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# EXTENSION OF NORMS RELATED TO FULLY *K*-ROTUND NORMS AND MIDPOINT LOCALLY UNIFORMLY ROTUND NORMS

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**Abstract:** In this paper, we prove that fully $k$-rotund (respectively, locally fully $k$-rotund, fully $\omega$-rotund, fully locally $\omega$-rotund, locally mean $k$-rotund and midpoint locally uniformly rotund) norms can be extended from a closed subspace of a Banach space to the whole space.

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## 1. Introduction

In [6] and [7], it is shown that if $Y$ is a closed subspace of a separable Banach space $X$ and if $Y$ admits a locally uniform rotund ($LUR$) norm, then this $LUR$ norm on $Y$ can be extended to a $LUR$ norm on $X$. It is shown in [4] that if a Banach space $X$ has an $LUR$ (or uniformly rotund ($UR$)) norm $|\cdot|$ and $Y$ is a reflexive subspace of $X$ with an equivalent $LUR$ (or $UR$) norm $\|\cdot\|$, then there exists on $X$ an equivalent $LUR$ (or $UR$) norm $|||\cdot|||$ extending $\|\cdot\|$. By using different techniques, it is shown in [10] and [11] that the condition that $Y$ is reflexive could be removed. In [1], it is shown that the extension holds for $k$-nearly uniform convex ($k$-UNC) norms. In this paper, by using the techniques

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in [1], we prove that fully $k$-rotund (respectively, locally fully $k$-rotund, fully $\omega$-rotund, fully locally $\omega$-rotund, locally mean $k$-rotund and midpoint locally uniformly rotund) norms can be extended from closed subspaces in any Banach spaces.

In the following, we list some definitions and notations. Let $(X, \|\cdot\|)$ be a Banach space. Let $B_X(\|\cdot\|) = \{x \in X : \|x\| \leq 1\}$ be the unit ball of $X$ and $S_X(\|\cdot\|) = \{x \in X : \|x\| = 1\}$ be the unit sphere of $X$. Throughout this paper, $k$ is a positive integer.

**Definition 1.1.** (see [5]) A Banach space $X$ is said to be fully $k$-rotund ($kR$) if for any sequence $(x_n)$ in $B_X(\|\cdot\|)$, $\lim_{n_1,\ldots,n_k\to\infty} \|x_{n_1} + \cdots + x_{n_k}\| = k$ implies that $(x_n)$ is a convergent sequence.

**Definition 1.2.** (see [9]) A Banach space $X$ is said to be locally fully $k$-rotund ($L-kR$) if for any $x \in S_X(\|\cdot\|)$ and every sequence $(x_n)$ in $B_X(\|\cdot\|)$, $\lim_{n_1,\ldots,n_k\to\infty} \|x + x_{n_1} + \cdots + x_{n_k}\| = k+1$ implies that $(x_n)$ converges to $x$ in norm.

**Definition 1.3.** (see [12]) A Banach space $X$ is said to be fully $\omega$-rotund ($\omega R$) if for any sequence $(x_n)$ in $B_X(\|\cdot\|)$, $\lim_{n_1,\ldots,n_k\to\infty} \|x_{n_1} + \cdots + x_{n_k}\| = k$ for each positive integer $k$ implies that $(x_n)$ is a convergent sequence.

**Definition 1.4.** (see [12]) A Banach space $X$ is said to be locally fully $\omega$-rotund ($L\omega R$) if for every $x \in S_X(\|\cdot\|)$ and every sequence $(x_n)$ in $B_X(\|\cdot\|)$, $\lim_{n_1,\ldots,n_k\to\infty} \|x + x_{n_1} + \cdots + x_{n_k}\| = k+1$ for each positive integer $k$ implies that $(x_n)$ converges to $x$ in norm.

**Definition 1.5.** (see [2]) A Banach space $X$ is said to be midpoint locally uniformly rotund ($MLUR$) if for any $x \in S_X(\|\cdot\|)$ and any sequence $(x_n)$ in $X$, $\lim_{n\to\infty} \|x \pm x_n\| = 1$ implies that $\lim_{n\to\infty} \|x_n\| = 0$.

**Definition 1.6.** (see [3]) A Banach space $X$ is said to be locally mean $k$-rotund ($LMkR$) if for any $x \in S_X(\|\cdot\|)$ and any $k$ sequences $(x_{i,n})$ in $B_X(\|\cdot\|)$, $i = 1, \ldots, k$,

$$\lim_{n\to\infty} \|x + x_{1,n} + \cdots + x_{k,n}\| = k+1$$

implies that $\frac{x_{1,n}+\cdots+x_{k,n}}{k} \to x$ as $n \to \infty$ in norm.

## 2. Main Results

We need the following lemma from [1].

**Lemma 2.1.** (see [1]) *Let $X$ be a Banach space, $k$ be a positive integer, $p$ be a positive convex function on a bounded convex set $C \subset X$ and $(x_n)$ be a sequence in $C$. Then, for any $k$ indices $n_1 \leq \cdots \leq n_k$, (a) $\Longleftrightarrow$ (b) $\Longrightarrow$ (c):*

*(a)* $\lim_{n_1,\ldots,n_k\to\infty}\left|p^2\left(\frac{1}{k}\Sigma_{i=1}^{k}x_{n_i}\right)-\frac{1}{k}\Sigma_{i=1}^{k}p^2\left(x_{n_i}\right)\right|=0.$
*(b)* $\lim_{n_i,n_j\to\infty}\left|p\left(x_{n_i}\right)-p\left(x_{n_j}\right)\right|=0$ *for any* $1\leq i\leq j\leq k$.
*(c)* $\left|p\left(x_{n_1}\right)-p\left(\frac{1}{k}\Sigma_{i=1}^{k}x_{n_i}\right)\right|\to 0$ *as* $n_1,\ldots,n_k\to\infty$.

**Theorem 2.1.** *Let $Y$ be a closed subspace of a Banach space $X$. Suppose that both $X$ and $Y$ admit $kR$ (respectively, $L-kR$, $\omega R$, $L\omega R$, $LMkR$, $MLUR$) norms $\|\cdot\|$ and $|\cdot|_Y$, respectively, then $|\cdot|_Y$ can be extended to a $kR$ (resp. $L-kR$, $\omega R$, $L\omega R$, $LMkR$, $MLUR$) norm on $X$, in other words, $|\cdot|_Y$ is a restriction of a $kR$ (resp. $L-kR$, $\omega R$, $L\omega R$, $LMkR$, $MLUR$) norm of $X$ to $Y$.*

*Proof.* First, we prove the case that both $\|\cdot\|$ and $|\cdot|_Y$ are $L-kR$ norms. The cases for $kR$, $\omega R$, $L\omega R$ and $LMkR$ norms are similar, so we omit the proof. Let $|\cdot|$ be any extension of $|\cdot|_Y$ onto $X$ such that $\|\cdot\|\leq\frac{1}{\sqrt{2}}|\cdot|$. Define a function $p(\cdot)$ on $B_X(|\cdot|)$ by

$$p^2(x)=|x|^2+q(x)+d(x,Y)^2,$$

where $q(x)=d(x,Y)^2e^{\|x\|2}$. Then $p$ is convex symmetric function. Let $B=\{x\in X:p(x)\leq 1\}$. Then $p(\cdot)\geq|\cdot|$ and $p^2(x)\leq(2+e)|x|^2$ for all $x\in B$. Let $|||\cdot|||$ denote the corresponding norm defined by $p$. Then $|||\cdot|||$ is an equivalent norm on $X$.

To show that $|||\cdot|||$ is $L-kR$, let $x\in S_X(|||\cdot|||)$ and $(x_n)$ be any sequence in $B_X(|||\cdot|||)$ such that

$$\lim_{n_1,\ldots,n_k\to\infty}|||x+x_{n_1}+\cdots+x_{n_k}|||=k+1.$$

Suppose that $(x_n)$ is not convergent to $x$. Then there exist a subsequence of $(x_n)$ which we label again as $(x_n)$ and $\varepsilon>0$ such that

$$|||x_n-x|||\geq\varepsilon, \tag{1}$$

for all $n\in\mathbf{N}$.

Let $x_{n_0}=x$. Since $\lim_{n_1,\ldots,n_k\to\infty}\frac{1}{k+1}|||x_{n_0}+x_{n_1}+\cdots+x_{n_k}|||=1$, we have

$$\lim_{n_1,\ldots,n_k\to\infty}\left|p^2\left(\frac{\sum_{i=0}^{k}x_{n_i}}{k+1}\right)-\frac{\sum_{i=0}^{k}p^2\left(x_{n_i}\right)}{k+1}\right|=0.$$

Hence, by convexity of $p$, we have

$$\lim_{n_1,\ldots,n_k\to\infty}\left|\left|\frac{\sum_{i=0}^{k}x_{n_i}}{k+1}\right|^2-\frac{\sum_{i=0}^{k}\left|x_{n_i}\right|^2}{k+1}\right|=0, \tag{2}$$

$$\lim_{n_1,\ldots,n_k\to\infty}\left|q\left(\frac{\sum_{i=0}^{k}x_{n_i}}{k+1}\right)-\frac{\sum_{i=0}^{k}q\left(x_{n_i}\right)}{k+1}\right|=0 \tag{3}$$

and

$$\lim_{n_1,\ldots,n_k\to\infty}\left|d^2\left(\frac{\sum_{i=0}^k x_{n_i}}{k+1},Y\right)-\frac{\sum_{i=0}^k d^2\left(x_{n_i},Y\right)}{k+1}\right|=0. \tag{4}$$

By applying Lemma 2.1 to (4), we have the following

$$\lim_{n_i\to\infty}\left(d\left(x_{n_0},Y\right)-d\left(x_{n_i},Y\right)\right)=0, \tag{5}$$

for all $1\le i\le k$ and

$$\lim_{n_1,\ldots,n_k\to\infty}\left(d\left(x_{n_0},Y\right)-d\left(\frac{\sum_{i=0}^k x_{n_i}}{k+1},Y\right)\right)=0. \tag{6}$$

Next, we consider two cases.

*Case 1.* Suppose $d\left(x_{n_0},Y\right)=0$. By (5), $\lim_{n_i\to\infty} d\left(x_{n_i},Y\right)=0$, for all $1\le i\le k$. Let $(\tilde{x}_{n_i})\subset Y$ with $|\tilde{x}_{n_i}|\le 1$, $i=1,\ldots,k$, such that $\lim_{n_i\to\infty}|x_{n_i}-\tilde{x}_{n_i}|=0$, $i=1,\ldots,k$. Since $(x_{n_i})$ is bounded, by (2), we have

$$\lim_{n_1,\ldots,n_k\to\infty}\left|\left|\frac{x_{n_0}+\sum_{i=1}^k \tilde{x}_{n_i}}{k+1}\right|_Y^2-\frac{|x_{n_0}|_Y^2+\sum_{i=1}^k|\tilde{x}_{n_i}|_Y^2}{k+1}\right|=0.$$

By Lemma 2.1 and passing to a subsequence if necessary, we have

$$\lim_{n_1,\ldots,n_k\to\infty}\left|\frac{x_{n_0}+\tilde{x}_{n_1}+\cdots+\tilde{x}_{n_k}}{k+1}\right|_Y=|x_{n_0}|_Y\,.$$

Since $|\cdot|_Y$ is $L-kR$, $(\tilde{x}_n)$ is convergent to $x_{n_0}=x$ and thus $(x_n)$ is convergent to $x$, contradicting (1).

*Case 2.* Suppose $d\left(x_{n_0},Y\right)=d>0$. By (3), we have

$$\lim_{n_1,\ldots,n_k\to\infty}\left|d\left(\frac{\sum_{i=0}^k x_{n_i}}{k+1},Y\right)e^{\left\|\frac{\sum_{i=0}^k x_{n_i}}{k+1}\right\|^2}-\frac{\sum_{i=0}^k d\left(x_{n_i},Y\right)e^{\|x_{n_i}\|^2}}{k+1}\right|=0,$$

which implies

$$\lim_{n_1,\ldots,n_k\to\infty}\left|e^{\left\|\frac{\sum_{i=0}^k x_{n_i}}{k+1}\right\|^2}-\frac{\sum_{i=0}^k e^{\|x_{n_i}\|^2}}{k+1}\right|=0.$$

Hence,

$$\lim_{n_1,\ldots,n_k\to\infty}\sum_{m=1}^{\infty}\frac{1}{m!}\left(\left\|\frac{\sum_{i=0}^k x_{n_i}}{k+1}\right\|^{2m}-\frac{\sum_{i=0}^k\|x_{n_i}\|^{2m}}{k+1}\right)=0.$$

Thus, we have

$$\lim_{n_1,\ldots,n_k\to\infty}\left(\left\|\frac{\sum_{i=0}^k x_{n_i}}{k+1}\right\|^2-\frac{\sum_{i=0}^k \|x_{n_i}\|^2}{k+1}\right)=0.$$

By Lemma 2.1 and passing to a subsequence if necessary, we have

$$\lim_{n_1,\ldots,n_k\to\infty}\left\|\frac{\sum_{i=0}^k x_{n_i}}{k+1}\right\|^2=\|x_{n_0}\|^2.$$

Since $\|\cdot\|$ is $L-kR$, $\lim_{n\to\infty}\|x_n-x_{n_0}\|=\lim_{n\to\infty}\|x_n-x\|=0$, contradicting (1).

For the case of $MLUR$ norms, we use the same extension as above. Suppose that both $\|\cdot\|$ and $|\cdot|_Y$ are $MLUR$ norms. Let $x\in S_X(|||\cdot|||)$ and $(x_n)$ be any sequence in $X$ such that

$$\lim_{n\to\infty}|||x\pm x_n|||=1$$

and $\lim_{n\to\infty}|||x_n|||\neq 0$. Without loss of generality, we may assume that there exists $\varepsilon>0$ such that

$$|||x_n|||\geq\varepsilon, \tag{7}$$

for all $n\in\mathbf{N}$. Since $\lim_{n\to\infty}|||x\pm x_n|||=|||x|||$, we have

$$\lim_{n\to\infty}|x\pm x_n|=|x|,\ \lim_{n\to\infty}q(x\pm x_n)=q(x)$$

and $\lim_{n\to\infty}d(x\pm x_n,Y)=d(x,Y)$. We consider two cases again.

*Case 1.* Suppose $d(x,Y)>0$. Since $\lim_{n\to\infty}q(x\pm x_n)=q(x)$, we have $\lim_{n\to\infty}||x\pm x_n||=||x||$. Since $||\cdot||$ is $MLUR$, $\lim_{n\to\infty}||x_n||=0$, contradicting (7).

*Case 2.* Suppose $d(x,Y)=0$. Then we may find $\left(\widetilde{\widetilde{x}}_n\right)\subset Y$ such that $\lim_{n\to\infty}\left|\widetilde{\widetilde{x}}_n-x_n\right|=0$. Hence, $\lim_{n\to\infty}|x\pm x_n|=|x|$ implies that $\lim_{n\to\infty}\left|x\pm\widetilde{\widetilde{x}}_n\right|=|x|$. Since $|\cdot|_Y$ is $MLUR$, $\lim_{n\to\infty}\left|\widetilde{\widetilde{x}}_n\right|=0$. Thus, $\lim_{n\to\infty}|x_n|=0$, contradicting (7).

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