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BOUNDEDNESS AND STABILITY PROPERTIES OF SOLUTIONS TO CERTAIN THIRD ORDER NONLINEAR NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract: In this paper we give criteria for the boundedness and global asymptotic stability of solutions to certain third order nonlinear non-autonomous differential equation

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(\dot{x}) + c(t)h(x) = p(t;x,\dot{x},\ddot{x})$$

when $p \equiv 0$ and when $p \neq 0$, with the use of a single complete Lyapunov function.

The results here include and improve some well known results in the literature.

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1. Introduction

Let $x(t)$ be any solution of some third order nonlinear non-autonomous differential equation of the form

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(\dot{x}) + c(t)h(x) = p(t;x,\dot{x},\ddot{x})\,, \tag{1.1}$$

where a, b, c, f, g, h and p are continuous and depend on the arguments displayed explicitly. In addition, they are such that existence, uniqueness and continuous dependence on initial condition is guaranteed.

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Equation of the form of the equation (1.1) and variants of it have been studied by many authors (see [6], [9], [10], [11], [12-13], [17]). Many more of these are summarized in [16]. In this direction, e.g. when $a = b = c = 1$, the autonomous case which have received great treatment and still receiving the attention of researchers (see [1-2], [3], [4], [5], [7-8], [14-15]) for results on boundedness and stability of solutions. All these are with the use of Lyapunov function.

In [11], the author with the use of the second Lyapunov method and further ideas from Yoshizawa [18] proved the boundedness (uniform ultimate boundedness) of the solution under the assumption that include the differentiability of $c(t)$ and $h(x)$.

In [13], the author treated the general non-autonomous system, he employed the Lyapunov method which he used to present some results on the global stability of the trivial solution of the system considered with example on second order equation.

In a system form our equation (1.1) can be expressed as a system of three-coupled first order equations by letting;

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -a(t)f(x,\dot{x})\ddot{x} - b(t)g(\dot{x}) - c(t)h(x) + p(t; x, \dot{x}, \ddot{x}). \end{aligned} \tag{1.2}$$

Since the second (direct) method of Lyapunov still remains one of the most effective method to study these concepts, in this study, we shall use a single complete Lyapunov function which enable us prove the global asymptotic stability of the trivial solution (when $p \equiv 0$) as well as uniform ultimate boundedness of the solutions without the assumption of differentiability on $c(t)$ and $h(x)$.

Notations and Definitions

Throughout this paper $K, K_0, \ldots K_{16}$ will denote finite positive constants whose magnitudes depend only on the functions ϕ, f and p as well as constants a, κ, β, Δ and δ but are independent of solutions of (1.1). K_i are not necessarily the same for each time they occur, but each $K_i, i = 1, 2 \ldots$ retains its identity throughout.

The functions $a(t), b(t), c(t), f(x,y), g(y), h(x)$ shall be expressed as a, b, c, f, g and h respectively. With I we denote the interval $0 \leq t < \infty$, and $S_r = \{X \in \Re^3 : \|X\| < r\}$, where $\|.\|$ will stand for an arbitrary norm in $\Re^3$.

Consider the system

$$\dot{X} = f(t, X), \tag{1.3}$$

where $X \in \Re^3$.

With $V(t, X)$ we denote arbitrary scalar function defined on open set $S \subset I \times \Re^3$ which we called a Lyapunov function. It is assumed that the function $V(t, X)$ have continuous partial derivatives with respect to all the arguments.

The total derivative of $V(= V(t, X))$ with respect to the system (1.3) is thus defined as

$$\dot{V}|_{(1.3)}(t, X) = \lim_{h \longrightarrow 0} \frac{V(t + h, X + hf(t, X)) - V(t, X)}{h} \tag{1.4}$$

which is given as

$$\dot{V}|_{(1.3)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial X}\frac{dX}{dt}. \tag{1.5}$$

We shall now state without proof some results which give the role of function V in the problem of stability and boundedness of differential systems.

Theorem A. (see [13], [18]) *Suppose that there exists a Lyapunov's function $V(t, X)$ defined on $I \times \Re^m (m = 3)$ satisfying the following conditions:*

i) $V(t, 0) \equiv 0$.

ii) $a(\|X\|) \leq V(t, X)$, where $a(\tau)$ is continuous in the interval I and positive definite.

iii) $\dot{V}|_{(1.3)} \leq 0$.

Then, the trivial solution of system (1.3) is stable.

Theorem B. (see [13], [18]) *Suppose that there exists a Lyapunov's function $V(t, X)$ defined on $I \times S_r$ satisfying the following conditions:*

i) $a(\|X\|) \leq V(t, X)$, where $a(\tau)$ is continuous and $a(\tau) \longrightarrow \infty$ as $\tau \longrightarrow \infty$.

ii) $\dot{V}|_{(1.3)} \leq 0$.

Then, the solutions of the system (1.3) are equi-bounded.

In [13], the following were also established:

Theorem C. (see [13]) *Suppose that there exists a continuous positive definite function $a(\tau)$ such that $a(\tau) \longrightarrow \infty$ as $\tau \longrightarrow \infty$, and that the following conditions are fulfilled:*

i) There exist a Lyapunov function $V(t, X)$ such that $a(\|X\|) \leq V(t, X)$.

ii) $\dot{V}|_{(1.3)} \leq -\lambda V(t, X) - \mu W(t, X)$, where λ, μ are continuous in I and $W(t, X)$ is a positive definite function.

Then the trivial solution $X \equiv 0$ of the system (1.3) is equi-asymptotically stable in the large.

Theorem D. (see [13]) *Suppose the functions a, b defined on S and $W(t, X)$ defined on $I \times \Re^m$ are positive definite and that the following two conditions hold:*

i) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, $a(\tau) \longrightarrow \infty$ as $\tau \longrightarrow \infty$.

ii) $\dot{V}|_{(1.3)} \leq -\lambda(t)\phi(V(t,X)) - \mu(t)W(t,X)$, *where* ϕ *is continuous and non-negative and is positive definite.*

Then the trivial solution $X \equiv 0$ *of the system (1.3) is asymptotically stable.*

2. Formulation of Results

The basic assumptions on the functions which appear in the equation (1.1) are given below:

i) $a, b, c,$ and p are continuous real valued functions on I with

$$0 < a_0 < a(t) \leq a_1 0 < b_0 < b(t) \leq b_1 0 < c_0 < c(t) \leq c_1$$

for all $t \in I$.

ii) $b > a\kappa^2$, $a > c\alpha^2$ and $b\beta(1 - b\beta) > ac$. α, β, κ are all positive constants.

iii) f is continuous for all $(x, y) \in \Re^2$ and $|f| \leq \kappa$.

iv) $h(x)$ is continuous for all x and $g(y)$ is continuous for all y with;

$$H_0 = \frac{h(x) - h(0)}{x} \in I_0 \quad (\text{say} \ \ \alpha) \ \ x \neq 0 \ \ \text{and} \quad h(0) = 0\,,$$

$$G_0 = \frac{g(y) - g(0)}{y} \in I_0 \quad (\text{say} \ \ \beta) \ \ y \neq 0 \ \ \text{and} \quad g(0) = 0\,.$$

v) $\sup\limits_{t\geq 0} \left| e^{-\mu t} \int_0^t p(s)ds \right| < \infty, \forall \quad \mu > 0.$

vi) $|p(t; x, \dot{x}, \ddot{x}| \leq (|x| + |y| + |z|)\, \phi(t)$ where $\phi(t)$ is a non-negative and continuous function of t and satisfies $\int_0^t \phi(s)ds \leq M < \infty, M > 0$ is a constant.

Remark. Assumptions (v) includes such cases as $|p(t)|$ bounded for all $t \geq 0$ (see [6]) or $\left|\int_0^t p(s)ds\right|$ bounded for all $t \geq 0$ (see [9], [17]), for the particular cases $a = b = c = 1$.

The main results of this paper with respect to the equation (1.1) are thus given below.

Theorem 2.1. *Suppose that assumptions (i) through (iv) hold. Then the trivial solution of the equation (1.1) is globally asymptotically stable.*

Theorem 2.2. *Suppose that assumptions (i) through (v) hold. Then* σ $(0 < \sigma < \infty)$ *depending only on* α, β, δ *and* κ *such that every solution of the equation (1.1) satisfies*

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \leq e^{-\frac{1}{2}\sigma t} \left\{ A_1 + A_2 \int_{t_0}^t |P(\tau)|\, e^{\frac{1}{2}\sigma\tau} d\tau \right\}^2$$

for all $t \geq t_0$, where the constant $A_1 > 0$, depends on $\alpha, \beta, \delta, \kappa$ as well as on $t_0, x(t_0), \dot{x}(t_0)$ and $\ddot{x}(t_0)$; and the constant $A_2 > 0$ depends on α, β, δ and κ only.

Theorem 2.3. *Suppose that assumptions (i) through (iv) and (vi) hold. Then, there exists a constant K_0 which depends on M, K_1, K_2 and t_0 such that every solution $x(t)$ of the equation (1.1) satisfies*

$$|x(t)| \leq K_0, \quad |\dot{x}(t)| \leq K_0, \quad |\ddot{x}(t)| \leq K_0$$

for sufficiently large t.

3. The Function $V(t; x, y, z)$

The main tool to prove the theorems in the last section shall be a function $V(t; x, y, z)$ which is defined as

$$2V = H(t)R(x, y, z)\,, \tag{3.1}$$

where

$$H(t) = \exp\left(-\int_0^t a(s)f(x, y)ds\right)\,,$$
$$R(x, y, z) = \left(\frac{x^2 + y^2 + z^2}{2}\right) + dxy + exz + jyz\,, \tag{3.2}$$

where the coefficients

$$d = \frac{(a\kappa)^2 + (c\alpha)^2 + ac[b\beta - 1]\alpha\kappa}{b\beta\kappa + c\alpha}, \quad e = \frac{a\kappa + b\beta\kappa[1 - b\beta]}{b\beta\kappa + c\alpha},$$

and

$$j = \frac{a\kappa + c\beta\kappa[b\beta - 1]\alpha}{b\beta\kappa + c\alpha}\,.$$

The following lemmas are to prove that $V(t; x, y, z)$ is indeed a Lyapunov function.

Lemma 3.1. *Subject to the assumptions of Theorem 2.1 there exist positive constants $K_i = K_i(\alpha, \beta, \kappa, \delta, \Delta), i = 1, 2$ such that*

$$K_1(x^2 + y^2 + z^2) \leq V(t; x, y, z) \leq K_2(x^2 + y^2 + z^2). \tag{3.3}$$

Proof. Clearly the function $V(t;x,y,z)$ satisfies the condition (i) of Theorem A, i.e.

$$V(t;0,0,0)\equiv 0.$$

By rearranging $R(x,y,z)$ the function (3.1) becomes that

$$\begin{aligned} V(t;x,y,z) = \frac{H(t)}{\Delta}\Big\{ & b\beta\kappa(x^2+y^2+z^2)+c\alpha\{(x+\frac{d'}{c\alpha}y)^2+(y+\frac{j'}{c\alpha}z)^2+(z+\frac{e'}{c\alpha}x)^2 \\ & +\{\frac{[b\beta\kappa\{(c\alpha)^2+[b\beta-1][2a\kappa-b\beta\kappa(b\beta-1)]-(a\kappa)^2}{(c\alpha)^2}\}x^2 \\ & +\{\frac{[(c\alpha)^2b\beta\kappa+2[(a\kappa)^2+(c\alpha)^2][ac\alpha\kappa(1-b\beta)]]-[(a\kappa)^2+(c\alpha)^2]-[ac(1-b\beta)]^2}{(c\alpha)^2}\}y^2 \\ & +\{\frac{c\beta\kappa\{c\alpha^2+\kappa[b\beta-1][2a-c\beta]-(a\alpha)^2}{(c\alpha)^2}\}z^2\Big\}, \end{aligned} \tag{3.4}$$

where $\Delta = b\beta\kappa + c\alpha$.

This implies that

$$\begin{aligned} R(x,y,z)\ge \Big\{ & \frac{[b\beta\kappa\{(c\alpha)^2+[b\beta-1][2a\kappa-b\beta\kappa(b\beta-1)]-(a\kappa)^2}{(c\alpha)^2}\}x^2 \\ & +\{\frac{[(c\alpha)^2b\beta\kappa+2[(a\kappa)^2+(c\alpha)^2][ac\alpha\kappa(1-b\beta)]]-[(a\kappa)^2+(c\alpha)^2]-[ac(1-b\beta)]^2}{(c\alpha)^2}\}y^2 \\ & +\{\frac{c\beta\kappa\{c\alpha^2+\kappa[b\beta-1][2a-c\beta]\}-(a\kappa)^2}{(c\alpha)^2}\}z^2\Big\}. \end{aligned} \tag{3.5}$$

Substituting inequality (3.5) into the equation (3.1) we have that

$$\begin{aligned} V(t;x,y,z)\ge H(t)\Big\{ & \frac{[b\beta\kappa\{(c\alpha)^2+[b\beta-1\}][2a\kappa-b\beta\kappa(b\beta-1)]-(a\kappa)^2}{(c\alpha)^2}\}x^2 \\ & +\{\frac{[(c\alpha)^2b\beta\kappa+2[(a\kappa)^2+(c\alpha)^2][ac\alpha\kappa(1-b\beta)]]-[(a\kappa)^2+(c\alpha)^2]-[ac(1-b\beta)]^2}{(c\alpha)^2}\}y^2 \\ & +\{\frac{c\beta\kappa\{c\alpha^2+\kappa[b\beta-1][2a-c\beta]\}-(a\kappa)^2}{(c\alpha)^2}\}z^2\Big\}. \end{aligned} \tag{3.6}$$

This can be written as

$$V(t;x,y,z)\ge K_1H(t)(x^2+y^2+z^2)\,, \tag{3.7}$$

where

$$\begin{aligned} K_1 = \frac{1}{\Delta(c_0\alpha)^2}\cdot\min\big\{ & [b_0\beta\kappa\{(c_0\alpha)^2+[b_0\beta-1][2a_0\kappa-b_0\beta\kappa(b_0\beta-1)]\} \\ & -(a_0\kappa)^2[(c_0\alpha)^2b_0\beta\kappa+2[(a_0\kappa)^2+(c_0\alpha)^2][a_0c_0\alpha\kappa(1-b_0\beta)]] \\ & -[(a_0\kappa)^2+(c_0\alpha)^2]-[a_0c_0(1-b_0\beta)]^2 \end{aligned}$$

$$C_0\beta\kappa\{c_0\alpha^2+\kappa[b_0\beta-1][2a_0-c_0\beta]\}-(a_0\kappa)^2\Big\}\,. \quad (3.8)$$

By the Schwartz inequality,

$$|xy|\le\frac{1}{2}|x^2+y^2|,$$

the equation (3.1) becomes

$$V(t;x,y,z)\le H(t)\left\{(1+d_1+e_1)x^2+(1+d_1+j_1)y^2+(1+e_1+j_1)z^2\right\}\,,$$

with

$$d_1=(a_1\kappa)^2+(c_1\alpha)^2+a_1c_1[b_1\beta-1]\alpha\kappa,\quad e_1=a_1\kappa+b_1\beta\kappa[1-b_1\beta]$$

and

$$j_1=a_1\kappa+c_1\beta\kappa[b_1\beta-1]\alpha\,.$$

Then

$$V(t;x,y,z)\le K_2H(t)(x^2+y^2+z^2), \quad (3.9)$$

where $K_2=\frac{1}{\Delta}\cdot\max\{(b_1\beta\kappa+c_1\alpha+d_1+e_1),$
$(b_1\beta\kappa+c_1\alpha+d+j),(b_1\beta\kappa+c_1\alpha+e_1+j_1)\}$.

Combining the inequalities (3.7) and (3.9) we have

$$K_1H(t)(x^2+y^2+z^2)\le V(t;x,y,z)\le K_2H(t)(x^2+y^2+z^2)\,.$$

Since $H(t)$ is non-negative by definition we have

$$K_1(x^2+y^2+z^2)\le V(t;x,y,z)\le K_2(x^2+y^2+z^2). \quad (3.10)$$

This proves the lemma. □

Lemma 3.2. *Suppose the conditions of Theorem 2.1 hold, then there exist a positive constant* $K_5=K_5(\alpha,\beta,\kappa,\delta,\Delta,)$ *such that for any solution* (x,y,z) *of the system (1.2)*

$$\dot V|_{(1.2)}\le -K_5H(t)(x^2+y^2+z^2)\,. \quad (3.11)$$

Proof. From the equations (1.1) and the system (1.2) we have

$$\dot V|_{(1.2)}=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}\dot x+\frac{\partial V}{\partial y}\dot y+\frac{\partial V}{\partial z}\dot z\,,$$

which gives

$$\dot{V}(t;x,y,z) = -H(t)[a(t)f(x,y) - a(0)f(x,y)]\{R(x,y,z)\} + H(t)\left\{\frac{\partial R(x,y,z)}{\partial x}\frac{dx}{dt} + \frac{\partial R(x,y,z)}{\partial y}\frac{dy}{dt} + \frac{\partial R(x,y,z)}{\partial z}\frac{dz}{dt}\right\}. \quad (3.12)$$

Simplifying the second term on the RHS of the equation (3.12), we have that

$$\frac{\partial R(x,y,z)}{\partial X}\frac{dX}{dt} = xy + yz + dy^2 + dxz + ez^2 + jyz + [ex + jy + z][-afz - bg - ch], \quad (3.13)$$

which reduces after some lengthy algebraic computations to

$$\frac{\partial R(x,y,z)}{\partial X}\frac{dX}{dt} = -ec\alpha x^2 - (jb\beta - d)y^2 - (a\kappa - e)^2. \quad (3.14)$$

Therefore

$$\dot{V}|_{(1.2)} = -H(t)[af(x,y) - a_0 f(x,y)]\{R(x,y,z)\} - H(t)\left\{ec\alpha x^2 + (jb\beta - d)y^2 + (a\kappa - e)^2\right\}. \quad (3.15)$$

This results in

$$\dot{V}|_{(1.2)} \leq -H(t)\kappa\left\{K_3(x^2+y^2+z^2) + K_4(x^2+y^2+z^2)\right\}, \quad (3.16)$$

where $K_3 = K_2$ and $K_4 = \max\{ec_1\alpha, jb_1\beta - d, a_1\kappa - e\}$. Therefore

$$\dot{V}|_{(1.2)} \leq -H(t)K_5\left\{(x^2+y^2+z^2)\right\} \quad (3.17)$$

with $K_5 = \kappa K_3 + K_4$. By the definition of H, we have

$$\dot{V}|_{(1.2)} \leq -K_6\left\{(x^2+y^2+z^2)\right\} \quad (3.18)$$

with $K_6 = K_5|H(t)|$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Suppose that the conditions of Theorem 2.2 hold, then there are positive constants* $K_j = K_j(\alpha,\beta,\kappa,\Delta,\delta)(j = 7,8)$ *such that for any solution* (x,y,z) *of system* (1.2),

$$\dot{V}|_{(1.2)} \equiv \frac{d}{dt}V|_{(1.2)}(t;x,y,z)$$

$$\leq -K_7(x^2+y^2+z^2) + K_8(|x|+|y|+|z|)\,|p(t;x,\dot{x},\ddot{x})|\,. \quad (3.19)$$

Proof. Following the same arguments as in Lemma 3.2 but this time with $p \neq 0$ set $p(t;x,y,z) = p(t)$ we have that,

$$\frac{\partial R(x,y,z)}{\partial x}\frac{dx}{dt} = xy + yz + dy^2 + dxz + ez^2 + jyz$$
$$+ [ex + jy + z][-afz - bg - ch + p(t)]\,, \quad (3.20)$$

which reduces after some lengthy algebraic computations to

$$-ec\alpha x^2 - (jb\beta - d)y^2 - (a\kappa - e)^2 + (ex + dy + z)p(t). \quad (3.21)$$

Therefore

$$\dot{V}|_{(1.2)} = -H(t)[af(x,y) - a_0 f(x,y)]\,\{R(x,y,z)\} - H(t)\left\{-ec\alpha x^2\right.$$
$$\left. + (jb\beta - d)y^2 + (a\kappa - e)^2\right\} + H(t)(ex + dy + z)p(t). \quad (3.22)$$

This results in

$$\dot{V}|_{(1.2)} \leq -H(t)\kappa\left\{K_3(x^2+y^2+z^2) + K_4(x^2+y^2+z^2)\right\}$$
$$+ H(t)K_7(|x|+|y|+|z|)p(t)\,, \quad (3.23)$$

where $K_7 = \max\{e, d, 1\}$. Therefore

$$\dot{V}|_{(1.2)} \leq -H(t)K_5\left\{(x^2+y^2+z^2)\right\} + H(t)K_7(|x|+|y|+|z|)p(t)\,. \quad (3.24)$$

We have

$$\dot{V}|_{(1.2)} \leq -H(t)K_5\left\{(x^2+y^2+z^2)\right\} + H(t)K_8(x^2+y^2+z^2)^{\frac{1}{2}}p(t)\,, \quad (3.25)$$

where $K_8 = \sqrt{3}K_7$.

Therefore

$$\dot{V}|_{(1.2)} \leq -K_6\left\{(x^2+y^2+z^2)\right\} + K_9(x^2+y^2+z^2)^{\frac{1}{2}}p(t) \quad (3.26)$$

with $K_9 = K_8|H(t)|$. By the definition of H, we have that

$$\dot{V}|_{(1.2)} \leq -K_6\left\{(x^2+y^2+z^2)\right\} + K_9(x^2+y^2+z^2)^{\frac{1}{2}}p(t). \quad (3.27)$$

This completes the proof of Lemma 3. □

From the proofs of the lemmas it is established that the function $V(t;x,y,z)$ is a Lyapunov function.

4. Proof of the Main Results

We now give the proofs of Theorems 2.1 to 2.3.

Proof of Theorem 2.1. From the proof of Lemmas 3.1 and 3.2 it is established that the trivial solution of the equation (1.1) is globally asymptotically stable, i.e every solution $(x(t), \dot{x}(t), \ddot{x}(t))$ of the system (1.2) satisfies $x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \longrightarrow 0$ as $t \longrightarrow \infty$. □

Proof of Theorem 2.2. Indeed from the inequality (3.27),

$$\frac{dV}{dt} \leq -K_6(x^2 + y^2 + z^2) + K_9(x^2 + y^2 + z^2)^{\frac{1}{2}} |p(t)| ,$$

and also from the inequality (3.7), we have

$$(x^2 + y^2 + z^2)^{\frac{1}{2}} \leq \left(\frac{2V}{K_1}\right)^{\frac{1}{2}} .$$

Thus the inequality (3.27) becomes

$$\frac{dV}{dt} \leq -K_{10}V + K_{11}V^{\frac{1}{2}} |P(t)| . \tag{4.1}$$

We note that $K_6(x^2 + y^2 + z^2) = K_6 \cdot \frac{V}{K_1}$ and

$$\frac{dV}{dt} \leq -K_{10}V + K_{11}V^{\frac{1}{2}} |P(t)| , \tag{4.2}$$

where $K_{10} = \frac{K_6}{K_2}$ and $K_{11} = \frac{K_9}{K_2^{\frac{1}{2}}}$.

These imply that

$$\dot{V} \leq -K_{10}V + K_{11}V^{\frac{1}{2}} |P(t)|$$

and this can be written as

$$\dot{V} \leq -2K_{12}V + K_{11}V^{\frac{1}{2}} |P(t)| , \tag{4.3}$$

where $K_{12} = \frac{1}{2}K_{10}$.

Therefore

$$\dot{V} + K_{12}V \leq -K_{12}V + K_{11}V^{\frac{1}{2}} |P(t)| \tag{4.4}$$

$$\leq K_{11}V^{\frac{1}{2}} \left\{|P(t)| - K_{13}V^{\frac{1}{2}}\right\} , \tag{4.5}$$

where $K_{13} = \frac{K_{12}}{K_{11}}$. Thus the inequality (4.5) becomes

$$\dot{V} + K_{12}V \leq K_{11}V^{\frac{1}{2}}V^*\,, \tag{4.6}$$

where

$$V^* = |P(t)| - K_{13}V^{\frac{1}{2}} \tag{4.7}$$

$$\leq V^{\frac{1}{2}}\,|P(t)|$$

$$\leq |P(t)|\,. \tag{4.8}$$

When $|P(t)| \leq K_{13}V^{\frac{1}{2}}$,

$$V^* \leq 0 \tag{4.9}$$

and when $|P(t)| \geq K_{13}V^{\frac{1}{2}}$,

$$V^* \leq |P(t)| \cdot \frac{1}{K_{13}}. \tag{4.10}$$

On substituting the inequality (4.9) into the inequality (4.5), we have,

$$\dot{V} + K_{12}V \leq K_{14}V^{\frac{1}{2}}\,|P(t)|\ ,$$

where $K_{14} = \frac{K_{10}}{K_{13}}$. This implies that

$$V^{-\frac{1}{2}}\dot{V} + K_{12}V^{\frac{1}{2}} \leq K_{14}\,|P(t)|\,. \tag{4.11}$$

Multiplying both sides of the inequality (4.11) by $e^{\frac{1}{2}K_{12}t}$ we have,

$$e^{\frac{1}{2}K_{12}t}\left\{V^{-\frac{1}{2}}\dot{V} + K_{12}V^{\frac{1}{2}}\right\} \leq e^{\frac{1}{2}K_{12}t}K_{11}\,|P(t)|\ , \tag{4.12}$$

i.e.

$$2\frac{d}{dt}\left\{V^{\frac{1}{2}}e^{\frac{1}{2}K_{12}t}\right\} \leq e^{\frac{1}{2}K_{12}t}K_{11}\,|P(t)|\,. \tag{4.13}$$

Integrating both sides of (4.13) from t_0 to t, gives

$$\left\{V^{\frac{1}{2}}e^{\frac{1}{2}K_{12}\gamma}\right\}_{t0}^{t} \leq \int_{t0}^{t}\frac{1}{2}e^{\frac{1}{2}K_{12}\tau}K_{14}\,|P(\tau)d\tau| \tag{4.14}$$

which implies that

$$\left\{V^{\frac{1}{2}}(t)\right\}e^{\frac{1}{2}K_{12}t} \leq V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_{12}t_0} + \frac{1}{2}K_{14}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_{12}\tau}d\tau,$$

or

$$V^{\frac{1}{2}}(t) \le e^{-\frac{1}{2}K_{12}t}\left\{V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_{12}t_0} + \frac{1}{2}K_{14}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_{12}\tau}d\tau\right\}.$$

Using (3.10) we have

$$K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)) \le e^{-\frac{1}{2}K_{12}t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)\right. \\ \left. +\ddot{x}^2(t_0))e^{\frac{1}{2}K_9t_0} + \frac{1}{2}K_{14}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_{12}\tau}d\tau\right\}^2, \quad (4.15)$$

for all $t \ge t_0$.

Thus,

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \le \frac{1}{K_1}\left\{e^{-\frac{1}{2}K_{12}t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)\right.\right. \\ \left.\left. +\ddot{x}^2(t_0))e^{\frac{1}{2}K_{12}t_0} + \frac{1}{2}K_{14}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_{12}\tau}d\tau\right\}^2\right\} \\ \le \left\{e^{-\frac{1}{2}K_{12}t}\left\{A_1 + A_2\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_{12}\tau}d\tau\right\}^2\right\}, \quad (4.16)$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, (x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))\}$ and $\{K_1, K_{14}\}$ respectively.

By substituting $K_{12} = \sigma$ in the inequality (4.16), we have

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \le \left\{e^{-\frac{1}{2}\sigma t}\left\{A_1 + A_2\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}\sigma\tau}d\tau\right\}^2\right\},$$

which completes the proof. □

Proof of Theorem 2.3. From the function V defined above and the conditions of Theorem 2.3, the conclusion of Lemma 3.1 can be obtained, as

$$V \ge K_1\left(x^2+y^2+z^2\right), \quad (4.17)$$

and since $p \ne 0$ we can revise the conclusion of Lemma 3.2, i.e.

$$\dot{V} \le -K_4(x^2+y^2+z^2) + K_5(|x|+|y|+|z|)\,|p(t)|\ .$$

Using the condition on $p(t;x,y,z)$ as stated in assumption (vi) we obtain

$$\dot{V} \le K_5(|x|+|y|+|z|)^2 r(t). \quad (4.18)$$

By using the Schwartz inequalities on (4.18), we have

$$\dot{V} \leq K_{15}(x^2 + y^2 + z^2)r(t), \tag{4.19}$$

where $K_{15} = 3K_5$. From inequalities (4.17) and (4.19) we have,

$$\dot{V} \leq K_{15}Vr(t). \tag{4.20}$$

Integrating equation (4.20) from 0 to t, we obtain

$$V(t) - V(0) \leq K_{16}\int_0^t V(s)r(s)ds. \tag{4.21}$$

where $K_{16} = \frac{K_{15}}{K_1} = \frac{3K_5}{K_1}$.

This gives,

$$V(t) \leq V(0) + K_{16}\int_0^t V(s)r(s)ds. \tag{4.22}$$

By Grownwall-Bellman inequality, the inequality (4.22) yields,

$$V(t) \leq V(0)exp\left(K_{16}\int_0^t r(s)ds\right). \tag{4.23}$$

This completes the proof of Theorem 2.3.

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