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ABOUT THE GENERALIZATION OF VORONOVSKAJA'S THEOREM FOR BERNSTEIN POLYNOMIALS OF TWO VARIABLES

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Abstract: In this paper, we study the uniform convergence and we will give the approximation theorems for Bernstein polynomials of two variables on a triangle.

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1. Introduction

In this section, we recall some results which we will use in this article.

Let the sets $\Delta_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, x, y \geq 0, x + y \leq 1\}$ and $\mathcal{F}(\Delta_2) = \{f | f : \Delta_2 \to \mathbb{R}\}$, $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let the operator $B_m : \mathcal{F}(\Delta_2) \to \mathcal{F}(\Delta_2)$, be defined for any function $f \in \mathcal{F}(\Delta_2)$ by

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$$(B_m f)(x,y) = \sum_{\substack{k,j=0\\ k+j\le m}} p_{m,k,j}(x,y) f\left(\frac{k}{m}\,\frac{j}{m}\right) \tag{1.1}$$

for any $(x,y)\in\Delta_2$, where

$$p_{m,k,j}(x,y) = \frac{m!}{k!\,j!(m-k-j)!}\, x^k y^j (1-x-y)^{m-k-j}. \tag{1.2}$$

The operators $(B_m)_{m\ge 1}$ are named the Bernstein bivariate polynomials (see [5]).

In the following, for $m\in\mathbb{N}$ and $i,l\in\mathbb{N}_0$, let the polynomials in x and y be defined by

$$\begin{aligned} T_{m,i,l}(x,y) &= \sum_{\substack{k,j=0\\ k+j\le m}} p_{m,k,j}(x,y)(k-mx)^i (j-my)^l \\ &= m^{i+l}\left(B_m(\cdot - x)^i (* - y)^l\right)(x,y) \end{aligned} \tag{1.3}$$

for any $(x,y)\in\Delta_2$, where "$\cdot$" and "$*$" stand for the first and the second variable.

In [11] the results contained in Lemma 1 – Lemma 3 exist.

Lemma 1. *For any $m,i,l\in\mathbb{N}$ and any $(x,y)\in\Delta_2$, we have*

$$\begin{aligned} T_{m,i+1,l}(x,y) &= x(1-x)\left[\frac{\partial T_{m,i,l}}{\partial x}(x,y) + miT_{m,i-1,l}(x,y)\right] \\ &\quad - xy\left[\frac{\partial T_{m,i,l}}{\partial y}(x,y) + mlT_{m,i,l-1}(x,y)\right] \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} T_{m,i,l+1}(x,y) &= y(1-y)\left[\frac{\partial T_{m,i,l}}{\partial y}(x,y) + mlT_{m,i,l-1}(x,y)\right] \\ &\quad - xy\left[\frac{\partial T_{m,i,l}}{\partial x}(x,y) + miT_{m,i-1,l}(x,y)\right]. \end{aligned} \tag{1.5}$$

Let $m\in\mathbb{N}$ and $(x,y)\in\Delta_2$. From (1.1) follows immediately that $T_{m,0,0}(x,y) = 1$, $T_{m,1,0}(x,y) = T_{m,0,1}(x,y) = 0$, $T_{m,1,1}(x,y) = -mxy$ and taking (1.4) and (1.5) into account, we obtain

$$\begin{aligned} &T_{m,2,1}(x,y) = -mxy(1-2x), \quad T_{m,1,2}(x,y) = -mxy(1-2y), \\ &T_{m,2,0}(x,y) = mx(1-x), \quad T_{m,0,2}(x,y) = my(1-y), \end{aligned}$$

$$T_{m,2,2}(x,y) = m^2xy(3xy - x - y + 1) + mxy(-6xy + 2x + 2y - 1),$$
$$T_{m,3,0}(x,y) = mx(1-x)(1-2x), \quad T_{m,0,3}(x,y) = my(1-y)(1-2y),$$
$$T_{m,3,1}(x,y) = -3m^2x^2y(1-x) + 6mx^2y(1-x) - mxy,$$
$$T_{m,1,3}(x,y) = -3m^2xy^2(1-y) + 6mxy^2(1-y) - mxy,$$
$$T_{m,4,0}(x,y) = 3m^2[x(1-x)]^2 - 6m[x(1-x)]^2 + mx(1-x),$$
$$T_{m,0,4}(x,y) = 3m^2[y(1-y)]^2 - 6m[y(1-y)]^2 + my(1-y).$$

Lemma 2. *Let $m, i, l \in \mathbb{N}_0$, $m \neq 0$ and $(x, y) \in \Delta_2$. The polynomial $T_{m,i,l}(x, y)$ can be written*

$$T_{m,i,l}(x,y) = m^{\left[\frac{i+l}{2}\right]} a_{i,l}(x,y) + Q_{m,i,l}(x,y)\,, \tag{1.6}$$

where $a_{i,l}(x, y)$ is a polynomial in x and y not depended of m, $Q_{m,i,l}(x, y)$ is polynomial in x, y and m, where the degree of m in $Q_{m,i,l}(x, y)$ is strictly smaller than $\left[\frac{i+l}{2}\right]$.

Lemma 3. *Let $i, l \in \mathbb{N}_0$. Then the constant $k_{i,l}$ exists, depending on i and l, such that the inequality*

$$|T_{m,i,l}(x,y)| \leq m^{\left[\frac{i+l}{2}\right]} k_{i,l} \tag{1.7}$$

holds for any $m \in \mathbb{N}$ and any $(x, y) \in \Delta_2$.

Let $m \in \mathbb{N}$ and $(x, y) \in \Delta_2$.

Taking Lemma 2 into account, we obtain

$$a_{1,1}(x,y) = -xy,\ a_{1,2}(x,y) = -xy(1-2y),\ a_{2,1}(x,y) = -xy(1-2x),$$
$$a_{2,0}(x,y) = x(1-x),\ a_{0,2}(x,y) = y(1-y),\ a_{2,2}(x,y) = xy(3xy - x - y + 1),$$
$$a_{3,0}(x,y) = x(1-x)(1-2x),\ a_{0,3}(x,y) = y(1-y)(1-2y),$$
$$a_{3,1}(x,y) = -3x^2y(1-x),\ a_{1,3}(x,y) = -3xy^2(1-y),$$
$$a_{4,0}(x,y) = 3\left[x(1-x)\right]^2,\ a_{0,4}(x,y) = 3\left[y(1-y)\right]^2.$$

Taking Lemma 3 into account, we have $T_{m,0,0}(x,y) = 1 = k_{0,0}$, and because $x(1-x) \leq \dfrac{1}{4}$ $y(1-y) \leq \dfrac{1}{4}$ for any $x, y \in [0,1]$, then $\dfrac{T_{m,2,0}(x,y)}{m} = x(1-x) \leq \dfrac{1}{4} = k_{2,0}$ and similarly $k_{0,2} = \dfrac{1}{4}$ and

$$\frac{T_{m,2,2}(x,y)}{m^2} = xy(3xy - x - y + 1) + \frac{1}{m}\, xy\left[2x(1-y) + 2y(1-x) - 2xy - 1\right]$$

$$\leq xy(3xy - x - y + 1) + \frac{1}{m} xy \left[2x(1-y) + 2y(1-x)\right]$$
$$\leq xy(3xy - x - y + 1) + xy \left[2x(1-y) + 2y(1-x)\right]$$
$$= x^2 y(1-y) + xy^2 + xy \leq \frac{9}{4} = k_{2,2}.$$

Let $I_1, I_2 \subset \mathbb{R}$ be given intervals and $f : I_1 \times I_2 \to \mathbb{R}$ a bounded function. The function $\omega_{total}(f; \cdot, *) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined for any $(\delta_1, \delta_2) \in [0,\infty) \times [0,\infty)$ by

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\left\{|f(x,y) - f(x',y')| : (x,y),(x',y') \in I_1 \times I_2, \; |x - x'| \leq \delta_1, |y - y'| \leq \delta_2\right\} \tag{1.8}$$

is called the first modulus of smoothness for the function f or total modulus of continuity for the function f.

The first order modulus of smoothness for bivariate functions has properties similar to the properties of the first order modulus of smoothness for univariate functions. Some of them are contained in Lemma 4.

Lemma 4. *The first order modulus of smoothness for the bounded function $f : I_1 \times I_2 \to \mathbb{R}$ has the following properties:*

(i) $\omega_{total}(f;\delta_1,\delta_2) \leq \omega_{total}(f;\delta_1',\delta_2')$ for any (δ_1,δ_2), $(\delta_1',\delta_2') \in [0,\infty)\times[0,\infty)$ such that $\delta_1 \leq \delta_1'$ and $\delta_2 \leq \delta_2'$;

(ii) $\omega_{total}(f;|t-x|,|\tau-y|) \leq \left(1+\delta_1^{-2}(t-x)^2\right)\left(1+\delta_2^{-2}(\tau-y)^2\right)\omega_{total}(f;\delta_1,\delta_2)$ for any $(\delta_1,\delta_2) \in (0,\infty) \times (0,\infty)$ and any $(t,\tau),(x,y) \in I_1 \times I_2$.

For further information on this measure of smoothness see for example [14].

The following theorem is demonstrated in [8].

Theorem 5. *Let $I_1, I_2 \subset \mathbb{R}$ be intervals, $(a,b) \in I_1 \times I_2$, $n \in \mathbb{N}$ and the function $f : I_1 \times I_2 \to \mathbb{R}$, f admits partial derivatives of order n, continuous in the neighborhood V of the point (a,b). According to Taylor's expansion theorem for the function f around (a,b), for $(x,y) \in V$, we have*

$$f(x,y) = \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{\partial}{\partial x}(x-a) + \frac{\partial}{\partial y}(y-b)\right)^k f(a,b) + \rho^n(x,y)\mu(x-a,y-b) \tag{1.9}$$

$$\left(\frac{\partial}{\partial x}(x-a) + \frac{\partial}{\partial y}(y-b)\right)^k f(a,b) \tag{1.10}$$

$$= \sum_{i=0}^{n} \binom{k}{i} \frac{\partial^k f}{\partial x^{k-i} \partial y^i} (a,b)(x-a)^{k-i}(y-b)^i,$$

$k \in \{0, 1, \ldots, n\}$, *$\mu$ is a bounded function with* $\lim\limits_{(x,y)\to(a,b)} \mu(x-a, y-b) = 0$ *and*

$$\rho(x,y) = \sqrt{(x-a)^2 + (y-b)^2}. \tag{1.11}$$

Then for any $\delta_1, \delta_2 > 0$, *we have*

$$|\mu(x-a, y-b)| \le \frac{1}{n!}\left(1+\delta_1^{-2}(x-a)^2\right) \times \left(1+\delta_2^{-2}(y-b)^2\right) \sum_{i=0}^{n} \binom{n}{i} \omega_{total}\left(\frac{\partial^n f}{\partial x^{n-i} \partial y^i}\, ; \delta_1, \delta_2\right). \tag{1.12}$$

2. Main Results

In the following, let s be an even natural number.

Theorem 6. *Let $f : \Delta_2 \to \mathbb{R}$ be a bivariate function. If $(x,y) \in \Delta_2$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x,y), then*

$$m^{\frac{s}{2}} \left| (B_m f)(x,y) - \sum_{i=0}^{s} \sum_{l=0}^{i} \binom{i}{l} \frac{1}{i! m^i} \frac{\partial^i f}{\partial t^{i-l} \partial \tau^l} (x,y) T_{m,i-l,l}(x,y) \right| \tag{2.1}$$

$$\le \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \left[\frac{T_{m,s-2l,2l}(x,y)}{m^{\frac{s}{2}}} + \frac{T_{m,s-2l+2,2l}(x,y)}{m^{\frac{s+2}{2}}} + \frac{T_{m,s-2l,2l+2}(x,y)}{m^{\frac{s+2}{2}}} + \frac{T_{m,s-2l+2,2l+2}(x,y)}{m^{\frac{s+4}{2}}} \right] \times \sum_{i=0}^{s} \binom{s}{i} \omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i} ; \frac{1}{\sqrt{m}} \frac{1}{\sqrt{m}} \right).$$

and

$$\lim_{m\to\infty} m^{\frac{s}{2}} \left[(B_m f)(x,y) - \sum_{i=0}^{s} \sum_{l=0}^{i} \binom{i}{l} \frac{1}{i! m^i} \frac{\partial^i f}{\partial t^{i-l} \partial \tau^l} (x,y) T_{m,i-l,l}(x,y) \right] = 0. \tag{2.2}$$

If f admits partial derivatives of order s continuous on Δ_2, then the convergence given in (2.2) *is uniform on Δ_2 and*

$$m^{\frac{s}{2}}\left|(B_m f)(x,y)-\sum_{i=0}^{s}\sum_{l=0}^{i}\binom{i}{l}\frac{1}{i!m^i}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)T_{m,i-l,l}(x,y)\right| \tag{2.3}$$
$$\leq \frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\big[k_{s-2l,2l}+k_{s-2l+2,2l}+k_{s-2l,2l+2}$$
$$+k_{s-2l+2,2l+2}\big]\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m}}\,\frac{1}{\sqrt{m}}\right),$$

for any $(x,y)\in\Delta_2$ and any $m\in\mathbb{N}$.

Proof. Let $m\in\mathbb{N}$. According to Taylor's Theorem for the function f around (x,y), we have

$$f(t,\tau)=\sum_{i=0}^{s}\frac{1}{i!}\left(\frac{\partial}{\partial t}(t-x)+\frac{\partial}{\partial\tau}(\tau-y)\right)^i f(x,y)+\rho^s(t,\tau)\mu(t-x,\tau-y),$$

and so,

$$f(t,\tau)=\sum_{i=0}^{s}\frac{1}{i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)(t-x)^{i-l}(\tau-y)^l \tag{2.4}$$
$$+\rho^s(t,\tau)\mu(t-x,\tau-y),$$

where μ is a bounded function and $\lim\limits_{(t,\tau)\to(x,y)}\mu(t-x,\tau-y)=0$.

If $t=\frac{k}{m}$ $\tau=\frac{j}{m}$ multiplying by $p_{m,k,j}(x,y)$ and summing after k and j, $k,j\geq 0$ with $k+j\leq m$, we obtain

$$(B_m f)(x,y)=\sum_{i=0}^{s}\sum_{l=0}^{i}\binom{i}{l}\frac{1}{i!m^i}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)T_{m,i-l,l}(x,y)$$
$$+\sum_{\substack{k,j=0\\k+j\leq m}}p_{m,k,j}(x,y)\rho^s\left(\frac{k}{m}\,\frac{j}{m}\right)\mu\left(\frac{k}{m}-x,\frac{j}{m}-y\right).$$

Hence,

$$m^{\frac{s}{2}}\left[(B_m f)(x,y)-\sum_{i=0}^{s}\sum_{l=0}^{i}\binom{i}{l}\frac{1}{i!m^i}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)T_{m,i-l,l}(x,y)\right] \tag{2.5}$$

$$= (R_m f)(x,y)\,,$$

where

$$(R_m f)(x,y) = m^{\frac{s}{2}} \sum_{\substack{k,j=0\\ k+j\le m}} p_{m,k,j}(x,y)\rho^s\left(\frac{k}{m}\,\frac{j}{m}\right)\mu\left(\frac{k}{m}-x,\frac{j}{m}-y\right). \tag{2.6}$$

From (2.6) it follows that

$$|(R_m f)(x,y)| \le m^{\frac{s}{2}} \sum_{\substack{k,j=0\\ k+j\le m}} p_{m,k,j}(x,y)\rho^s\left(\frac{k}{m}\,\frac{j}{m}\right)\left|\mu\left(\frac{k}{m}-x,\frac{j}{m}-y\right)\right|. \tag{2.7}$$

According to (1.12), for any $\delta_1, \delta_2 > 0$ we have

$$\left|\mu\left(\frac{k}{m}-x,\frac{j}{m}-y\right)\right| \le \frac{1}{s!}\left(1+\delta_1^{-2}\left(\frac{k}{m}-x\right)^2\right)\left(1+\delta_2^{-2}\left(\frac{j}{m}-y\right)^2\right)\cdot$$
$$\times\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right)$$

and taking

$$\rho^s\left(\frac{k}{m}\,\frac{j}{m}\right) = \sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\left(\frac{k}{m}-x\right)^{s-2l}\left(\frac{j}{m}-y\right)^{2l}$$

into account, from (2.7), we obtain

$$\begin{aligned}
&|(R_m f)(x,y)|\\
&\le m^{\frac{s}{2}}\frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\sum_{\substack{k,j=0\\ k+j\le m}} p_{m,k,j}(x,y)\Bigg[\frac{1}{m^s}(k-mx)^{s-2l}(j-my)^{2l}\\
&+\delta_1^{-2}\frac{1}{m^{s+2}}(k-mx)^{s-2l+2}(j-my)^{2l}+\delta_2^{-2}\frac{1}{m^{s+2}}(k-mx)^{s-2l}(j-my)^{2l+2}\\
&+\delta_1^{-2}\delta_2^{-2}\frac{1}{m^{s+4}}(k-mx)^{s-2l+2}(j-my)^{2l+2}\Bigg]\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right)
\end{aligned}$$

or

$$|(R_m f)(x,y)| \le \frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\Bigg[\frac{T_{m,s-2l,2l}(x,y)}{m^{\frac{s}{2}}}+\delta_1^{-2}\frac{1}{m}\frac{T_{m,s-2l+2,2l}(x,y)}{m^{\frac{s+2}{2}}}$$

$$+\,\delta_2^{-2}\frac{1}{m}\frac{T_{m,s-2l,2l+2}(x,y)}{m^{\frac{s+2}{2}}}+\delta_1^{-2}\delta_2^{-2}\frac{1}{m^2}\frac{T_{m,s-2l+2,2l+2}(x,y)}{m^{\frac{s+4}{2}}}\Bigg]$$

$$\times\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right).$$

Considering $\delta_1=\delta_2=\dfrac{1}{\sqrt{m}}$ the inequality above becomes

$$|(R_m f)(x,y)|\le\frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\Bigg[\frac{T_{m,s-2l,2l}(x,y)}{m^{\frac{s}{2}}}+\frac{T_{m,s-2l+2,2l}(x,y)}{m^{\frac{s+2}{2}}}\tag{2.8}$$

$$+\frac{T_{m,s-2l,2l+2}(x,y)}{m^{\frac{s+2}{2}}}+\frac{T_{m,s-2l+2,2l+2}(x,y)}{m^{\frac{s+4}{2}}}\Bigg]$$

$$\times\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m}}\,\frac{1}{\sqrt{m}}\right).$$

From (2.5) and (2.8), (2.1) follows.

Taking (1.6) into account and considering the fact that

$$\lim_{m\to\infty}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m}}\,\frac{1}{\sqrt{m}}\right)=\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};0,0\right)=0,$$

$i\in\{0,1,\dots,s\}$, it results from (2.8) that

$$\lim_{m\to\infty}(R_m f)(x,y)=0.\tag{2.9}$$

From (2.5) and (2.9), (2.2) follows.

Taking (1.7) into account, then (2.8) becomes

$$|(R_m f)(x,y)|\le\frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\big[k_{s-2l,2l}+k_{s-2l+2,2l}+k_{s-2l,2l+2}\tag{2.10}$$

$$+\,k_{s-2l+2,2l+2}\big]\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m}}\,\frac{1}{\sqrt{m}}\right),$$

for any non zero natural numbers and for any $(x,y)\in\Delta_2$. Thus, the convergence from (2.2) is uniform on Δ_2. From (2.5) and (2.10), (2.3) follows. $\square$

Corollary 7. *Let $f : \Delta_2 \to \mathbb{R}$ be a bivariate function, which admits partial derivatives of order s continuous in a neighborhood of the point $(x, y) \in \Delta_2$.*

If $s = 0$ then

$$\lim_{m\to\infty}(B_m f)(x,y) = f(x,y), \tag{2.11}$$

if $s \geq 2$ then

$$\lim_{m\to\infty} m^{\frac{s}{2}}\left[(B_m f)(x,y) - \sum_{i=0}^{s-1}\sum_{l=0}^{i}\binom{i}{l}\frac{1}{i!m^i}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)T_{m,i-l,l}(x,y)\right] \tag{2.12}$$
$$= \frac{1}{s!}\sum_{l=0}^{s}\binom{s}{l}\frac{\partial^s f}{\partial t^{s-l}\partial\tau^l}(x,y)a_{s-l,l}(x,y)$$

and

$$\lim_{m\to\infty} m^{\frac{s}{2}}\left[(B_m f)(x,y) - \sum_{i=0}^{s-2}\sum_{l=0}^{i}\binom{i}{l}\frac{1}{i!m^i}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)T_{m,i-l,l}(x,y)\right] \tag{2.13}$$
$$= \frac{1}{(s-1)!}\sum_{l=0}^{s-1}\binom{s-1}{l}\frac{\partial^{s-1} f}{\partial t^{s-1-l}\partial\tau^l}(x,y)a_{s-1-l,l}(x,y)$$
$$+ \frac{1}{s!}\sum_{l=0}^{s}\binom{s}{l}\frac{\partial^s f}{\partial t^{s-l}\partial\tau^l}(x,y)a_{s-l,l}(x,y).$$

If f admits partial derivatives of order s continuous on Δ_2, then the convergences given in (2.11) - (2.13) *are uniform on Δ_2.*

Proof. It results from Lemma 2 and Theorem 6. □

Theorem 8. *Let $f : \Delta_2 \to \mathbb{R}$ be a bivariate function.*

If $(x, y) \in \Delta_2$ and f is a continuous function in a neighborhood of the point (x, y), then

$$\lim_{n\to\infty}(B_m f)(x,y) = f(x,y). \tag{2.14}$$

If f is a continuous function on Δ_2, then the convergence given in (2.14) *is uniform on Δ_2 and*

$$|(B_m f)(x,y) - f(x,y)| \leq \frac{13}{4}\,\omega_{total}\left(f; \frac{1}{\sqrt{m}}\,\frac{1}{\sqrt{m}}\right), \tag{2.15}$$

for any $(x, y) \in \Delta_2$ and $m \in \mathbb{N}$.

Proof. It results from Theorem 6 for $s = 0$. □

Theorem 9. *Let $f : \Delta_2 \to \mathbb{R}$ be a bivariate function, which admits partial derivatives of second order continuous in a neighborhood of the point $(x, y) \in \Delta_2$. Then*

$$\lim_{m\to\infty} m\left[(B_m f)(x,y) - f(x,y)\right] \tag{2.16}$$
$$= \frac{1}{2}\left[x(1-x)\frac{\partial^2 f}{\partial t^2}(x,y) - 2xy\frac{\partial^2 f}{\partial t\partial\tau}(x,y) + y(1-y)\frac{\partial^2 f}{\partial \tau^2}(x,y)\right]$$

and if f admits partial derivatives of second order continuous on Δ_2, then the convergence given in (2.16) *is uniform on Δ_2.*

Proof. It results from Corollary 7 for $s = 2$. □

Remark 10. The result from (2.16) is demonstrated in [11].

Theorem 11. *Let $f : \Delta_2 \to \mathbb{R}$ be a bivariate function, which admits partial derivatives of four order continuous in a neighborhood of the point $(x, y) \in \Delta_2$.*

Then

$$\lim_{m\to\infty} m^2 \Bigg\{ (B_m f)(x,y) - f(x,y) \tag{2.17}$$
$$- \frac{1}{2m}\left[x(1-x)\frac{\partial^2 f}{\partial t^2}(x,y) - 2xy\frac{\partial^2 f}{\partial t\partial\tau}(x,y) + y(1-y)\frac{\partial^2 f}{\partial \tau^2}(x,y)\right]\Bigg\}$$
$$= \frac{1}{6}\Bigg[x(1-x)(1-2x)\frac{\partial^3 f}{\partial t^3}(x,y) - 3xy(1-2x)\frac{\partial^3 f}{\partial t^2\partial\tau}(x,y)$$
$$- 3xy(1-2y)\frac{\partial^3 f}{\partial t\partial\tau^2}(x,y) + y(1-y)(1-2y)\frac{\partial^3 f}{\partial \tau^3}(x,y)\Bigg]$$
$$+ \frac{1}{24}\Bigg[3x^2(1-x)^2\frac{\partial^4 f}{\partial t^4}(x,y) - 12x^2y(1-x)\frac{\partial^4 f}{\partial t^3\partial\tau}(x,y)$$
$$+ 6xy(3xy - x - y + 1)\frac{\partial^4 f}{\partial t^2\partial\tau^2}(x,y)$$
$$- 12xy^2(1-y)\frac{\partial^4 f}{\partial t\partial\tau^3}(x,y) + 3y^2(1-y)^2\frac{\partial^4 f}{\partial \tau^4}(x,y)\Bigg]$$

and if f admits partial derivatives of fourth order continuous on Δ_2, then the convergence given in (2.17) *is uniform in Δ_2.*

Proof. It results from Corollary 7 for $s = 4$. □

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