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THE NUMBER OF CRITICAL COLORINGS FOR SOME RAMSEY NUMBERS

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Abstract: We enumerate all critical $(K_4, K_5 - e; 18)$ and $(K_5, K_4 - e; 15)$-colorings. This gives a good starting point for an attack on the problem of improving the bounds on the Ramsey number $R(K_5, K_5 - e)$, which is one of the three remaining open cases for graphs on at most five vertices.

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1. Introduction

In this paper all graphs are undirected, finite and simple. K_m denotes the complete graph on m vertices, $K_m - e$ denotes the graph K_m without one edge. For given graphs G_1, G_2, the *two color Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that if we arbitrarily color the edges of K_n with 2 colors, say red and blue, then there is a red copy of G_1 or a blue copy of G_2. 2-coloring of K_n is called a $(G_1, G_2; n)$-coloring, if it contains neither a red G_1 nor a blue G_2. A coloring $(G_1, G_2; n)$ is said to be *critical* if $n = R(G_1, G_2) - 1$. For any $v \in V(G)$ of a 2-colored G, by $r(v)$ and $b(v)$ we denote the number of red and blue edges incident to v, respectively. The open red neighbourhood

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of v is $N_r(v) = \{u \in V(G) : uv \in E(G)$ and uv has red color$\}$; the open blue neighbourhood $N_b(v)$ of v is defined analogously. By $N_r(v, S)$ and $N_b(v, S)$ we denote the open red and blue neighbourhood of v in the set $S \subseteq V(G)$, respectively, that is, $N_r(v, S) = N_r(v) \cap S$, $N_b(v, S) = N_b(v) \cap S$.

In 1980 Bolze and Harborth [1] proved that $R(K_5, K_4 - e) = 16$ and in the proof they presented only one critical $(K_5, K_4 - e; 15)$-coloring. The value 19 for $R(K_5 - e, K_4)$ was obtained by Exoo, Harborth and Mengersen in 1988 [2], and they also gave only one critical coloring of type $(K_4, K_5 - e; 18)$. In this paper, by using combinatorial properties and computer algorithms, we enumerate all critical colorings for these two cases. All the computations needed for these results were done by algorithms implemented independently by the authors; for generating graphs and checking isomorphism of colorings the utility program *Nauty*[1] [3] was used.

2. The Number of Critical $(K_5, K_4 - e; 15)$-Colorings

Our main result is the following

Theorem 1. *There exist exactly* 13 $(K_5, K_4 - e; 15)$*-colorings.*

Since $R(K_4, K_4 - e) = 11$ and $R(K_5, K_3 - e) = 9$ [5], we can assume that $r(v) \leq 10$ and $b(v) \leq 8$ for all vertices v of a 2-colored K_{15} – otherwise, $r(v) \geq 11$ and $b(v) \geq 9$ lead immediately to red K_5 and blue $K_4 - e$, respectively. Next, it is impossible that $r(v) = 7$ for all vertices v since $7 \cdot 15$ cannot count twice the number of red edges. The proof of Theorem 1 is divided into several lemmas.

Lemma 2. *Let R_1 be a set of all* 2*-colorings of K_{15} which contain a vertex v such that $r(v) = 6$. Then the set R_1 contains no $(K_5, K_4 - e; 15)$-coloring.*

Proof. Let us consider a vertex $v \in V(K_{15})$ such that $r(v) = 6$. Assume that a $(K_5, K_4 - e; 15)$-coloring exists. Then the graph $K_8 = K_{15}[N_b(v)]$ contains exactly 4 vertex disjoint blue edges. Subsequently, any vertex connected to v in red must have at least one red edge to each of the 4 blue edges of that K_8. However, this guarantees a red K_5 contradicting the assum-pion. □

Lemma 3. *Let R_2 be a set of all* 2*-colorings of K_{15} which contain a vertex v such that $r(v) = 7$. Then the set R_2 contains no $(K_5, K_4 - e; 15)$-coloring.*

Proof. The proof of this lemma is very similar to that of Lemma 2. Let us consider a vertex $v \in V(K_{15})$ such that $r(v) = 7$, and assume that there exists a $(K_5, K_4 - e; 15)$-coloring. Then the graph $K_7^{(1)} = K_{15}[N_b(v)]$ contains

[1] For more details concerning *Nauty* see the last section.

exactly 3 vertex disjoint blue edges; let $w \in V(K_7^{(1)})$ be the vertex not incident to any of these edges (clearly, there exists such w). Subsequently, any vertex of $K_7^{(2)} = K_{15}[N_r(v)]$ must have at least one red edge to each of the 3 blue edges of $K_7^{(1)}$. By Lemma 2, there exists a vertex $u \in V(K_7^{(2)})$ connected to w in red. This leads to a red K_5, contradiction. □

Lemma 4. *Let R_3 be a set of all 2-colorings of K_{15} all of whose vertices have property that $r(v) = 8$. Then the set R_3 contains one $(K_5, K_4 - e; 15)$-coloring.*

Proof. Let the vertices of graph K_{15} be labeled $v, A, B, \ldots, H, 1, 2, \ldots, 6$ and all vertices $x \in V(K_{15})$ have the property that $r(x) = 8$. Assume that $N_r(v) = \{A, \ldots, H\}$ and $N_b(v) = \{1, 2, \ldots, 6\}$. It is easy to see that there are two $(K_5, K_3 - e; 6)$-colorings: the first one has only two blue edges, say $\{1, 2\}$ and $\{3, 4\}$, in the latter case, the coloring has three blue edges, say $\{1, 2\}, \{3, 4\}, \{5, 6\}$.

In the first case, for each $u \in \{1, 2, 3, 4\}$ we have that $|N_b(u, \{A, \ldots, H\})| = 4$ and for each $u \in \{5, 6\}$ we have $|N_b(u, \{A, \ldots, H\})| = 5$. To avoid a blue $K_4 - e$, any vertex of the graph $K_8 = K_{15}[N_r(v)]$ must have at least one red edge to each of the blue edges of $K_6 = K_{15}[N_b(v)]$. Next, each vertex $w \in \{A, \ldots, H\}$ must have a blue edge to at least one of the vertices $5, 6$ – otherwise, we obtain a red K_5. Consequently, we can assume the edges $\{A, 5\}$, $\{B, 5\}$, $\{C, 5\}$, $\{D, 5\}$, $\{E, 5\}$, $\{D, 6\}$, $\{E, 6\}$, $\{F, 6\}$, $\{G, 6\}$, $\{H, 6\}$ are colored blue. Then two complete graphs on vertex-set $\{A, B, C, D, E\}$ and $\{D, E, F, G, H\}$ cannot contain either a blue $K_3 - e$ or a red K_4. Thus these graphs contain exactly 2 vertex disjoint blue edges and avoid a blue $K_4 - e$ on $\{D, E, 5, 6\}$, the edge $\{D, E\}$ is colored red. Consider now a vertex $u \in \{D, E\}$. Then $|N_b(u, \{A, \ldots, H\})| = 2$ – otherwise, blue degree at least 3 in K_8 implies that there exists blue $K_3 - e$ in either $K_8[\{A, B, C, D, E\}]$ or $K_8[\{D, E, F, G, H\}]$; clearly for all other vertices $u \in \{A, B, C, F, G, H\}$ we have $|N_b(u, \{A, \ldots, H\})| \geq 3$. Without loss of generality we can assume the edges $\{D, A\}$, $\{D, F\}$, $\{E, B\}$, $\{E, G\}$ to be blue. Then $\{A, F\}$, $\{B, G\}$ are red, $\{F, B\}$, $\{F, C\}$, $\{A, G\}$, $\{A, H\}$, $\{G, C\}$, $\{B, H\}$, $\{C, H\}$ are blue and the remaining edges of K_8 are red. The blue induced subgraph of K_8 contains the cycle C_8: $A - D - F - C - G - E - B - H - A$ and three vertex disjoint edges: $\{F, B\}$, $\{H, C\}$, $\{A, G\}$. Now, we have to determine the edges between $V(K_8)$ and $\{1, 2, 3, 4\}$. Observe that for each $w \in \{A, \ldots, H\}$ we have $|N_b(w, \{1, 2, 3, 4\})| = 2$, so $N_b(w, \{1, 2, 3, 4\}) = s$, where $s \in \mathcal{S} = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$. Let us consider the red triangle ABC. To avoid a red K_5, the vertices A, B, C must have different blue neighbourhoods in $\{1, 2, 3, 4\}$. The same property holds for the vertices of the red triangle FGH. We obtain that $N_b(A,\{1,2,3,4\}) = s_1$, $N_b(B,\{1,2,3,4\}) = s_2$, $N_b(C,\{1,2,3,4\}) = s_3$, and then $N_b(H,\{1,2,3,4\}) = s_4$, $N_b(G,\{1,2,3,4\}) = s_2$, $N_b(F,\{1,2,3,4\}) = s_1$, where s_1, s_2, s_3, s_4 are different elements of $\mathcal{S}$. Finally, $N_b(D,\{1,2,3,4\}) = s_3$ and $N_b(E,\{1,2,3,4\}) = s_4$ or $N_b(D,\{1,2,3,4\}) = s_4$ and $N_b(E,\{1,2,3,4\}) = s_3$. Clearly, we obtain many different good colorings of type $(K_5, K_4 - e; 15)$, but all of them are isomorphic.

In the latter case, for each $u \in \{1,\dots,6\}$ we have that $|N_b(u,\{A,\dots,H\})| = 4$, and consequently, this implies that for all vertices y of K_8, $b(y) = 3$ and $r(y) = 4$ in K_8. To avoid a blue $K_4 - e$, any vertex of K_8 must have at least one red edge to each of the blue edges of K_6 – without loss of generality we can assume the edges $\{A,1\}$, $\{B,1\}$, $\{C,1\}$, $\{D,1\}$, $\{E,2\}$, $\{F,2\}$, $\{G,2\}$, $\{H,2\}$ to be blue. Then two complete graphs on vertex-set $\{A,B,C,D\}$ and $\{E,F,G,H\}$ cannot contain either a blue $K_3 - e$ or a red K_4, thus we can assume that $\{A,B\}$, $\{G,H\}$ are blue, and this enforces $\{A,C\}$, $\{A,D\}$, $\{B,C\}$, $\{B,D\}$, $\{E,G\}$, $\{E,H\}$, $\{F,G\}$, $\{F,H\}$ to be red. Then vertex C must have three other blue edges in graph K_8. We have to consider three subcases: (i) $\{C,D\}$, $\{C,F\}$, $\{C,G\}$ are blue, $\{C,E\}$, $\{C,H\}$ are red; (ii) $\{C,D\}$, $\{C,E\}$, $\{C,F\}$ are blue, $\{C,G\}$, $\{C,H\}$ are red; (iii) $\{C,E\}$, $\{C,F\}$, $\{C,G\}$ are blue, $\{C,D\}$, $\{C,H\}$ are red.

Subcase (i) forces $\{D,G\}$, $\{D,F\}$ to be red. Since $|N_b(D,\{A,\dots,H\})| = 3$, the edges $\{D,E\}$, $\{D,H\}$ are blue. Next, as $|N_b(A,\{E,G,H\})| = 2$, then $\{A,E\}$ is blue, $\{B,E\}$ red and similarily, $\{B,F\}$ is blue and $\{A,F\}$ red. Therefore $\{E,F\}$ must have blue color. To avoid a red K_4 on vertex-set $\{B,C,E,H\}$, $\{B,H\}$ is blue, and $\{B,G\}$, $\{A,H\}$ are red and $\{A,G\}$ blue. Thus we have a good $(K_4, K_4 - e; 8)$-coloring of K_8, whose induced blue subgraph consists of the cycle C_8: $A - G - H - B - F - C - D - E - A$ and four vertex disjoint edges $\{A,B\}$, $\{C,G\}$, $\{D,H\}$ and $\{E,F\}$. Observe that for each vertex $w \in \{A,\dots,H\}$ we have $|N_b(w,\{1,\dots,6\})| = 3$, so in order to avoid a red K_5 and a blue $K_4 - e$ we have 8 different possibilities of coloring of the edges joining vertex w to vertices $\{1,\dots,6\}$. If any two vertices of $\{A,\dots,H\}$ have different blue neighbourhoods in K_6, then we obtain $(K_5, K_4 - e; 15)$-coloring. Note that such coloring is isomorphic to that presented in the first case. Indeed, the following function $u \to w$, where u, w denotes the vertices of K_{15} colored in the first and second way, respectively: $3 \to G$, $C \to H$, $5 \to B$, $E \to F$, $4 \to C$, $H \to D$, $6 \to E$, $D \to A$, $1 \to v$, $A \to 1$, $G \to 2$, $B \to 3$, $F \to 4$, $v \to 5$ and $2 \to 6$ is the required isomorphism.

In subcase (ii), $\{E,F\}$ and $\{E,D\}$ must be red. Since $\{E,G\}$ and $\{E,H\}$ are red, $\{E,A\}$ and $\{E,B\}$ are blue, thus vertices $A, B, E, 1$ induce a blue $K_4 - e$, thus this subcase does not lead to $(K_5, K_4 - e; 15)$-coloring. Subcase

(iii) forces $\{E, F\}$ to be red, $\{D, E\}$, $\{D, F\}$ to be blue. To avoid a blue $K_4 - e$ on $\{A, B, H, 1\}$, $\{D, H\}$ is blue. Then $\{D, G\}$ is red. Without loss of generality we can assume the edges $\{A, G\}$, $\{A, E\}$ to be blue. Then $\{B, G\}$, $\{A, H\}$, $\{A, F\}$, $\{B, E\}$ are red and $\{B, H\}$, $\{B, F\}$ are blue. It is easy to see that we get a $(K_4, K_4 - e; 8)$-coloring of K_8 which is isomorphic to that described in subcase (i), and this leads to $(K_5, K_4 - e; 15)$-coloring which is isomorphic to the coloring presented above.

Thus we obtained only one non-isomorphic $(K_5, K_4 - e; 15)$-coloring. □

Lemma 5. *Let R_4 be a set of all 2-colorings of K_{15} which contain a vertex v such that $r(v) = 10$. Then the set R_4 contains 3 $(K_5, K_4 - e; 15)$-colorings.*

Proof. Let us consider a vertex $v \in V(K_{15})$ such that $r(v) = 10$: we obtain two subgraphs $G' = K_{15}[N_r(v)]$ and $H' = K_{15}[N_b(v)]$. In order to avoid either a red K_5 or a blue $K_4 - e$, G' does not contain either a red K_4 or a blue $K_4 - e$ and H' does not contain either a red K_5 or a blue $K_3 - e$. Next, let us recall that Exoo et al [2] proved that there are only five $(K_4, K_4 - e; 10)$-colorings, and it is easy to check that we have only three $(K_5, K_3 - e; 4)$-colorings (with 2, 1 and without blue edges). Thus, as $K_{15} = (G' + H') + \{v\}$, all $(K_5, K_4 - e; 15)$-colorings can be obtained by taking all possible appropriately colored pairs (G', H') and determining the edges between G' and H'; clearly there are exactly 15 such pairs. The above method for generating colorings results in 3 nonisomorphic colorings (the testing was done by using the program *Nauty*). □

Lemma 6. *Let R_5 be a set of all 2-colorings of K_{15} which contain a vertex v such that $r(v) = 9$. Then the set R_5 contains 11 $(K_5, K_4 - e; 15)$-colorings.*

Proof. The proof of this lemma is very similar to that of Lemma 5. Let us consider a vertex $v \in V(K_{15})$ such that $r(v) = 9$: we obtain two subgraphs $G' = K_{15}[N_r(v)]$ and $H' = K_{15}[N_b(v)]$. In order to avoid either a red K_5 or a blue $K_4 - e$, G' does not contain either a red K_4 or a blue $K_4 - e$ and H' does not contain either a red K_5 or a blue $K_3 - e$. By using a simple computer algorithm, we easily obtained all $(K_4, K_4 - e; 9)$-colorings. We have produced the results gathered in the 10-th column of Table 5. It is easy to check that we have two $(K_5, K_3 - e; 5)$-colorings (with one or two blue edges). Similarily to the proof of Lemma 5, as $K_{15} = (G' + H') + \{v\}$, all $(K_5, K_4 - e; 15)$-colorings can be obtained by taking all appropriately colored pairs (G', H') and determining the edges between G' and H'. There are 196 such pairs. As a result of these computations, only five pairs have given us good colorings. Finally, 11 nonisomophic colorings were found. Testing of isomorphism was also done by using the program *Nauty*. □

Proof of Theorem 1. Notice that by Handshake Lemma the case $r(v) = 9$ for

graph 1	graph 2	graph 3	graph 4
10001000111000	10001000111000	00000111	0110000
0000001001011	0000001000111	010101001	00010101
100100100110	100010001011	1001010	0001010
00010010101	00100110100	100100	0100101
0000111111	0000111111	10101010	10101010
110000111	110100110	01010101	0010101
01011001	01011001	10110	011010
1101010	1010101	0101	1100
110100	101010	10101	00101
00001	00001	1010	01010
0010	0010	00	10
100	100	01	01
00	00	10	10
0	0	0	1

Table 1:

all vertices v of K_{15} is impossible, thus bearing in mind the results of Lemmas 1-5, we get fifteen $(K_5, K_4 - e; 15)$-colorings. Once again, the program *Nauty* was used resulting in 13 nonisomorphic colorings. □

Let us mention that Lemmas 1-5 were verified using computer support, and upper triangles of adjacency matrices of all 13 colorings are presented in Tables 4-5 (with 1 we denote the edge with blue color).

3. Critical $(K_4, K_5 - e; 18)$-Colorings

Theorem 7. *There exist* 6 $(K_4, K_5 - e; 18)$*-colorings.*

Proof. The proof is based on computer calculations, whose idea is as follows. First, since $R(K_4, K_4 - e) = 11$ and $R(K_3, K_5 - e) = 11$ [5], we have that $r(v) \leq 10$ and $b(v) \leq 10$ for all vertices v of a 2-colored K_{18}, more precisely, $r(v), b(v) \in \{7, 8, 9, 10\}$. Hence, we need all $(K_3, K_5 - e; n_1)$ and $(K_4, K_4 - e; n_2)$-colorings, where $7 \leq n_1, n_2 \leq 10$; these colorings for arbitrary n are presented in Tables 1 and 2. Observe the nonexistence of $(K_3, K_5 - e; 10)$-colorings for $16 \leq r \leq 19$. Next, the proof follows by similar arguments as the proofs of Lemmas 5 and 6.

Let us consider a vertex $v \in V(K_{18})$ such that $r(v) = i$ and $b(v) = 17 - i$, where $i \in \{7, 8, 9, 10\}$. We obtain two subgraphs $G' = K_{18}[N_r(v)]$ and $H' =$

graph 5	graph 6	graph 7	graph 8	graph 9
000101011	00010011	1110000	0110000	1010000
00010101	00011001	1001100	10010101	000100101
0100101	010101	000011	0001010	00011010
101010101	101010101	1010101010	0100101	1010010
0010101	0101011	001010101	10101010	001101
0110101	010101	10101	0010101	10101
11000	11000	01010	011010	01010
001010	01010	10010	1100	1100
01010	0010	0101	00101	0101
100	010	0101	01010	001
010	10	110	10	10
10	010	00	01	010
10	0	0	10	1
0	0	1	1	0

Table 2:

graph 10	graph 11	graph 12	graph 13
10011000	10011000	11000100	10011000
0010010101	01000101	0001001	10001010
010001010	01001010	001010101	0000101
110100	110100	00101010	101001
0010101	00101010	101010	1001010
010010	0100101	010100	01001010
0011	0011	10001	010101
101010	101010	0010	010101
10101	10101	0101	0110
1010	101	010	100
0101	010	01	010
10	10	10	10
0	0	01	01
1	1	0	1

Table 3:

$K_{18}[N_b(v)]$. In order to avoid either a red K_4 or a blue $K_5 - e$, G' does not contain either a red K_3 or a blue $K_5 - e$ and H' does not contain either a red K_4 or a blue $K_4 - e$. Thus, as $K_{18} = (G' + H') + \{v\}$, we enumerated all $(K_4, K_5 -$

number of red edges r	number of vertices n: 1	2	3	4	5	6	7	8	9	10	total
0	1	1	1	1							4
1		1	1	1							3
2			1	2	2						5
3				2	3	1					6
4				1	4	4					9
5					2	7					9
6					1	7	5				13
7						4	8				12
8						2	12	2			16
9						1	8	5			14
10							4	12			16
11							1	12			13
12							1	10	1		12
13								4	1		5
14								2	3		5
15								1	1	1	3
16								1	1		2
17											0
18											0
19											0
20										1	1
total	1	2	3	7	12	26	39	49	7	2	148

Table 4: Statistics of $(K_3, K_5 - e; n)$-colorings

$e; 18)$-colorings by taking all good colored pairs (G', H') and determining edges between G' and H'. We used the algorithm which is based on backtracking, and we obtained:

Case 1. $r(v) = 10$ and $b(v) = 7$. Only 164 pairs were considered and no $(K_4, K_5 - e; 18)$-coloring was obtained.

Case 2. $r(v) = 9$ and $b(v) = 8$ .896 pairs were considered resulting in 4 nonisomorphic $(K_4, K_5 - e; 18)$-colorings.

Case 3. $r(v) = 8$ and $b(v) = 9$. A large set of 4802 pairs was considered, however, it resulted only in 6 nonisomorphic $(K_4, K_5 - e; 18)$-colorings.

Case 4. $r(v) = 7$ and $b(v) = 10$ .195 pairs were considered, and we obtained 2 nonisomorphic $(K_4, K_5 - e; 18)$-colorings.

number of blue edges b	number of vertices n										total
	1	2	3	4	5	6	7	8	9	10	
0	1	1	1								3
1		1	1	1							3
2			1	2	1						4
3			1	3	3	1					8
4				2	5	2					9
5					5	7	1				13
6					3	12	4				19
7						11	11	1			23
8						5	20	3			28
9						2	26	8	1		37
10							15	20	1		36
11							5	34	3		42
12								37	7		44
13								20	12		32
14								5	20		25
15									27		27
16									18		18
17									6		6
18									3	1	4
19										1	1
20										2	2
21										1	1
total	1	2	4	8	17	40	82	128	98	5	385

Table 5: Statistics of $(K_4, K_4 - e; n)$-colorings

Finally, all obtained colorings were tested in respect to isomorphisms, and exactly six good $(K_4, K_5 - e; 18)$-colorings were found. □

Upper triangles of adjacency matrices of all graphs on 18 vertices which does not contain a red K_4 and a blue $K_5 - e$, row by row excluding the diagonal are presented Table 6 and Table 7 (with 1 we denote the edge with blue color).

There is some hope that our results on critical colorings will be helpful in determining a new upper bounds for some larger Ramsey numbers, in particular $R(K_5, K_5 - e)$ and $R(K_4, K_6 - e)$.

graph 1	graph 2	graph 3
111110000	10101010010101	101010010
110100010101	010101010101	0101010100101
101010010101	101010010101	10011001010
00100110	010010101	010101001
1011010	1111000	1101100
010110010	1010010	01010101
10101001	001010	010101010
0110101	1010100	001101
1101010	0101100	0101010
1010	10101	1100
01100	010101	101010
0011	101	0101
110	10101	10101
101	011	10
101	11	11
1	10	1
1	0	1

Table 6:

4. Nauty

Nauty, written by Brendan D. McKay, is a well known isomorphism testing program. It is a set of very efficient C language procedures for canonical labelling of graphs, generating small graphs without isomorphs. *Nauty* may be obtained from http://cs.anu.edu.au/people/bdm/. It is free for educational and research applications.

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[3] B.D. McKay, *Nauty Users' Guide, Version 2.2* (2004), source code available at http://cs.anu.edu.au/people/bdm/nauty.

graph 4	graph 5	graph 6
111110000	1101010010	101001100110
11000110	01010100101	00101010101
1010101010	01010100101	1001101011
001100110	11001010	0101010101
101010101	010010101	101010101
0101010101	010010101	1001010101
01011001	111000	10011010
01101010	1010100	01011010
010101	0101100	11100
1101010	1010101	010101
1010	010101	0110
0100	10101	0101
011	101	110
101	110	00
01	10	01
1	1	1
1	1	0

Table 7:

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