \text{ad}_\tau$ respectively defines a new DGLA structure on g with respect to the old bracket.

Example 1. If g is a DGLA or L_∞ algebra, then the corresponding coalgebra perturbation ∂ in $\text{Coder}(S^c(sg))$ is a solution of the ME in the DGA $\text{End}(S^c(sg))$, where the differential is $D = \text{ad}_d$.

Example 2. (Gauge Theory) Let ξ be a principal bundle with structure group G and Lie algebra g . There is a graded Lie algebra structure on the $\text{ad}(\xi)$ -valued de Rham forms induced by g . Given a connection A and an $\text{ad}(\xi)$ -valued 1-form η , the sum $A + \eta$ is again a connection, and its curvature is

$$F_{A+\eta} = F_A + d_A\eta + \frac{1}{2}[\eta, \eta].$$

In particular, $F_A = F_{A+\eta}$ if and only if

$$d_A\eta + \frac{1}{2}[\eta, \eta] = 0$$

(the Maurer-Cartan equation). Here d_A is the covariant derivative of the connection A . When A is a flat connection (zero curvature) then there exists a DGLA structure on the $\text{ad}(\xi)$ -valued differential forms ($d_A^2 = 0$) and $F_{A+\eta}$ is also flat iff the MCE is satisfied (then the covariant derivative for $A + \eta$ is d_τ).

4. Twisting Cochain

The notion of a twisting cochain generalizes that of a connection in differential geometry.

4.1. Differential on Hom

If (C, d_C) and (A, d_A) are chain complexes, the following differential D makes $\text{Hom}(C, A)$ into a chain complex: $D\phi = d_A\phi \pm \phi d_C$.

4.2. Cup Product and Cup Bracket

Proposition 4. For any differential graded coalgebra C and a differential graded associative algebra (DGA) A , the chain complex $(\text{Hom}(C, A), D)$ becomes a DGA via the cup (convolution) product $a \smile b$ defined by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{a \otimes b} A \otimes A \xrightarrow{\mu} A.$$

The coaugmentation and augmentation maps η and ϵ on C and A respectively define an augmentation map on $(\text{Hom}(C, A), D)$.

Proposition 5. For any differential graded coalgebra C and a DGLA g , the chain complex $(\text{Hom}(C, g), D)$ becomes a DGLA via the cup bracket $[a, b]$ defined by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{a \otimes b} g \otimes g \xrightarrow{[,] } g.$$

Example 3. If g is a Lie algebra, the cup bracket on $\text{Hom}(S^c[sg], g)$ is defined as above. For example, if τ and κ are maps $S^c_1[g] \rightarrow g$, then $[\tau, \kappa]$ may be nonzero only on $S^c_2[sg]$. In this case, we compute

$$\Delta(xy) = 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1 \quad (x, y \in sg),$$

and

$$[\tau, \kappa](xy) = [\tau(x), \kappa(y)] + [\tau(y), \kappa(x)]. \tag{5}$$

Example 4. The Hochschild complex of an associative algebra (A, μ) (where $\mu^2 = 0$): Let

$$C^\bullet(A) = \text{Hom}(T^c A, A) = \text{Hom}\left(\bigoplus_{n \geq 0} A^{\otimes n}, A\right),$$

with differential $D = \text{ad}_\mu \in \text{Der}(C^\bullet(A))$. The cup product $x \smile y = \{\mu\}\{x, y\}$ is the composition

$$T^c A \xrightarrow{\Delta} T^c A \otimes T^c A \xrightarrow{x \otimes y} A \otimes A \xrightarrow{\mu} A;$$

if x is an n -linear map and y is an m -linear map, then

$$(x \smile y)(a_1 \otimes \cdots \otimes a_{n+m}) = x(a_1 \otimes \cdots \otimes a_n) \cdot y(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

Remark 3. The differential D above is an inner derivation and not derived from differentials on A and $T^c A$. Still, it is a derivation of the cup product.

4.3. Twisting Cochain

Definition 5. Given a coaugmented DG coalgebra C and an augmented DGA A , a *twisting cochain* is a homogeneous morphism $t : C \rightarrow A$ of degree -1 such that $\epsilon\tau = 0$ and $\tau\eta = 0$, and which satisfies $Dt = t \smile t$.

In other words, a twisting cochain is a solution of the master equation on $\text{Hom}(C, A)$ with the usual differential D induced from those of C, A and the product is the cup product.

Definition 6. Given a DG cocommutative coalgebra C and a DGLA g , a *Lie algebra twisting cochain* $t : C \rightarrow g$ is a homogeneous map of degree -1 whose composition with the coaugmentation map is zero, and which satisfies

$$Dt = \frac{1}{2}[t, t] \tag{6}$$

($[,]$ being the cup bracket).

Recall that a DGLA structure on a graded chain complex (g, d) is given by a perturbation ∂ of the corresponding codifferential on $S^c[sg]$. Moreover, the piece $d\partial + \partial d = 0$ of $(d + \partial)^2 = 0$ says that the bracket is a chain map and the piece $\partial^2 = 0$ says that the bracket satisfies the Jacobi identity. Let us denote the symmetric coalgebra with the codifferential ∂ by $S^c_{[,]}[sg]$. Quillen’s notation $C[g]$ for the same DG coalgebra reminds us that this is the Koszul or Chevalley-Eilenberg complex that computes the homology of g without any regard for the additional differential d on g .

Example 5. For any DGLA g , its *universal Lie algebra twisting cochain*

$$\tau_g : S^c_{[,]}[sg] \rightarrow g$$

is given by

$$\begin{aligned} \tau_g(sx) &= x \quad \text{for } x \in g, \\ \tau_g(y) &= 0 \quad \text{for } y \in S^c_k[sg], k \neq 1. \end{aligned}$$

That is, an element with tensor degree one goes to its desuspension and everything else goes to zero. Clearly, the composition $\tau_g\eta$ is zero, as $\tau_g = 0$ on constants. Next, we show that τ_g satisfies the equation (6), but we note that in this construction the differential on g itself is taken to be zero. On the left-hand side, we have

$$D\tau_g(x_1 \wedge \cdots \wedge x_n) = \tau_g\partial(x_1 \wedge \cdots \wedge x_n)$$

which is zero if $n \neq 2$ and is equal to $[x_1, x_2]$ if $n = 2$. Meanwhile, on the right-hand side, we have

$$\frac{1}{2}[\tau_g, \tau_g](x_1 \wedge \cdots \wedge x_n)$$

which is zero if $n \neq 2$ and is equal to

$$\frac{1}{2}\{[\tau_g(x_1), \tau_g(x_2)] \pm [\tau_g(x_2), \tau_g(x_1)]\} = [x_1, x_2]$$

if $n = 2$.

Remark 4. The universal property of the universal LA twisting cochain is that every Lie algebra twisting cochain factors through this one: that is, whenever C is a coalgebra and $\tau : C \rightarrow g$ is a twisting cochain, then $\tau_g \circ c(\tau) = \tau$, where $c(\tau) : C \rightarrow S_{[\cdot, \cdot]}^c[sg]$ is the unique coalgebra map induced by τ .

Using HPT, we will construct formal solutions

$$\tau \in \text{Hom}(S_D^c[sH(g)], g)$$

of the master equation. Once we make the choice of a contraction, we will obtain explicit inductive formulas for D and τ .

5. Homological Perturbation Theory (HPT)

“HPT is concerned with transferring various kinds of algebraic structure through a homotopy equivalence”. Also: “HPT is a set of techniques for the transference of structures from one object to another up to homotopy” (Real [14]).

5.1. Contraction

Definition 7. Let (M, d_M) and (N, d_N) be chain complexes, $\pi : N \rightarrow M$ and $\nabla : M \rightarrow N$ be chain maps, and $h \in \text{End}(N)$ be a morphism (possibly preserving some extra structure) of degree 1. Then a *contraction*

$$(M \begin{matrix} \xrightarrow{\pi} \\ \xleftarrow{\nabla} \end{matrix} N, h) \tag{7}$$

of N onto M is a collection of the above data satisfying

$$\pi \nabla = \text{id}_M,$$

$$D(h) = \text{ad}_{d_N}(h) = \nabla\pi - \text{id}_N$$

$$\pi h = 0, \quad h\nabla = 0, \quad hh = 0.$$

Another way to describe this structure is to say that M is a *strong deformation retract (SDR)* of N (also called *Eilenberg-Zilber data*). The properties on the last line are referred to as the *annihilation properties* or *side conditions*. Note that the first line makes π surjective (projection) and ∇ , injective (inclusion). The map h is also known as the *homotopy operator* between $\nabla\pi$ and id_N :

$$\nabla\pi - \text{id}_N = D(h) = d_N h + h d_N$$

($D = \text{ad}_{d_N}$ is the induced differential on $\text{Hom}(N, N)$).

Often *filtered contractions* are considered.

Remark 5. Lambe and Stasheff [12] noticed that the side conditions on h are not restrictive: if $\pi h = 0$ and $h\nabla = 0$ are not satisfied, then we can replace h by $h' = D(h)hD(h)$. Now if $h^2 = 0$ is not satisfied either, we replace h' by $h'' = h'd_N h'$, which finally gives us an operator h'' satisfying the side conditions.

Lemma 2. *Given a contraction (7), we have a (not necessarily direct) sum*

$$N = \text{Im}(\nabla) + \text{Im}(h) + \text{Im}(d_N).$$

Proof. Each $x \in N$ can be written as

$$x = \nabla\pi(x) - h d_N(x) - d_N h(x). \tag{8}$$

Lemma 3. (see [14]) *Given a contraction (7), we have*

$$\text{Im}(\nabla) + \text{Im}(h) = \text{Im}(\nabla) \oplus \text{Im}(h) = \text{Ker}(h).$$

Proof. We have

$$\text{Im}(\nabla) \subset \text{Ker}(h) \quad \text{and} \quad \text{Im}(h) \subset \text{Ker}(h)$$

since $h\nabla = 0$ and $h^2 = 0$. Conversely, by (8), each $x \in \text{Ker}(h)$ can be written as

$$x = \nabla\pi(x) - h d_N(x) \in \text{Im}(\nabla) + \text{Im}(h).$$

That the sum is direct can be seen as follows: let $x \in \text{Im}(\nabla) \cap \text{Im}(h)$. Then we have $x = \nabla(y) = h(z)$ for some $y \in M$ and $z \in N$. Rewriting the decomposition of x in $\text{Ker}(h)$, we obtain

$$x = \nabla\pi(x) - h d_N(x) = \nabla\pi h(z) - h d_N \nabla(y) = 0 - h d_N \nabla(y) \quad (\pi h = 0)$$

$$= h \nabla d_M(y) \quad (\nabla \text{ chain map}) = 0 \quad (h\nabla = 0). \quad \square$$

Corollary 1. *For any contraction (7), we have*

$$H(N, h) \cong \text{Im}(\nabla) \cong M.$$

Lemma 4. *Given any contraction (7), we have*

$$\text{Im}(h) + \text{Im}(d_N) = \text{Im}(h) \oplus \text{Im}(d_N).$$

Moreover, if $d_M \equiv 0$, then

$$\text{Im}(h) \oplus \text{Im}(d_N) = \text{Ker}(\pi).$$

Proof. Say $x \in \text{Im}(h) \cap \text{Im}(d_N)$. Then $x = h(y) = d_N(z)$ for some $y, z \in N$, so that by (8) we obtain

$$x = \nabla\pi h(y) - hd_Nd_N(z) - d_Nhh(y) = 0.$$

It is always true that $\text{Im}(h) \subset \text{Ker}(\pi)$ as $\pi h = 0$. If $d_M \equiv 0$, we further have the result

$$\pi d_N(x) = -d_M\pi x = 0,$$

so that altogether

$$\text{Im}(h) \oplus \text{Im}(d_N) \subset \text{Ker}(\pi).$$

Conversely, for $x \in \text{Ker}(\pi)$, we see that (even without the condition $d_M = 0$)

$$\text{Ker}(\pi) \subset \text{Im}(h) + \text{Im}(d_N)$$

since

$$x = -hd_N(x) - d_Nh(x) \quad \forall x \in \text{Ker}(\pi)$$

due to (8). □

Lemma 5. *For any contraction (7) with $d_M = 0$ we have*

$$\text{Im}(\nabla) + \text{Im}(d_N) = \text{Im}(\nabla) \oplus \text{Im}(d_N) = \text{Ker}(d_N).$$

Proof. The given sum is direct: let $x \in \text{Im}(\nabla) \cap \text{Im}(d_N)$. Then $x = \nabla(y) = d_N(z)$ and by (8)

$$x = \nabla\pi d_N(z) - hd_Nd_N(z) - d_Nh\nabla(y) = \nabla\pi d_N(z) = -\nabla d_M\pi(z) = 0.$$

Clearly, we have $\text{Im}(d_N) \subset \text{Ker}(d_N)$. Also $\text{Im}(\nabla) \subset \text{Ker}(d_N)$, because

$$d_N\nabla(x) = -\nabla d_M(x) = 0.$$

Conversely, by (8),

$$\text{Ker}(d_N) \subset \text{Im}(\nabla) + \text{Im}(d_N)$$

since we can write

$$x = \nabla\pi(x) - d_N h(x)$$

(no condition on d_M) for $x \in \text{Ker}(d_N)$. □

Corollary 2. *For any contraction (7) with $d_M = 0$ we have*

$$H(N, d_N) \cong \text{Im}(\nabla) \cong M.$$

Proposition 6. *For any contraction (7) with $d_M = 0$ we have*

1. $N = \text{Im}(\nabla) \oplus \text{Im}(h) \oplus \text{Im}(d_N)$, where

$$\begin{aligned} \text{Im}(\nabla) \oplus \text{Im}(h) &= \text{Ker}(h), \\ \text{Im}(h) \oplus \text{Im}(d_N) &= \text{Ker}(\pi), \\ \text{Im}(\nabla) \oplus \text{Im}(d_N) &= \text{Ker}(d_N), \end{aligned}$$

2. $\text{Im}(h) \xrightarrow{d_N} \text{Im}(d_N)$ is an isomorphism with inverse $\text{Im}(d_N) \xrightarrow{-h} \text{Im}(h)$, and

3. $\text{Im}(\nabla) \xrightarrow{\pi} M$ is an isomorphism with inverse $M \xrightarrow{\nabla} \text{Im}(\nabla)$; we also have

$$\text{Im}(\nabla) \cong M = \text{Im}(\pi) \cong H(N, d_N) \cong H(N, h).$$

Remark 6. This is a Hodge-type decomposition reminiscent of the case of a compact orientable Riemannian manifold M without boundary. If

$$* : \Omega^r(M) \rightarrow \Omega^{\dim(M)-r}$$

is the ‘‘Hodge star operator’’ (an isomorphism) and

$$d : \Omega^{r-1}(M) \rightarrow \Omega^r(M)$$

is the de Rham differential, then we define a ‘‘partial inverse’’ d^\dagger (the adjoint exterior derivative operator) to $-d$ by $d^\dagger = \pm * d*$. The commutator of d and d^\dagger is called the ‘‘Laplace-Beltrami operator’’: $\Delta = dd^\dagger + d^\dagger d$. Then there exists a unique decomposition of the algebra of de Rham forms as follows:

$$\Omega^r(M) = \text{Harm}^r(M) \oplus d(\Omega^{r-1}(M)) \oplus d^\dagger(\Omega^{r+1}(M)),$$

where the ‘‘harmonic forms’’ are given by $\text{Harm}^r(M) = \text{Ker}(\Delta)$. In the case of our general contraction with $d_M = 0$, the operators h and d_N replace d and d^\dagger respectively. What do we know about Δ here? We have

$$\Delta = D(h) = hd_N + d_N h = (h + d_N)^2 = \nabla\pi - \text{id}_N.$$

The kernel of this operator is equal to $\nabla(M)$, as we have

$$(\nabla\pi - \text{id}_N)(x) = 0 \Leftrightarrow \nabla\pi(x) = x \Leftrightarrow x \in \nabla(M),$$

or $\text{Ker}(\Delta) = \text{Im}(\nabla)$. So is there an analog of the Hodge star operator? If we define an isomorphism $* = h + d_N + \text{Id}_{\nabla(M)}$ (where the last operator is zero on the remaining direct summands), then we have $*^{-1} = -h - d_N + \text{Id}_{\nabla(M)}$, and

$$\begin{aligned} *d_N*^{-1} &= (h + d_N + \text{Id}_{\nabla(M)})d_N(-h - d_N + \text{Id}_{\nabla(M)}) \\ &= -hd_Nh = -h\text{Id}_{\text{Im}(d_N)} = h. \end{aligned}$$

Remark 7. The operator d^\dagger is more like the BV operator than the (even) Laplacian, which is not square-zero. Another similar case is Q (BRST operator) and b_0 (anti-ghost operator), for which we have $Qb_0 + b_0Q = L_0$ (the degree operator which is zero on the cohomology).

Proof. We only need to prove (2) and part of (3). First, we want to show that $-d_Nh = \text{id}_{\text{Im}(d_N)}$. But then for $x = d_N(y)$, we have

$$\begin{aligned} -d_Nh(x) &= [-d_Nh]d_N(y) = [hd_N - \nabla\pi + \text{id}_N]d_N(y) \\ &= -\nabla[\pi d_N](y) + d_N(y) = -\nabla[-d_M\pi]d_N(y) + x = x. \end{aligned}$$

Similarly, we would like to have $-hd_N = \text{id}_{\text{Im}(h)}$. If $x = h(y)$, then

$$\begin{aligned} -hd_N(x) &= [-hd_N]h(y) = [d_Nh - \nabla\pi + \text{id}_N]h(y) \\ &= -\nabla\pi h(y) + h(y) = x. \end{aligned}$$

Finally, we have $\pi\nabla = \text{id}_M$ and $\nabla\pi\nabla = \nabla\text{id}_M = \nabla$, which shows the isomorphism between $\nabla(M)$ and $\pi(N)$. \square

Example 6. Let (g, d) be a chain complex. Assume that the underlying ring is a field. Then there exists a contraction

$$(H(g) \underset{\nabla}{\overset{\pi}{\cong}} g, h) \tag{9}$$

of chain complexes, where the differential on $H(g)$ is zero: we can write g as a linear sum

$$g = G \oplus \text{Ker}(d) = G \oplus \text{Im}(d) \oplus H(g)$$

by choosing arbitrary representatives of the homology classes, etc. Let us show the decomposition of an element x of g by

$$x = x_G + x_{\text{Im}(d)} + x_{H(g)}.$$

Then π is the projection of g onto $H(g)$ and ∇ is the inclusion map of $H(g)$ into g . Note that as vector spaces G and $\text{Im}(d)$ are isomorphic via d : let $x, y \in G$. Then

$$dx = dy \Rightarrow d(x - y) = 0 \Rightarrow x - y \in \text{Ker}(d) \cap G = \{0\}$$

and $d : G \rightarrow \text{Im}(d)$ is one-to-one as well as onto. We define h to be the inverse of $-d$ on $\text{Im}(d)$ and zero on the rest of g . The linear map h is square-zero and increases degree by one. Moreover,

$$\begin{aligned} (dh + hd)(x) &= (dh + hd)(x_G + x_{\text{Im}(d)} + x_{H(g)}) \\ &= dh(x_{\text{Im}(d)}) + hd(x_G) = -x_{\text{Im}(d)} - x_G \end{aligned}$$

and

$$\begin{aligned} (\nabla\pi - \text{id}_g)(x_G + x_{\text{Im}(d)} + x_{H(g)}) \\ = x_{H(g)} - (x_G + x_{\text{Im}(d)} + x_{H(g)}) = -x_{\text{Im}(d)} - x_G. \end{aligned}$$

In comparison with the last corollary, we have

$$\begin{aligned} (N, d_N) &= (g, d), & (M, d_M) &= (H(g, d), 0), \\ \text{Im}(\nabla) &= H(g, d), & \text{Im}(h) &= G, \\ \text{Im}(d_N) &= \text{Im}(d). \end{aligned}$$

Example 7. (The *Tensor Trick*) Any contraction (7) of chain complexes induces a filtered contraction

$$(T^c[M] \begin{array}{c} \xrightarrow{T^c\pi} \\ \xleftarrow{T^c\nabla} \end{array} T^c[N], T^c h)$$

of coaugmented differential graded coalgebras. Here is how: the projection

$$P_N : T^c[N] \rightarrow N$$

followed by the surjective chain map $\pi : N \rightarrow M$ gives us a linear map

$$\begin{aligned} \pi \circ P_N : T^c[N] &\rightarrow M, \\ (\pi \circ P_N)(x_1 \otimes \cdots \otimes x_k) &= \begin{cases} \pi(x_1) & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which can then be made into a unique coalgebra map

$$T^c\pi : T^c[N] \rightarrow T^c[M]$$

with the usual formula

$$T^c\pi(x_1 \otimes \cdots \otimes x_k) = \sum_{i=1}^k x_1 \otimes \cdots \otimes \pi(x_i) \otimes \cdots \otimes x_k.$$

Next, the morphisms $T^c\pi$ and $T^c\nabla$ pass to the corresponding morphisms on the coalgebras $S^c[N]$ and $S^c[M]$ respectively, and $S^c h$ is obtained from $T^c h$ by symmetrization, to yield a contraction

$$(S^c[M] \underset{S^c\nabla}{\overset{S^c\pi}{\rightleftarrows}} S^c[N], S^c h).$$

In particular, the contraction (9) induces

$$(S^c[sH(g)] \underset{S^c\nabla}{\overset{S^c\pi}{\rightleftarrows}} S^c[sg], S^c h), \tag{10}$$

which is a filtered contraction of coaugmented DG coalgebras. (Warning: $S^c\pi$ and $S^c\nabla$ are morphisms of coalgebras but $S^c h$ is not a coalgebra morphism, although it is somewhat compatible with the coalgebra structure, being a homotopy of coalgebra maps. One has to be careful when defining a homotopy of cocommutative coalgebras.)

5.2. The First Main Theorem

Assume that ∂ is the codifferential corresponding to an sh-Lie algebra structure on (g, d) . Since the corresponding multilinear map on g has other components than the binary bracket, we will denote the symmetric coalgebra on sg with codifferential ∂ by $S^c_\partial[sg]$ and not by $S^c_{[,] [sg]$. Given two sh-Lie algebras (g_1, ∂_1) and (g_1, ∂_2) , an *sh-morphism* or *sh-Lie map* from g_1 to g_2 is a morphism $S^c_{\partial_1}[sg_1] \rightarrow S^c_{\partial_2}[sg_2]$ of DG coalgebras.

Theorem 1. *Given a DGLA g and a contraction of chain complexes such as (9), the data determine*

- (i) *a differential D on $S^c[sH(g)]$ (a coalgebra perturbation of the zero differential) turning the latter into a coaugmented DG coalgebra, hence endowing $H(g)$ with an sh-Lie algebra structure,*
- (ii) *a Lie algebra twisting cochain*

$$\tau : S^c_D[sH(g)] \rightarrow g$$

Chain complex	Bracket(s)	Symmetric coalgebra	Coderivation	Property
(g, d) graded chain complex	$[,]$ generic bilinear bracket	$(S^c[sg], d)$ DG coalgebra; induced diff. d	∂ coderivation induced by $[,]$	$\partial^2 = 0$
(g, d) DGLA	$[,]$ Lie bracket d derivation	$(S^c_{[,]}[sg], d) = C[g]$ (generalized Koszul) Chevalley-Eilenberg complex	∂ coderivation induced by $[,]$	$(d + \partial)^2 = 0$
$(g, 0)$ Lie algebra	$[,]$ Lie bracket	$(S^c[sg], 0) = \Lambda^\bullet(g)$ Koszul complex for homology	∂ codifferential induced by $[,]$	$\partial^2 = 0$
(g, d) L_∞ algebra; $d = \ell_1$	ℓ_2, ℓ_3, \dots higher brackets	$(S^c_\partial[sg], d)$	∂ codiff. induced by ℓ_2, ℓ_3, \dots	$(d + \partial)^2 = 0$
$(H(g), 0)$ homology of DGLA (g, d) with given contraction	$[,]$ induced Lie bracket on $H(g)$	$(S^c_D[sH(g)], 0)$	D codiff. defined in the proof	$D^2 = 0$

with adjoint $\bar{\tau}$, written

$$\bar{\tau} = (S^c\nabla)_\partial : S^c_D[sH(g)] \rightarrow C[g],$$

that induces an isomorphism on the homology, and

(iii) an extension of $(S^c\nabla)_\partial$ to a new contraction

$$(S^c_D[sH(g)]) \xrightarrow[(S^c\nabla)_\partial]{(S^c\pi)_\partial} S^c_\partial[sg], (S^c d)_\partial$$

of filtered chain complexes (not necessarily of coalgebras).

Notes on Notation. While the induced bracket on $H(g)$ is a strict graded Lie bracket, the differential D may involve meaningful terms of higher order. Let us introduce a table for the notation used in [8] for different types of chain complexes and the corresponding symmetric coalgebras.

Sketch of the Proof. We obtain the differential D and the twisting cochain τ on $S^c[sH(g)]$ as infinite series by induction: for $b \geq 1$, write S^c_b for the homogeneous degree- b component of $S^c[sH(g)]$. Then D, τ for $b \geq 2$ are given by

$$\tau = \tau^1 + \tau^2 + \dots, \quad \tau^1 = \nabla\tau_{H(g)}, \quad \tau^j : S^c_j \rightarrow g, \quad j \geq 1, \quad (11)$$

$$\begin{aligned} \tau^b &= \frac{1}{2}h([\tau^1, \tau^{b-1}] + \dots + [\tau^{b-1}, \tau^1]), \\ D &= D^1 + D^2 + \dots, \end{aligned} \tag{12}$$

where D^{b-1} is the coderivation of $S^c[sH(g)]$ determined by

$$\tau_{H(g)}D^{b-1} = \frac{1}{2}\pi([\tau^1, \tau^{b-1}] + \dots + [\tau^{b-1}, \tau^1]) : S_b^c \rightarrow H(g).$$

That is, the coderivation followed by projection onto the degree-one subspace $sH(g)$ of $S^c[sH(g)]$ is given by the above formula. In the notation of Subsection 2.3, we have $f_b = \tau_{H(g)}D^{b-1}$ and $D = \hat{f}$. For example (dropping the symbol s for elements of $sH(g)$), we have $\tau^1(x) = x \in H(g) \subset g$, and $\tau^2(xy) = h[x, y]$ by (5). Let us also compute two terms of D :

$$\tau_{H(g)}D^1(xy) = \pi[x, y] \in H(g),$$

and

$$\tau_{H(g)}D^2(xyz) = \frac{1}{2}\pi([x, h[y, z]] + [h[x, y], z]),$$

etc. We can see why τ is a LA twisting cochain: since τ satisfies

$$\tau = h\left(\frac{1}{2}[\tau, \tau]\right),$$

we obtain

$$-d\tau = \frac{1}{2}[\tau, \tau]$$

in case of the particular SDR we constructed, and the last equation is the master equation (the differential on $H(g)$ being zero). The sums (11) and (12) are infinite, but when either one is applied to a specific element in some subspace of finite filtration degree, only finitely many terms will be nonzero (the summand D^1 is the ordinary Cartan-Chevalley-Eilenberg differential for the classifying coalgebra of the graded Lie algebra $H(g)$). The proof that D is indeed a coalgebra differential and τ is a twisting cochain “will be given elsewhere”. A “spectral sequence argument” shows that $\bar{\tau}$ induces an isomorphism on the homology. □

Remark 8. If $\nabla H(g)$ happens to be a Lie subalgebra of g , then $[\tau^1, \tau^1]$ will have values in $H(g)$ and $\tau^2 = (1/2)h[\tau^1, \tau^1]$, as well as the remaining τ^j , will be zero. Similarly, we will have $D = D^1$.

Corollary 3. *Under the hypotheses of Theorem 1,*

$$\tau : S_D^c[sH(g)] \rightarrow g, \tag{13}$$

viewed as an element of degree -1 of the DGLA $\text{Hom}(S_D^c[sH(g)], g)$, satisfies the master equation (2).

The twisting cochain (13) is our most general solution of the master equation. The other solutions of the master equation can be derived from it.

6. Corollaries and the Second Main Theorem

6.1. Other Corollaries of Theorem 1

Corollary 4. *Under the hypotheses of Theorem 1, suppose in addition that there is a differential \tilde{D} on $S^c[sH(g)]$ turning the latter into a coaugmented DG coalgebra in such a way that $(S^c\pi)\partial = \tilde{D}(S^c\pi)$. Then $D = \tilde{D}$ and $(S^c\pi)_\partial$ may be taken to be $S^c\pi$. In particular, when $(S^c\pi)\partial$ is zero, then the differential D on $S^c[sH(g)]$ is necessarily zero, that is, the new contraction in Theorem 1 has the form*

$$(S^c[sH(g)] \xrightarrow{(S^c\nabla)_\partial} S^c\pi) \xrightarrow{S^c\partial} S_\partial^c[sg], (S^c h)_\partial.$$

For example, this is the case when the composite

$$g \otimes g \xrightarrow{[\cdot, \cdot]} g \xrightarrow{\pi} H(g)$$

is zero.

Corollary 5. *Under the hypotheses of Theorem 1, suppose in addition that there is a differential \tilde{D} on $S^c[sH(g)]$ turning the latter into a coaugmented DG coalgebra in such a way that $\partial(S^c\nabla) = (S^c\nabla)\tilde{D}$. Then $D = \tilde{D}$ and $(S^c\nabla)_\partial = S^c\nabla$. In particular, when $\partial(S^c\nabla)$ is zero, then the differential D on $S^c[sH(g)]$ is necessarily zero, that is, the new contraction in Theorem 1 has the form*

$$(S^c[sH(g)] \xrightarrow{(S^c\nabla)_\partial} S^c\nabla) \xrightarrow{S^c\partial} S_\partial^c[sg], (S^c h)_\partial.$$

For example, this is the case when the composite

$$H(g) \otimes H(g) \xrightarrow{\nabla \otimes \nabla} g \otimes g \xrightarrow{[\cdot, \cdot]} g$$

is zero.

6.2. The Second Main Theorem

Theorem 2. *Given a DGLA g , a DGL subalgebra m of g , and a contraction*

$$(H(g) \begin{matrix} \xrightarrow{\pi} \\ \cong \\ \xrightarrow{\nabla} \end{matrix} m, h)$$

of chain complexes so that the composite

$$m \otimes m \xrightarrow{[\cdot, \cdot]} m \xrightarrow{\pi} H(g)$$

is zero, then the induced bracket on $H(g)$ is zero, that is, $H(g)$ is Abelian as a graded Lie algebra, and the data determine a solution $\tau \in \text{Hom}(S^c[sH(g)], g)$ of the master equation (2) in such a way that the following hold:

- (i) *The composite $\pi\tau$ coincides with the universal twisting cochain $S^c[sH(g)] \rightarrow H(g)$ for the Abelian Lie algebra $H(g)$, and*
- (ii) *the values of τ lie in m .*

7. Differential Gerstenhaber and BV Algebras

7.1. Differential Gerstenhaber Algebras

Definition 8. *A Gerstenhaber (or G -)algebra consists of:*

- *A graded commutative and associative algebra (A, μ) (μ suppressed), and*
- *A graded Lie bracket (the Gerstenhaber or G -bracket) $[\cdot, \cdot] : A \otimes A \rightarrow A$ of degree -1 , such that*
- *For each homogeneous element $a \in A$, bracketing with a is a derivation of the Lie bracket of degree $|a| - 1$.*

That is, we want $\text{ad}(a)$ to commute with the bracket for all $a \in A$.

Definition 9. *A differential G -algebra is a Gerstenhaber algebra $(A, [\cdot, \cdot])$ with a differential d of degree $+1$ on A which is a derivation of the multiplication on A .*

We want $[d, \mu] = 0$ and $[d, d] = 0$ in Gerstenhaber's composition bracket notation.

Definition 10. *A differential G -algebra is called strict if the differential d is a derivation of the G -bracket as well.*

We want d to commute with the bracket.

Let $(A, [,], d)$ be a strict differential G -algebra. We will for the moment ignore all the extra structure on A except for the G -bracket and the differential. As such, A is a DGLA, and we will change the notation to g to emphasize that. We will use the grading

$$g_1 = A^0, g_0 = A^1, g_{-1} = A^2, \dots, g_{-n} = A^{n+1}, \dots$$

so that the graded bracket and the differential on g are now “ordinary”: namely,

$$[,] : g_j \otimes g_k \rightarrow g_{j+k}, \quad d : g_j \rightarrow g_{j-1}.$$

Consider a contraction of g onto $H(g)$ as in (9). Let ∂ denote the operator (“perturbation of d ”) on $S_\partial^c[sg]$ corresponding to the Lie bracket on g . By the main theorem (Theorem 1), we can transfer it to the symmetric coalgebra of the homology: there exists a new contraction

$$(S_D^c[sH(g)] \stackrel{(S^c\pi)_\partial}{=} S_\partial^c[sg], (S^c d)_\partial) \stackrel{(S^c\nabla)_\partial}{\leftarrow}$$

of not only filtered chain complexes but of *filtered differential graded coalgebras*. The twisting cochain τ of (13) of Corollary 1 is now an element of

$$\text{Hom}(S_D^c[sH(g)], s^{-1}g)$$

of degree -2 , satisfying the master equation

$$D\tau = \frac{1}{2}[\tau, \tau],$$

where D is the Hom-differential and the graded cup bracket on the right-hand side refers to the one induced by the graded coalgebra structure on $S^c[sH(g)]$ and the graded Lie algebra structure on g .

7.2. Differential BV Algebras

Definition 11. Let $(A, [,], \Delta)$ be a G -algebra with an additional operator Δ on A of degree -1 . If Δ satisfies the condition

$$[a, b] = (-1)^{|a|} \left(\Delta(ab) - (\Delta a)b - (-1)^{|a|} a(\Delta b) \right),$$

then it is said to be a *generator* of the G -algebra. In this case, (A, Δ) is called a *weak Batalin-Vilkovisky (BV-) algebra*. If, moreover, Δ is exact (i.e. $\Delta^2 = 0$),

then (A, Δ) is simply called a *Batalin-Vilkovisky (BV-) algebra*. Koszul has shown that Δ behaves as a derivation for the G -bracket:

$$\Delta[x, y] = [\Delta x, y] - (-1)^{|x|}[x, \Delta y] \quad \forall x, y \in A.$$

With respect to the original graded commutative and associative product on A , we can only say that Δ is a second order differential operator, or $\Phi_{\Delta}^3(a, b) = 0$, where Φ_{Δ}^r are r -linear operators used to define higher order differential operators.

Now let us denote the bracket on a (weak) BV-algebra (A, Δ) by $[\ , \]$.

Definition 12. If d is a differential of degree $+1$ that endows $(A, [\ , \])$ with a differential G -algebra structure, such that $\Delta d + d\Delta = 0$, then the triple (A, Δ, d) is called a *(weak) differential BV-algebra*.

Proposition 7. For any weak differential BV-algebra (A, Δ, d) , the differential d behaves as a derivation of the G -bracket $[\ , \]$:

$$d[x, y] = [dx, y] - (-1)^{|x|}[x, dy].$$

That is, $(A, [\ , \], d)$ is a differential G -algebra.

Note: $(A, [\ , \], \Delta)$ is not a differential G -algebra unless Δ is exact.

Example 8. (see [7]) Let V be a \mathbf{Z}_2 -graded finite dimensional vector space and sV^* be its suspended-graded dual. If $\{x_i\}$ is a basis for V consisting of homogeneous elements, then the dual basis $\{x_i^*\}$ has the property that x_i and x_i^* always have opposite parities. Then the algebra $\mathbb{C}[[x_1, \dots, x_n, x_1^*, \dots, x_n^*]]$ of formal power series has the following BV operator (Laplacian):

$$\Delta = \sum_{i=1}^n \frac{\partial}{\partial x_i^*} \frac{\partial}{\partial x_i}.$$

Since the underlying algebra is graded commutative, the composition of two derivations is a second order differential operator by any definition. Moreover $\Delta^2 = 0$, which makes a BV-algebra out of this data.

7.3. Formality

7.3.1. Formality of Differential Graded P -Algebras

Recall that our ground ring is a field of characteristic zero. Let P be a differential graded operad and (A, d) be an algebra over P . We often want to know

to which extent the cohomology of a space reflects the underlying topological or geometrical properties of that space.

Definition 13. The P -algebra A is called *formal* if there exists a strongly homotopy P -algebra map $(H(A), 0) \xrightarrow{F} (A, d)$ which induces an isomorphism in homology.

7.3.2. Examples

Example 9. (Commutative DG Associative Algebras) A smooth manifold M is called *formal* if the commutative associative DG algebra of de Rham forms on M is formal in the sense of the above Definition. Examples are compact Kähler manifolds, Lie groups, and complete intersections. Poisson manifolds (proof by Sharygin and Talalaev).

Example 10. (DGLA's.) The Hochschild complex for the algebra $A = C^\infty(M)$ of smooth functions on a Poisson manifold M (Kontsevich).

7.4. Differential BV Algebras and Formality

Definition 14. We will say a differential BV-algebra (A, Δ, d) ($A = g$ as a Lie algebra) *satisfies the statement of the Kählerian Formality Lemma* (or the $\partial\bar{\partial}$ Lemma) if the maps

$$\left(\text{Ker}(\Delta), d|_{\text{Ker}(\Delta)} \right) \hookrightarrow (g, d), \quad \left(\text{Ker}(\Delta), d|_{\text{Ker}(\Delta)} \right) \xrightarrow{\text{proj}} H(g, \Delta)$$

are isomorphisms on the homology, where $H(g, \Delta)$ is endowed with the zero differential.

Remark 9. If the statement of the K.F.L. is satisfied, then proj can be extended to a contraction

$$\left((H(g, d), 0) \xrightleftharpoons[\nabla]{\pi} \left(\text{Ker}(\Delta), d|_{\text{Ker}(\Delta)} \right), h \right).$$

Since we have

$$H(\text{Ker}(\Delta), d) \cong H(g, d)$$

and

$$H(\text{Ker}(\Delta), d) \cong H(g, \Delta),$$

now we can have a contraction of $\text{Ker}(\Delta)$ onto $H(\text{Ker}(\Delta)) = H(g)$ as we did with g and $H(g)$.

Theorem 3. *Let (A, Δ, d) be a differential BV-algebra satisfying the statement of the Kählerian Formality Lemma and extend the projection proj to a contraction*

$$\left((H(g, d), 0) \underset{\nabla}{\overset{\pi}{\rightleftharpoons}} m, h \right),$$

where

$$m = \left(\text{Ker}(\Delta), d|_{\text{Ker}(\Delta)} \right)$$

and $\pi = \text{proj}$. Then $H(g)$ is abelian as a graded Lie algebra and the data determine a solution

$$\tau \in \text{Hom}(S^c[sH(g)], g)$$

of the master equation $d\tau = \frac{1}{2}[\tau, \tau]$ in such a way that the following hold:

- The values of τ lie in m , that is, the composite

$$\Delta \circ \tau : S^c[sH(g)] \rightarrow g$$

is zero;

- The composite $\pi\tau$ coincides with the universal twisting cochain for the abelian graded Lie algebra $H(g)$; so that

- For $k \geq 2$, the values of the component τ_k of τ on $S_k^c[sH(g)]$ lie in $\text{Im}(\Delta)$.

Proof. Follows from Theorem 2. □

Let us add the following condition to the ones in Theorem 3: suppose that A consists of a single copy of the ground ring F (necessarily generated by the unit 1 of A) and that $\Delta(1) = 0$. Then 1 generates a central copy of the ground ring in g (the ground ring commutes with all elements of A), and we may write

$$g = F \oplus \tilde{g}$$

as a direct sum of differential graded Lie algebras. Here \tilde{g} is the uniquely determined complement of F . Why unique? We have $g_1 = A_0 = F$ and we may take

$$\tilde{g} = \bigoplus_{n \geq 0} g_{-n} = \bigoplus_{n \geq 0} A^{n+1},$$

where \tilde{g} will be closed under the (degree-zero) Lie bracket:

$$[\cdot, \cdot] : g_j \otimes g_k \rightarrow g_{j+k},$$

with $j + k \leq 0$ if $j, k \leq 0$.

Corollary 6. *Assume that the hypotheses of Theorem 3, the abovementioned conditions ($A_0 = F$, $\Delta(1) = 0$), and the condition $H_1(g) \neq 0$ hold. Then the contraction of the Theorem can be chosen in such a way that*

$$\text{Im}(\tau_k) \subset \tilde{g} \quad \text{for } k \geq 2.$$

The statements of the main theorems have an interpretation in the context of deformation theory, as explained below.

8. Deformation Theory

Given a DGLA g , the construction of the universal solution τ from Theorems 2 and 3 relies on a chosen contraction. This provides a formal solution of the master equation (MCE), with a perturbed differential \mathcal{D} on $S^c[sHg]$ in the direction of (starting with) the Lie bracket induced on homology, endowing the former with a dg-coalgebra structure and a twisting cochain:

$$\tau : S_{\mathcal{D}}^c[sHg] \rightarrow g.$$

The moduli space interpretation of the set of solutions is along the lines of Schlesinger-Stasheff, see [15]. Since our focus is on the construction of solutions of the MCE, the reader is referred to the original text [8]. Additional details in terms of deformation functors, tangent cohomology, and the Kuranishi functor can be found in [13]. The relation between the latter functor and the construction of a twisting cochain corresponding to a contraction will be investigated elsewhere.

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