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LAGUERRE-TYPE ORTHOGONAL POLYNOMIALS: ELECTROSTATIC INTERPRETATION

Herbert Dueñas[1,2], Francisco Marcellán[2 §]

[1]Departamento de Matemáticas
Universidad Nacional de Colombia
Ciudad Universitaria, Bogotá, COLOMBIA
e-mail: haduenasr@unal.edu.co

[2]Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad 30, Leganés, 28911, SPAIN
e-mail: pacomarc@ing.uc3m.es

Abstract: In this contribution we study the second order linear differential equation satisfied by polynomials orthogonal with respect to the linear functional

$$\langle \widetilde{\mu}, p \rangle = \int_0^{+\infty} p(x) x^{\alpha} e^{-x} dx + M p(0)\,,$$

where $\alpha > -1$, $M \in \mathbb{R}_+$, and p is a polynomial with real coefficients. We also find some results concerning the distribution of their zeros. Finally, an electrostatic interpretation of the zeros in terms of a logarithmic potential with an external field is presented.

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1. Introduction

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of complex numbers and μ a linear functional defined in the linear space $\mathbb{P}$ of the polynomials with complex coefficients, such that:

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[§]Correspondence author

$$\langle \mu, x^n \rangle = \mu_n, \quad n = 0, 1, 2 \dots .$$

Then we will say that μ is a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Moreover μ_n is called the *moment of order* n of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if:

(i) The degree of P_n is n.

(ii) $\langle \mu, P_n(x)P_m(x) \rangle = 0$, $m \neq n$.

(iii) $\langle \mu, P_n^2(x) \rangle \neq 0$, $n = 0, 1, 2, \dots$

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is a sequence of *monic orthogonal polynomials.* It is clear that for every sequence of orthogonal polynomials there exists the corresponding family of monic orthogonal polynomials. In the sequel we will work with monic polynomials.

The next theorem, whose proof you can find in [3], gives us necessary and sufficient conditions for the existence of a sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1. *Let μ be a moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $[\mu_{i+j}]_{i,j\in\mathbb{N}}$ are non singular.*

A moment functional such that there exists the correspondient sequence of orthogonal polynomials is said to be *regular* or *quasi-definite.*

Now we show the three term recurrence formula that a sequence of monic orthogonal polynomials satisfies and that we will use in the next section. The proof is given in [3].

Theorem 2. *If μ is a regular moment functional and $\{P_n\}_{n\geq 0}$ the corresponding sequence of monic orthogonal polynomials, then there exist sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$, with $\gamma_n \neq 0$ for every $n \in \mathbb{N}$, such that:*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \tag{1}$$

with $P_0(x) = 1, P_1(x) = x - \beta_0$.

If $\phi(x)$ is a complex polynomial, we define the moment functional $\phi\mu$, the left multiplication by a polynomial ϕ, and $D\mu$, the usual distributional derivative of μ, as follows:

$$\langle \phi\mu, p(x) \rangle = \langle \mu, \phi(x)\, p(x) \rangle, \;\; \langle D\mu, p(x) \rangle = -\langle \mu, p'(x) \rangle .$$

We will say that a sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ is *classical*, if there exist polynomials ϕ and ψ with $\deg\phi \leq 2$, $\deg\psi = 1$, so that μ satisfies the Pearson differential equation:

$$D(\phi\mu) = \psi\mu.$$

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) are used in the la literature taking into account their applications in mathematical physics. For example in the study of problems that involve hypergeometric differential equations (see [7], [10] and [11]).

The next theorem summarizes some properties of classical orthogonal polynomials. The proof is given in [8].

Theorem 3. *Let μ be a regular moment functional and $\{P_n\}_{n\geq 0}$ the correspondient sequence of monic orthogonal polynomials.*

1. $\{P_n\}_{n\geq 0}$ is classical if and only if there exist sequences a_n, b_n, c_n, with $c_n \neq 0$ for every $n \in \mathbb{N}$, such that:

$$\phi(x)P_n'(x) = a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x) \tag{2}$$

with $\deg\phi \leq 2$.

2. $\{P_n\}_{n\geq 0}$ is classical if and only if for every $n \in \mathbb{N}$, P_n is an eigenfunction of the differential operator

$$\mathcal{L} = \phi D^2 + \psi D\,, \tag{3}$$

where $\deg\phi \leq 2$, $\deg\psi = 1$, *and D denotes the standard derivative operator.*

3. $\{P_n\}_{n\geq 0}$ is classical if and only if there exist sequences r_n and s_n such that:

$$P_n(x) = Q_n(x) + r_nQ_{n-1}(x) + s_nQ_{n-2}(x),\ n \geq 2, \tag{4}$$

with $Q_k(x) = \dfrac{(P_{k+1}(x))'}{k+1}$, $k \geq 0$.

The polynomials $\{P_n\}_{n\geq 0}$ are *semiclassical orthogonal polynomials* if there exist polynomials ϕ and ψ with $\deg\psi \geq 1$, so that the corresponding moment functional μ satisfies $D(\phi\mu) = \psi\mu$.

Given a moment functional μ, if we add a Dirac mass, we obtain a new moment functional $\widetilde{\mu}$. The next theorem, whose proof can be founded in [9], characterizes the sequence of monic orthogonal polynomials $\left\{\widetilde{P}_n\right\}_{n\geq 0}$ corresponding to $\widetilde{\mu}$.

Theorem 4. *Let $\{P_n\}_{n\geq 0}$ be the sequence of monic orthogonal polyno-*

mials corresponding to the moment functional μ. We introduce the moment functional $\widetilde{\mu} = \mu + M\delta(x-a)$. Then:

1. $\widetilde{\mu}$ is regular if and only if $1 + MK_n(a,a) \neq 0$, for every $n \in \mathbb{N}$, where $K_n(x,a) = \sum_{j=0}^{n} \frac{P_j(a)}{\langle \mu, P_j^2 \rangle} P_j(x)$.

2. For every $n \in \mathbb{N}$,

$$\widetilde{P}_n(x) = P_n(x) - \frac{MP_n(a)}{1 + MK_{n-1}(a,a)} K_{n-1}(x,a). \tag{5}$$

As we mentioned before, a relevant class of classical orthogonal polynomials are the Laguerre polynomials. For these polynomials his moment functional satisfies the Pearson equation with $\phi(x) = x$ and $\psi(x) = -x + \alpha + 1, -\alpha \notin \mathbb{N}$. The next proposition summarizes some properties of the Laguerre polynomials $L_n^\alpha(x)$

Proposition 5. *Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials.*

1. For every $n \in \mathbb{N}$,

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+1+\alpha)\, L_n^\alpha(x) + n\,(n+\alpha)\, L_{n-1}^\alpha(x) \tag{6}$$

with $L_0^\alpha(x) = 1, L_1^\alpha(x) = x - (\alpha+1)$.

2. For every $n \in \mathbb{N}$,

$$x\,(L_n^\alpha(x))' = nL_n^\alpha(x) + n\,(n+\alpha)\, L_{n-1}^\alpha(x) \tag{7}$$

3. For every $n \in \mathbb{N}$, $L_n^\alpha(x)$ satisfies the differential equation

$$xy'' + (\alpha + 1 - x)\, y' = \lambda_n y \tag{8}$$

with $\lambda_n = -n$.

4. For every $n \in \mathbb{N}$,

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x) \tag{9}$$

5. For every $n \in \mathbb{N}$,

$$K_n(x,0) = C_n L_n^{\alpha+1}(x),\; C_n = \frac{L_n^\alpha(0)}{\langle \mu, 1\rangle\, \gamma_1\gamma_2\ldots\gamma_n}.$$

The aim of the present contribution is to find a second order linear differential equation (holonomic equation) for the so-called Laguerre-type orthogonal polynomials, i.e. orthogonal polynomials with respect the moment functional $\widetilde{\mu} = \mu + M\delta(x)$, where μ is the Laguerre moment functional. As a consequence, an electronic interpretation of their zeros in terms of potential theory is presented.

2. The Holonomic Equation

In order to find the second order linear differential equation satisfied by the polynomials $\widetilde{L}_n^{\alpha}(x)$, associated with the functional $\widetilde{\mu} = \mu + M\delta(x)$, we will first use Proposition 5.

$$\begin{aligned}
\widetilde{L}_n^{\alpha}(x) & \\
&= L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x) - \frac{ML_n^{\alpha}(0)}{1+MK_{n-1}(0,0)} \frac{L_{n-1}^{\alpha}(0)}{\langle \mu, 1\rangle \gamma_1\gamma_2 \dots \gamma_{n-1}} L_{n-1}^{\alpha+1}(x) \\
&= L_n^{\alpha+1}(x) + \left(n - \frac{ML_n^{\alpha}(0)}{1+MK_{n-1}(0,0)} \frac{L_{n-1}^{\alpha}(0)}{\langle \mu, 1\rangle \gamma_1\gamma_2 \dots \gamma_{n-1}}\right) L_{n-1}^{\alpha+1}(x), \\
& \qquad n = 1, 2 \dots
\end{aligned}$$

Some computation yields:

$$\begin{aligned}
&\frac{ML_n^{\alpha}(0)}{1+MK_{n-1}(0,0)} \frac{L_{n-1}^{\alpha}(0)}{\langle \mu, 1\rangle \gamma_1\gamma_2 \dots \gamma_{n-1}} \\
&\qquad = \frac{ML_n^{\alpha}(0)L_{n-1}^{\alpha}(0)}{\langle \mu, 1\rangle \gamma_1\gamma_2 \dots \gamma_{n-1} + ML_{n-1}^{\alpha}(0)L_{n-1}^{\alpha+1}(0)}.
\end{aligned}$$

Writing the Laguerre polynomials in terms of the Gauss hypergeometric (see [1]) function we have:

$$L_n^{\alpha}(x) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ {}_1F_1\left(-n, \alpha+1, x\right),$$

with

$${}_1F_1\left(-n, \alpha+1, x\right) = \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}$$

and

$$(\alpha)_k = \alpha\left(\alpha+1\right) \dots \left(\alpha+k-1\right),\ k \geq 1,\ (\alpha)_0 = 1.$$

Therefore:

$$L_n^{\alpha}(0) = \frac{\Gamma(n+\alpha+1)\,(-1)^n}{\Gamma(\alpha+1)} = (-1)^n\,(\alpha+1)_n.$$

On the other hand, from (6)

$$\gamma_1\gamma_2 \dots \gamma_n = n!\,(\alpha+1)_n\,,\ n \geq 1.$$

Therefore, we have

$$\frac{ML_n^{\alpha}(0)L_{n-1}^{\alpha}(0)}{\langle \mu, 1\rangle \gamma_1\gamma_2\dots\gamma_{n-1} + ML_{n-1}^{\alpha}(0)L_{n-1}^{\alpha+1}(0)}$$
$$= \frac{M(-1)^n(\alpha+1)_n(-1)^{n-1}(\alpha+1)_{n-1}}{\langle \mu, 1\rangle (n-1)!(\alpha+1)_{n-1} + M(-1)^{n-1}(\alpha+1)_{n-1}(-1)^{n-1}(\alpha+2)_{n-1}}$$
$$= \frac{-M(\alpha+1)_n}{\langle \mu, 1\rangle (n-1)! + M(\alpha+2)_{n-1}}$$

and, as a consequence,

$$n + \frac{M(\alpha+1)_n}{\langle \mu, 1\rangle (n-1)! + M(\alpha+2)_{n-1}}$$
$$= \frac{\Gamma(\alpha+1)n! + nM(\alpha+2)_{n-1} + M(\alpha+1)_n}{\Gamma(\alpha+1)(n-1)! + M(\alpha+2)_{n-1}}$$
$$= \frac{\Gamma(\alpha+1)n! + M(\alpha+2)_n}{\Gamma(\alpha+1)(n-1)! + M(\alpha+2)_{n-1}} = \frac{\alpha_n}{\alpha_{n-1}},$$

with $\alpha_n = \Gamma(\alpha+1)n! + M(\alpha+2)_n$. Thus, we have:

$$\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha+1}(x) + \frac{\alpha_n}{\alpha_{n-1}}L_{n-1}^{\alpha+1}(x), \tag{10}$$

and choosing

$$d_n = \frac{\alpha_n}{\alpha_{n-1}},$$

we get

Proposition 6. *If $\{L_n^{\alpha}\}_{n\geq 0}$ are the Laguerre monic orthogonal polynomials and $\left\{\widetilde{L}_n^{\alpha}\right\}_{n\geq 0}$ the corresponding sequence of orthogonal polynomials associated with the moment functional $\widetilde{\mu} = \mu + M\delta(x)$, then:*

$$\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha+1}(x) + d_nL_{n-1}^{\alpha+1}(x). \tag{11}$$

We are going to use the last proposition in order to obtain the second order linear differential equation that $\widetilde{L}_n^{\alpha}(x)$ satisfies. From (8) we have:

$$\phi\left(L_{n-1}^{\alpha+1}(x)\right)'' + \psi\left(L_{n-1}^{\alpha+1}(x)\right)' - \lambda_{n-1}L_{n-1}^{\alpha+1}(x) = 0$$

and

$$\phi\left(L_n^{\alpha+1}(x)\right)'' + \psi\left(L_n^{\alpha+1}(x)\right)' - \lambda_nL_n^{\alpha+1}(x) = 0$$

with $\phi(x) = x$ and $\psi(x) = \alpha + 2 - x$. Adding to the last identity the previous one multiplied d_n we get:

$$\phi\left(\left(L_n^{\alpha+1}(x)\right)'' + d_n\left(L_{n-1}^{\alpha+1}(x)\right)''\right) + \psi\left(\left(L_n^{\alpha+1}(x)\right)' + d_n\left(L_{n-1}^{\alpha+1}(x)\right)'\right) - \lambda_n L_n^{\alpha+1}(x) - \lambda_{n-1} d_n L_{n-1}^{\alpha+1}(x) = 0,$$

$$\phi\left(\widetilde{L}_n^{\alpha}(x)\right)'' + \psi\left(\widetilde{L}_n^{\alpha}(x)\right)' - \lambda_n L_n^{\alpha+1}(x) - \lambda_{n-1} d_n L_{n-1}^{\alpha+1}(x) = 0,$$

$$\phi\left(\widetilde{L}_n^{\alpha}(x)\right)'' + \psi\left(\widetilde{L}_n^{\alpha}(x)\right)' - \lambda_n L_n^{\alpha+1}(x) - \lambda_n d_n L_{n-1}^{\alpha+1}(x) - \lambda_{n-1} d_n L_{n-1}^{\alpha+1}(x) + \lambda_n d_n L_{n-1}^{\alpha+1}(x) = 0,$$

$$\phi\left(\widetilde{L}_n^{\alpha}(x)\right)'' + \psi\left(\widetilde{L}_n^{\alpha}(x)\right)' - \lambda_n \widetilde{L}_n^{\alpha}(x) = d_n L_{n-1}^{\alpha+1}(x)\left(\lambda_{n-1} - \lambda_n\right).$$

Taking into account $\lambda_{n-1} - \lambda_n = -(n-1) + n = 1$, we deduce

$$\phi\left(\widetilde{L}_n^{\alpha}(x)\right)'' + \psi\left(\widetilde{L}_n^{\alpha}(x)\right)' - \lambda_n \widetilde{L}_n^{\alpha}(x) = d_n L_{n-1}^{\alpha+1}(x). \tag{12}$$

If we derivate in each side of (11) and multiply by x we can write:

$$x\left(\widetilde{L}_n^{\alpha}(x)\right)' = x\left(L_n^{\alpha+1}(x)\right)' + d_n x\left(L_{n-1}^{\alpha+1}(x)\right)'.$$

Using (7) in $L_n^{\alpha+1}(x)$ and $L_{n-1}^{\alpha+1}(x)$, respectively, and replacing in the last expression:

$$x\left(\widetilde{L}_n^{\alpha}(x)\right)' = \left(nL_n^{\alpha+1}(x) + n(n+\alpha+1)L_{n-1}^{\alpha+1}(x)\right) + d_n\left((n-1)L_{n-1}^{\alpha+1}(x) + (n-1)(n+\alpha)L_{n-2}^{\alpha+1}(x)\right).$$

Therefore:

$$x\left(\widetilde{L}_n^{\alpha}(x)\right)' = nL_n^{\alpha+1}(x) + \left(n(n+\alpha+1) + d_n(n-1)\right)L_{n-1}^{\alpha+1}(x) + d_n(n-1)(n+\alpha)L_{n-2}^{\alpha+1}(x). \tag{13}$$

From (6) we have:

$$xL_{n-1}^{\alpha+1}(x) = L_n^{\alpha+1}(x) + (2n+\alpha)L_{n-1}^{\alpha+1}(x) + (n-1)(n+\alpha)L_{n-2}^{\alpha+1}(x).$$

The elimination of $L_{n-2}^{\alpha+1}(x)$ and replacing in (13), yields

$$x\left(\widetilde{L}_n^{\alpha}(x)\right)' = nL_n^{\alpha+1}(x) + \left(n(n+\alpha+1) + d_n(n-1)\right)L_{n-1}^{\alpha+1}(x) + d_n\left((x-(2n+\alpha))L_{n-1}^{\alpha+1}(x) - L_n^{\alpha+1}(x)\right) = (n-d_n)L_n^{\alpha+1}(x) + \left(n(n+\alpha+1) + d_n(n-1) + d_n(x-(2n+\alpha))\right)L_{n-1}^{\alpha+1}(x).$$

Now, taking into account:

$$n-d_n = n - \frac{\Gamma(\alpha+1)\,n! + M(\alpha+2)_n}{\Gamma(\alpha+1)(n-1)! + M(\alpha+2)_{n-1}} =$$
$$\frac{M\left(n(\alpha+2)_{n-1} - (\alpha+2)_n\right)}{\alpha_{n-1}} = \frac{-M(\alpha+2)_{n-1}(\alpha+1)}{\alpha_{n-1}} = -\frac{M(\alpha+1)_n}{\alpha_{n-1}}$$

and

$$n(n+\alpha+1) + d_n(n-1) + d_n(x-(2n+\alpha))$$
$$= d_n x + (n+\alpha+1)(n-d_n) = d_n\left(x - \frac{M(\alpha+1)_{n+1}}{\alpha_n}\right),$$

then:

$$x\left(\widetilde{L}_n^\alpha(x)\right)' = -\frac{M(\alpha+1)_n}{\alpha_{n-1}} L_n^{\alpha+1}(x) + d_n\left(x - \frac{M(\alpha+1)_{n+1}}{\alpha_n}\right) L_{n-1}^{\alpha+1}(x).$$

If we denote $m_n = -\frac{M(\alpha+1)_n}{\alpha_{n-1}}$ and $f_n(x) = d_n\left(x - \frac{M(\alpha+1)_{n+1}}{\alpha_n}\right)$, then:

$$x\left(\widetilde{L}_n^\alpha(x)\right)' = m_n L_n^{\alpha+1}(x) + f_n(x) L_{n-1}^{\alpha+1}(x). \tag{14}$$

Now, we use (11) and (14) in order to write $L_{n-1}^{\alpha+1}(x)$ in terms of $\widetilde{L}_n^\alpha(x)$ and $\left(\widetilde{L}_n^\alpha(x)\right)'$. Thus:

$$L_{n-1}^{\alpha+1}(x) = \frac{m_n \widetilde{L}_n^\alpha(x) - x\left(\widetilde{L}_n^\alpha(x)\right)'}{m_n d_n - f_n(x)}. \tag{15}$$

Replacing it in (12):

$$\phi\left(\widetilde{L}_n^\alpha(x)\right)'' + \psi\left(\widetilde{L}_n^\alpha(x)\right)' - \lambda_n \widetilde{L}_n^\alpha(x) = d_n\left(\frac{m_n \widetilde{L}_n^\alpha(x) - x\left(\widetilde{L}_n^\alpha(x)\right)'}{m_n d_n - f_n(x)}\right)$$

and, as a consequence,

$$\phi\left(\widetilde{L}_n^\alpha(x)\right)'' + \left(\psi + \frac{d_n x}{m_n d_n - f_n(x)}\right)\left(\widetilde{L}_n^\alpha(x)\right)'$$
$$- \left(\frac{d_n m_n}{m_n d_n - f_n(x)} - n\right)\widetilde{L}_n^\alpha(x) = 0. \tag{16}$$

But,

$$m_n d_n - f_n(x) = d_n\left(-\frac{M(\alpha+1)_n}{\alpha_{n-1}} - x + \frac{M(\alpha+1)_{n+1}}{\alpha_n}\right)$$
$$= d_n\left(-x + \frac{M(\alpha+1)_n}{\alpha_n}\left((\alpha+n+1) - d_n\right)\right) = d_n$$

$$\times\left(-x+\frac{M\left(\alpha+1\right)_n}{\alpha_n}\left(\left(\alpha+n+1\right)-\frac{\Gamma\left(\alpha+1\right)n!+M\left(\alpha+2\right)_n}{\Gamma\left(\alpha+1\right)\left(n-1\right)!+M\left(\alpha+2\right)_{n-1}}\right)\right)$$
$$=d_n\left(-x+\frac{M\left(\alpha+1\right)_n}{\alpha_n}\frac{\Gamma\left(\alpha+2\right)\left(n-1\right)!}{\alpha_{n-1}}\right)=d_n\left(-x+s_n\right),$$

where $s_n=\frac{M(\alpha+1)_n}{\alpha_n}\frac{\Gamma(\alpha+2)(n-1)!}{\alpha_{n-1}}$. Thus (16) can be written like:

$$x\left(\widetilde{L}_n^\alpha(x)\right)''+\left(\alpha+2-x+\frac{x}{s_n-x}\right)\left(\widetilde{L}_n^\alpha(x)\right)'$$
$$-\left(\frac{m_n}{s_n-x}-n\right)\widetilde{L}_n^\alpha(x)=0$$

and, finally,

$$x\left(s_n-x\right)\left(\widetilde{L}_n^\alpha(x)\right)''+\left(x^2-\left(\alpha+1+s_n\right)x+\left(\alpha+2\right)s_n\right)\left(\widetilde{L}_n^\alpha(x)\right)'$$
$$-\left(nx-\frac{M\left(\alpha+1\right)_n}{\alpha_{n-1}}-\frac{M\left(\alpha+1\right)_n\Gamma\left(\alpha+2\right)n!}{\alpha_n\alpha_{n-1}}\right)\widetilde{L}_n^\alpha(x)=0\,,$$

assuming that:

$$\alpha_n=n!\Gamma\left(\alpha+1\right)+M\left(\alpha+2\right)_n\,,\qquad d_n=\frac{\alpha_n}{\alpha_{n-1}}\,,$$
$$m_n=-\frac{M\left(\alpha+1\right)_n}{\alpha_{n-1}}\,,\qquad f_n(x)=d_n\left(x-\frac{M\left(\alpha+1\right)_{n+1}}{\alpha_n}\right)\,,$$
$$s_n=\frac{M(n-1)!\Gamma\left(\alpha+2\right)\left(\alpha+1\right)_n}{\alpha_n\alpha_{n-1}}\,.$$

So, we have proved:

Theorem 7. *If* $\{L_n^\alpha\}_{n\geq0}$ *are the Laguerre monic orthogonal polynomials and* $\left\{\widetilde{L}_n^a\right\}_{n\geq0}$ *are the monic orthogonal polynomials associated with the moment functional* $\widetilde{\mu}=\mu+M\delta(x)$*, then:*

$$A(x;n)\left(\widetilde{L}_n^\alpha(x)\right)''+B(x;n)\left(\widetilde{L}_n^\alpha(x)\right)'-C(x;n)\widetilde{L}_n^\alpha(x)=0, \tag{17}$$

with $A(x;n)=x\left(s_n-x\right)$, $B(x;n)=x^2-\left(\alpha+1+s_n\right)x+\left(\alpha+2\right)s_n$ *and* $C(x;n)=nx-\frac{M(\alpha+1)_n}{\alpha_{n-1}}-\frac{M(\alpha+1)_n\Gamma(\alpha+2)n!}{\alpha_n\alpha_{n-1}}$.

For an alternative proof see [6].

3. The Zeros

Let us consider the Laguerre monic orthogonal polynomials $\{L_n^\alpha\}_{n\geq 0}$ and μ_n the corresponding moment of order n. Taking into account $\widetilde{L}_n^\alpha(x)$ can be written as:

$$\widetilde{L}_n^\alpha(x) = \frac{1}{\Delta_{n-1}(\widetilde{\mu})}\begin{vmatrix} \mu_0+M & \mu_1 & \cdots & \mu_n \\ \mu_1 & & \cdots & \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix},$$

with

$$\Delta_{n-1}(\widetilde{\mu}) = \begin{vmatrix} \mu_0+M & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & & \cdots & \mu_n \\ \vdots & \vdots & & \vdots \\ \mu_{n-2} & \mu_{n-1} & \cdots & \mu_{2n-3} \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix},$$

we get: $\widetilde{L}_n^\alpha(x) = \frac{1}{\Delta_{n-1}(\widetilde{\mu})}\left(\Delta_{n-1}(\mu)\, L_n^\alpha(x) + Mx\Delta_{n-2}\left(x^2\mu\right) R_{n-1}(x)\right)$, where $\{R_n\}_{n\geq 0}$ is the sequence of monic orthogonal polynomials with respect to the moment functional $x^2\mu$. Thus:

$$\widetilde{L}_n^\alpha(x) = \frac{\Delta_{n-1}(\mu)\, L_n^\alpha(x) + Mx\Delta_{n-2}\left(x^2\mu\right) R_{n-1}(x)}{\Delta_{n-1}(\mu) + M\Delta_{n-2}\left(x^2\mu\right)}, \tag{18}$$

and, as a consequence:

$$\lim_{M\to\infty} \widetilde{L}_n^\alpha(x) = xR_{n-1}(x) = xL_{n-1}^{\alpha+2}(x).$$

Let $\left\{x_{n,k}^{(M)}\right\}_{k=1}^n$ be the zeros of the polynomial $\widetilde{L}_n^\alpha(x)$ such that $x_{n,1}^{(M)} < x_{n,2}^{(M)} < \cdots < x_{n,n}^{(M)}$. Fixed n and $1 \leq k \leq n$, we have that $x_{n,k}^{(M)}$ in a decreasing continuous function in M. Moreover, we have the next proposition, whose proof can be founded in [2].

Proposition 8. *For $M > 0$:*

1. *The zeros of $\widetilde{L}_n^\alpha(x)$ are real, simple, and nonnegative.*

2. Taking into account $d_n > 0$, we get:

$$0 < x_{n,1}^{(M)} < x_{n,1}^{\alpha+1} \quad \text{and} \quad x_{n-1,j-1}^{\alpha+1} < x_{n,j}^{(M)} < x_{n,j}^{\alpha+1},\ j = 2, \ldots, n\,,$$

where $\left\{x_{n,j}^{\alpha+1}\right\}_{j=1}^{n}$ are the zeros of the polynomial $L_n^{\alpha+1}(x)$ and $\left\{x_{n,j}^{(M)}\right\}_{j=1}^{n}$ the zeros of $\widetilde{L}_n^{\alpha}(x)$, respectively.

Furthermore:

$$\lim_{M\to\infty} x_{n,1}^{(M)} = 0, \qquad \lim_{M\to\infty} x_{n,k}^{(M)} = x_{n-1,k-1}^{\alpha+2}, \quad k = 2, \ldots, n\,,$$

where $\left\{x_{n-1,k}^{\alpha+2}\right\}_{k=1}^{n-1}$ are the zeros of $L_{n-1}^{\alpha+2}(x)$. From (18) we get:

$$\widetilde{L}_n^{\alpha}(0) = \frac{\Delta_{n-1}(\mu)\, L_n^{\alpha}(0)}{\Delta_{n-1}(\mu) + M\Delta_{n-2}(x^2\mu)}.$$

On the other hand, since $\widetilde{L}_n^{\alpha}(0) = (-1)^n\, x_{n,1}^{(M)} x_{n,2}^{(M)} \ldots x_{n,n}^{(M)}$ and $L_n^{\alpha}(0) = (-1)^n\, x_{n,1}^{\alpha} x_{n,2}^{\alpha} \ldots x_{n,n}^{\alpha}$, we get:

$$(-1)^n\, x_{n,1}^{(M)} x_{n,2}^{(M)} \ldots x_{n,n}^{(M)} = \frac{\Delta_{n-1}(\mu)\,(-1)^n\, x_{n,1}^{\alpha} x_{n,2}^{\alpha} \ldots x_{n,n}^{\alpha}}{\Delta_{n-1}(\mu) + M\Delta_{n-2}(x^2\mu)}.$$

Multiplying by M in both sides of the above expression and taking the limit when $M \to \infty$, we obtain:

$$\lim_{M\to\infty} M x_{n,1}^{(M)} x_{n,2}^{(M)} \ldots x_{n,n}^{(M)} = \lim_{M\to\infty} \frac{M\Delta_{n-1}(\mu)\, x_{n,1}^{\alpha} x_{n,2}^{\alpha} \ldots x_{n,n}^{\alpha}}{\Delta_{n-1}(\mu) + M\Delta_{n-2}(x^2\mu)},$$

i.e.

$$\lim_{M\to\infty} M x_{n,1}^{(M)} x_{n-1,1}^{\alpha+2} \ldots x_{n-1,n-1}^{\alpha+2} = \frac{\Delta_{n-1}(\mu)\, x_{n,1}^{\alpha} x_{n,2}^{\alpha} \ldots x_{n,n}^{\alpha}}{\Delta_{n-2}(x^2\mu)}\,,$$

so, we have:

$$\begin{aligned}
\lim_{M\to\infty} M x_{n,1}^{(M)} &= \frac{\Delta_{n-1}(\mu)}{\Delta_{n-2}(x^2\mu)} \frac{x_{n,1}^{\alpha} x_{n,2}^{\alpha} \ldots x_{n,n}^{\alpha}}{x_{n-1,1}^{\alpha+2} \ldots x_{n-1,n-1}^{\alpha+2}} = -\frac{\Delta_{n-1}(\mu)}{\Delta_{n-2}(x^2\mu)} \frac{L_n^{\alpha}(0)}{L_{n-1}^{\alpha+2}(0)} \\
&= -\frac{\Delta_{n-1}(\mu)}{\Delta_{n-2}(x^2\mu)} \frac{(-1)^n (\alpha+1)_n}{(-1)^{n-1} (\alpha+3)_{n-1}} = \frac{\Delta_{n-1}(\mu)\,(\alpha+1)_n}{\Delta_{n-2}(x^2\mu)\,(\alpha+3)_{n-1}} \\
&= \frac{(n-1)!\Gamma(\alpha+2)}{(\alpha+3)_{n-3}} = \frac{\alpha(\alpha+1)(n-1)!\Gamma(\alpha+3)}{(\alpha)_n},\ n \geq 1,
\end{aligned}$$

therefore we have the next

Theorem 9. *Let $\{x_{n,k}\}_{k\geq 1}$ be the zeros of $L_n^{\alpha}(x)$ and $\left\{x_{n,k}^{(M)}\right\}_{k\geq 1}$ be the* *zeros of* $\widetilde{L}_n^{\alpha}(x)$. *Then, for* $n \geq 1$

$$\lim_{M\to\infty} M x_{n,1}^{(M)} = \frac{\alpha\,(\alpha+1)\,(n-1)!\Gamma\,(\alpha+3)}{(\alpha)_n}. \tag{19}$$

4. Electrostatic Model

Assume that $\left\{x_{n,k}^{(M)}\right\}_{k\geq 1}$ are the zeros of $\widetilde{L}_n^{\alpha}(x)$, and evaluate (17) in every zero, thus:

$$A(x_{n,k}^{(M)};n)\left(\widetilde{L}_n^{\alpha}(x_{n,k}^{(M)})\right)'' + B(x_{n,k}^{(M)};n)\left(\widetilde{L}_n^{\alpha}(x_{n,k}^{(M)})\right)' = 0$$

and, as a consequence,

$$\frac{\left(\widetilde{L}_n^{a}(x_{n,k}^{(M)})\right)''}{\left(\widetilde{L}_n^{a}(x_{n,k}^{(M)})\right)'} = -\frac{B(x_{n,k}^{(M)};n)}{A(x_{n,k}^{(M)};n)}.$$

On the other hand, from (16),

$$\begin{aligned}\frac{B(x_{n,k}^{(M)};n)}{A(x_{n,k}^{(M)};n)} &= \frac{\left(m_n d_n - f_n(x_{n,k}^{(M)})\right)\psi\left(x_{n,k}^{(M)}\right) + d_n x_{n,k}^{(M)}}{\left(m_n d_n - f_n(x_{n,k})\right)\phi(x_{n,k})} \\ &= \frac{\psi\left(x_{n,k}^{(M)}\right)}{\phi\left(x_{n,k}^{(M)}\right)} + \frac{d_n}{\left(m_n d_n - f_n(x_{n,k}^{(M)})\right)}.\end{aligned}$$

Then:

$$\frac{\left(\widetilde{L}_n^{a}(x_{n,k}^{(M)})\right)''}{\left(\widetilde{L}_n^{a}(x_{n,k}^{(M)})\right)'} = \frac{d_n}{\left(f_n(x_{n,k}^{(M)}) - m_n d_n\right)} - \frac{\psi\left(x_{n,k}^{(M)}\right)}{\phi\left(x_{n,k}^{(M)}\right)}, 1 \leq k \leq n. \tag{20}$$

But

$$\frac{\left(\widetilde{L}_n^{a}(x_{n,k}^{(M)})\right)''}{\left(\widetilde{L}_n^{a}(x_{n,k}^{(M)})\right)'} = -2\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{x_{n,j}^{(M)} - x_{n,k}^{(M)}}$$

and, as a consequence, (20) becomes:

$$-2\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{x_{n,j}^{(M)} - x_{n,k}^{(M)}} = \frac{d_n}{\left(f_n\left(x_{n,k}^{(M)}\right) - m_n d_n\right)} - \frac{\psi\left(x_{n,k}^{(M)}\right)}{\phi\left(x_{n,k}^{(M)}\right)},$$

or, equivalently,

$$\sum_{\substack{j=1\\j\neq k}}^{n}\frac{1}{x_{n,j}^{(M)}-x_{n,k}^{(M)}}-\frac{1}{2\left(s_n-x_{n,k}^{(M)}\right)}-\frac{\alpha+2-x_{n,k}^{(M)}}{2x_{n,k}^{(M)}}=0,\ 1\leq k\leq n. \qquad (21)$$

The last expression means that the zeros of the polynomial $\widetilde{L}_n^{\alpha}(x)$ solve the next balanced electrostatic problem (see [4] and [5] for other examples).

Let us consider n unitary charges located in the positive real line under a logarithmic interaction with an external field:

$$\varphi(x)=-\frac{1}{2}\ln\left(x^{\alpha+2}e^{-x}\right)+\frac{1}{2}\ln|x-s_n|\,.$$

The equation (21) means that the gradient of the total energy

$$E(X)=-\sum_{1\leq k<j\leq n}\ln|x_k-x_j|+\sum_{j=1}^{n}\varphi\left(x_j\right)$$

with $X=(x_1,x_2,\ldots,x_n)$ vanishes at $\left(x_{n,1}^{(M)},x_{n,2}^{(M)},\ldots,x_{n,n}^{(M)}\right)$. In other words, it is a critical point. The analysis of the equilibrium problem remains open.

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