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EXISTENCE AND UNIQUENESS FOR NEUTRAL EQUATIONS WITH DELAY DEPENDANT ON A SOLUTION AND ITS DERIVATIVE

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Abstract: The comparison method is used to prove an existence and uniqueness of solutions of neutral equations with delay dependent on the solution and its derivative. Particular cases of these equations are neutral equations with state dependent delays.

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1. Introduction

Let $\mathbb{R}_+ = [0, +\infty)$. For any metric spaces U and W we denote by $C(U, W)$ the class of all continuous functions from U to W and let the symbol $\mathbb{X}^n$, $n \in \mathbb{N}$, means the carthesian product of n factors $\mathbb{X}$. Let E be an arbitrary Banach space with the norm $\| \cdot \|$, and let $a > 0$, $h > 0$, $I_t = [0, t]$, $t \in [0, a]$. Given the functions

$$f : I_a \times C^{2p}([-h, 0], E) \to E,$$
$$\Psi, \Theta : I_a \times C([-h, 0], E) \times C([-h, 0], E) \to I_a^p,$$
$$\Psi = (\Psi_1, \dots, \Psi_p), \quad \Theta = (\Theta_1, \dots, \Theta_p),$$
$$\alpha, \beta, \gamma, \delta : I_a \to I_a^p,$$
$$\alpha = (\alpha_1, \dots, \alpha_p),\ \beta = (\beta_1, \dots, \beta_p),\ \gamma = (\gamma_1, \dots, \gamma_p),\ \delta = (\delta_1, \dots, \delta_p),$$
$$\varphi : [-h, 0] \to E,$$

where $p \in \mathbb{N}$. For the function $y : [-h, a] \to E$ and $t \in I_a$ we define Hale's

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operator in the following way

$$y_t(\tau) = y(t+\tau), \quad \tau \in [-h, 0].$$

We consider the problem

$$x'(t) = f(t, x^{\star}_{\Psi(t, x_{\alpha(t)}, x'_{\beta(t)})}, (x')^{\star}_{\Theta(t, x_{\gamma(t)}, x'_{\delta(t)})}), \quad t \in I_a, \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-h, 0], \tag{2}$$

where

$$x^{\star}_{\Psi(t, x_{\alpha(t)}, x'_{\beta(t)})} = (x_{\Psi_1(t, x_{\alpha_1(t)}, x'_{\beta_1(t)})}, \ldots, x_{\Psi_p(t, x_{\alpha_p(t)}, x'_{\beta_p(t)})}),$$

$$(x')^{\star}_{\Theta(t, x_{\gamma(t)}, x'_{\delta(t)})} = ((x')_{\Theta_1(t, x_{\gamma_1(t)}, x'_{\delta_1(t)})}, \ldots, (x')_{\Theta_p(t, x_{\gamma_p(t)}, x'_{\delta_p(t)})}),$$

and the symbol $x_{\Psi_i(t, x_{\alpha_i(t)}, x'_{\beta_i(t)})}$, $i = 1, \ldots, p$, is the restriction of x to the set $[\Psi_i(t, x_{\alpha_i(t)}, x'_{\beta_i(t)}) - h, \Psi_i(t, x_{\alpha_i(t)}, x'_{\beta_i(t)})]$, $t \in I_a$, and this restriction is shifted to the set $[-h, 0]$. The same convention is applied to symbols: $x'_{\Psi_i(t, x_{\alpha_i(t)}, x'_{\beta_i(t)})}$, $x_{\Theta_i(t, x_{\alpha_i(t)}, x'_{\beta_i(t)})}$, $x'_{\Theta_i(t, x_{\alpha_i(t)}, x'_{\beta_i(t)})}$, $i = 1, \ldots, p$.

Put $x'(t) = z(t)$ for $t \in [0, a]$. Then

$$(Vz)(t) = \varphi(0) + \int_0^t z(s)ds, \tag{3}$$

and the Cauchy problem (1), (2) is equivalent to the following one

$$z(t) = f(t, (Vz)^{\star}_{\Psi(t, (Vz)_{\alpha(t)}, z_{\beta(t)})}, z^{\star}_{\Theta(t, (Vz)_{\gamma(t)}, z_{\delta(t)})}), \quad t \in I_a, \tag{4}$$

$$z(t) = \varphi'(t), \quad t \in [-h, 0], \tag{5}$$

where

$$(Vz)^{\star}_{\Psi(t, (Vz)_{\alpha(t)}, z_{\beta(t)})} = ((Vz)_{\Psi_1(t, (Vz)_{\alpha_1(t)}, z_{\beta_1(t)})}, \ldots, (Vz)_{\Psi_p(t, (Vz)_{\alpha_p(t)}, z_{\beta_p(t)})}),$$

$$z^{\star}_{\Theta(t, (Vz)_{\gamma(t)}, z_{\delta(t)})} = (z_{\Theta_1(t, (Vz)_{\gamma_1(t)}, z_{\delta_1(t)})}, \ldots, z_{\Theta_p(t, (Vz)_{\gamma_p(t)}, z_{\delta_p(t)})}).$$

The investigation for differential functional equations can be found in many papers (see for example [2], [3], [15]-[25]), and the initial problem (1), (2) in some particular cases have been considered in papers, see [1], [4]-[14], [23] (see also Remark 5). We consider the special case of equation (1) in [7] (for $p = 1$).

If the function f does not depend on the last p variables and the function Ψ does not depend on the last variable, then (1) is the equation with state dependent delays. Such equations with $p = 1$ were considered in many papers, for example in [10]-[14].

If functions Ψ and Θ do not depend on functional variables, then we get a differential equation with a retarded argument of the netral type. Theorems on the existence of solutions of such equations we can find in papers [1], [5], [6], [15], [17], [20]-[24], [26], and in many others.

In this paper we prove a theorem on the existence and uniqueness of a solution of equation (1), under conditions involving some relation between Lipschitz constants of the function f, and the estimations imposed of the functions Ψ, and Θ. The proof of the existence and uniqueness theorem is based on the comparison method. A general formulation of the comparison method can be found in [25].

2. Lemmas

For the functions k, σ, l, $\zeta : I_a \to \mathbb{R}^p_+$, where

$$k(t) = (k_1(t), \ldots, k_p(t)), \quad l(t) = (l_1(t), \ldots, l_p(t)),$$

$$\sigma(t) = (\sigma_1(t), \ldots, \sigma_p(t)), \quad \zeta(t) = (\zeta_1(t), \ldots, \zeta_p(t)),$$

and $t \in I_a$ we put

$$(Kg)(t) = \sum_{i=1}^{p} k_i(t) \int\limits_0^{\sigma_i(t)} g(s)ds, \quad (Lg)(t) = \sum_{i=1}^{p} l_i(t) g(\zeta_i(t)), \quad t \in I_a.$$

We define $L^{n+1} = LL^n$, $n = 1, 2, \ldots$, $L^0 = J$, where J denotes the identity operator. For $n = 0, 1, \ldots$, $t \in I_a$ we put

$$l_1^i(t) = l_i(t), \quad \zeta_1^i(t) = \zeta_i(t), \quad i = 1, \ldots, p,$$

$$l_{n+1}^{i_1,\ldots,i_n}(t) = l_{i_{n+1}}(t) l_n^{i_1,\ldots,i_n}(\zeta_{i_{n+1}}(t)), \quad i_1, \ldots, i_n = 1, \ldots, p,$$

$$\zeta_{n+1}^{i_1,\ldots,i_{n+1}}(t) = \zeta_n^{i_1,\ldots,i_n}(\zeta_{i_{n+1}}(t)), \quad i_1, \ldots, i_{n+1} = 1, \ldots, p,$$

and

$$(L^n g)(t) = \sum_{i_1=1}^{p} \ldots \sum_{i_n}^{p} l_n^{i_1,\ldots,i_n} g(\zeta_n^{i_1,\ldots,i_n}(t)), \quad Mg = \sum_{n=0}^{+\infty} L^n g.$$

Lemma 1. *Suppose that* $k,\ \sigma,\ l,\ \zeta \in C(I_a, \mathbb{R}_+^p)$, $\alpha,\ \beta,\ \gamma,\ \delta \in C(I_a, I_a^p)$, $H \in C(I_a, \mathbb{R}_+)$ *are nondecreasing functions,* $\alpha_i(t),\ \beta_i(t),\ \gamma_i(t),\ \delta_i(t) \in [0,t]$, $t \in I_a$, $i = 1, \ldots, p$ *and*

$$MH < +\infty, \quad \bar{s} = M(\sum_{i=1}^{p} k_i(t)\sigma_i(t)) < +\infty, \quad \sup_{t\in(0,a]} \frac{\bar{s}(t)}{t} < +\infty. \tag{6}$$

Then:

(a) There exists a nondecreasing function $\bar{g} \in C(I_a, \mathbb{R}_+)$, *which is unique solution of equation*

$$g = MKg + MH \tag{7}$$

in the class $P(I_a, \mathbb{R}_+)$ *of upper semicontinuous functions defined on* I_a;

(b) The function $\bar{g}$ *is a nondecreasing and unique solution of the equation*

$$g = Kg + Lg + H \tag{8}$$

in the class

$$P(I_a, \mathbb{R}_+, \bar{g}) = \{g \in P(I_a, \mathbb{R}_+) : \ \|g\|_\star < +\infty, \},$$

where

$$\|g\|_\star = \inf\{c \in \mathbb{R}_+ : \ g(t) \le c\bar{g}(t), t \in I_a\};$$

(c) The function $g = 0$ *is the unique solution of the inequality*

$$g \le Kg + Lg \tag{9}$$

in the class $P(I_a, \mathbb{R}_+, \bar{g})$.

Proof. At first we prove (a). If the function g is a solution of (7), then $g \in C(I_a, \mathbb{R}_+)$. We show that g is unique solution of this equation in the space $C(I_a, \mathbb{R}_+)$. Let

$$\|g\|_\Lambda = \sup_{t\in I_a} e^{-\Lambda t} g(t),$$

where $g \in C(I_a, \mathbb{R}_+)$, and $\Lambda > \Lambda_0 = \sup_{t\in(0,a]} \frac{\bar{s}(t)}{t}$. We show that the operator MK is a contraction. Using the estimation $e^{\varepsilon t} - 1 \le \varepsilon e^t$, $\varepsilon \in I_1$, $t \in \mathbb{R}_+$, we get

$$\|MKg\|_\Lambda$$
$$\le \sup_{t\in I_a} \sum_{i=1}^{p} \sum_{n=0}^{+\infty} \sum_{i_1}^{p} \cdots \sum_{i_n}^{p} l_n^{i_1,\ldots,i_n}(t) k_i(\zeta_n^{i_1,\ldots,i_n}(t)) \int_0^{\sigma(\zeta_n^{i_1,\ldots,i_n}(t))} e^{\Lambda s}[e^{-\Lambda s}]ds$$

$$\leq \frac{\|g\|_\Lambda}{\Lambda} \sup_{t\in I_a} \sum_{i=1}^{p}\sum_{n=0}^{+\infty}\sum_{i_1}^{p}\dots\sum_{i_n}^{p} l_n^{i_1,\dots,i_n}(t)k_i(\zeta_n^{i_1,\dots,i_n}(t))[e^{\Lambda\sigma_i(\zeta_n^{i_1,\dots,i_n}(t))}-1]$$

$$\leq \frac{\|g\|_\Lambda}{\Lambda} \sup_{t\in (0,a]} \sum_{i=1}^{p}\sum_{n=0}^{+\infty}\sum_{i_1}^{p}\dots\sum_{i_n}^{p} l_n^{i_1,\dots,i_n}(t)k_i(\zeta_n^{i_1,\dots,i_n}(t))[e^{\Lambda\frac{\sigma_i(\zeta_n^{i_1,\dots,i_n}(t))}{t}t}-1]$$

$$\leq \frac{\|g\|_\Lambda}{\Lambda} \sup_{t\in (0,a]} \sum_{i=1}^{p}\sum_{n=0}^{+\infty}\sum_{i_1}^{p}\dots\sum_{i_n}^{p} l_n^{i_1,\dots,i_n}(t)k_i(\zeta_n^{i_1,\dots,i_n}(t))\frac{\sigma_i(\zeta_n^{i_1,\dots,i_n}(t))}{t}e^{\Lambda t}$$

$$\leq \frac{\|g\|_\Lambda}{\Lambda} \sup_{t\in (0,a]} \frac{\bar{s}(t)}{t} \leq \frac{\Lambda_0}{\Lambda}\|g\|_\Lambda .$$

Therefore MK is a contraction and the estimation (a) is a consequence of Banach fixed point theorem.

Now we show (b). We show that any solution of (7) is also a solution of (8). Let $\bar{g}$ be a solution of (7). Using the equality $LMg = Mg - g$ we get

$$\begin{aligned} K\bar{g} + L\bar{g} + H &= K\bar{g} + L(MK\bar{g} + MH) + H = K\bar{g} + LMK\bar{g} + LMH + H \\ &= K\bar{g} + MK\bar{g} - K\bar{g} + MH - H + H = MK\bar{g} + MH = \bar{g}. \end{aligned}$$

We observe that for any solution $\bar{g}$ of equation (7) we have

$$L^n\bar{g} = L^nMK\bar{g} + L^nMH = \sum_{i=n}^{+\infty} L^iK\bar{g} + \sum_{i=n}^{+\infty} L^iH,$$

hence we get

$$L^n\bar{g} \to 0 \quad \text{if} \quad n \to +\infty.$$

If $\tilde{g} \in P(I_a, \mathbb{R}_+, \bar{g})$ is a solution of equation (8), then by induction we obtain

$$\tilde{g} = \sum_{i=0}^{n-1} L^iK\tilde{g} + \sum_{i=0}^{n-1} L^iH + L^n\tilde{g}, \quad n = 1, 2, \dots . \tag{10}$$

Since $\tilde{g} \in P(I_a, \mathbb{R}_+, \bar{g})$, then for some $c \geq 0$ we have $0 \leq \tilde{g} \leq c\bar{g}$, now according to $L^n\tilde{g} \leq cL^n\bar{g}$, we infer $L^n\tilde{g} \to 0$ if $n \to +\infty$. If we let $n \to +\infty$ in relation (10) we get $\tilde{g} = MK\tilde{g} + MH$, i.e. $\tilde{g}$ is the solution of (7), but this equation has only the solution $\bar{g}$, thus $\tilde{g} = \bar{g}$, and (b) is proved.

Now we prove (c). If $g \in P(I_a, \mathbb{R}_+, \bar{g})$ is the solution of inequality (9) then by induction we get

$$g \leq \sum_{i=0}^{n-1} L^iKg + L^ng, \quad n = 1, 2, \dots$$

For some $c \in \mathbb{R}_+$ we have $g \le c\bar{g}$. Therefore g satisfies the inequality $g \le MKg$.

Because of $\|MK\|_\chi < 1$ we get that $g = 0$ is the unique solution of (9) in the class $C(I_a, \mathbb{R}_+)$ with the norm $\|\cdot\|_\chi$. Thus $g = 0$ is the unique solution of (9) in the class with the supremum norm. The lemma is proved. □

Remark 1. If assumptions of Lemma 1 are satisfied for $\bar{H} \in C(I_a, \mathbb{R}_+)$, where $\bar{H}(t) \le H(t)$, $t \in I_a$, then the suitable solution $\tilde{g}$ of equation (7) with $\bar{H}$ instead of H established in Lemma 1, is the unique solution of the equation (8) with H replaced by $\bar{H}$ in the class $P(I_a, \mathbb{R}_+, \bar{g})$.

This fact follows immediately from the part (b) of the proof of Lemma 1.

Remark 2. If $p = 1$, and $k(t) = k_0(t)$, $l(t) = l_0(t)$, $\sigma(t) = \sigma_0(t)$, $\zeta(t) = \zeta_0(t)$, where k_0, l_0, σ_0, $\zeta_0 \in C(I_a, \mathbb{R}_+)$, then the sequences $\{\sigma_n\}$, and $\{\zeta_n\}$ are defined by relations

$$\sigma_0(t) = t, \quad \sigma_{n+1}(t) = \sigma(\sigma_n(t)), \quad n = 0, 1, \ldots, \quad t \in I_a,$$

$$\zeta_0(t) = 1, \quad \zeta_{n+1}(t) = \zeta(t)\zeta_n(\sigma(t)), \quad n = 0, 1, \ldots, \quad t \in I_a,$$

and the conditions (6) have the form

$$MH < +\infty, \quad \bar{s} = M(k_0(t)\sigma_0(t)) < +\infty, \quad \sup_{t\in(0,a]} \frac{\bar{s}(t)}{t} < +\infty.$$

In the space $C([-h, 0], E)$ we define the norm

$$\|v\|_0 = \sup_{\tau\in[-h,0]} |v(\tau)|,$$

where $v \in C([-h, 0], E)$. We also define some functional spaces

$$\Omega([-h, a], E, \bar{g}) \\ = \left\{u \in C([-h, a], E) : \ u\,|_{[-h,0]}, \ \|u(s)\| \le \bar{g}(t), \ s \in [-h, t], \ t \in I_a\right\},$$

where $\bar{g}$ is defined in Lemma 1.

Assumption H_1. Suppose that:

(i) There exist nondecreasing functions $\bar{k}$, $\bar{l}$, μ, ν, χ, $\rho \in C(I_a, \mathbb{R}^p_+)$, $\bar{k} = (\bar{k}_1, \ldots, \bar{k}_p)$, $\bar{l} = (\bar{l}_1, \ldots, \bar{l}_p)$, $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_p)$, $\bar{\nu} = (\bar{\nu}_1, \ldots, \bar{\nu}_p)$, $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_p)$, $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_p)$, such that

$$\|f(t, u_1, \ldots, u_p, v_1, \ldots, v_p) - f(t, \bar{u}_1, \ldots, \bar{u}_p, \bar{v}_1, \ldots, \bar{v}_p)\| \\ \le \sum_{i=1}^{p} [\bar{k}_i(t)\|u_i - \bar{u}_i\|_0 + \bar{l}_i(t)\|v_i - \bar{v}_i\|_0],$$

$$|\Psi_i(t,y,w)-\Psi_i(t,\bar y,\bar w)|\le \nu_i(t)\|y-\bar y\|_0+\mu_i(t)\|w-\bar w\|_0, \quad i=1,\dots,p,$$
$$|\Theta_i(t,y,w)-\Theta_i(t,\bar y,\bar w)|\le \chi_i(t)\|y-\bar y\|_0+\rho_i(t)\|w-\bar w\|_0, \quad i=1,\dots,p,$$

where $t\in I_a$, u_i, $\bar u_i$, v_i, $\bar v_i$, y, $\bar y$, w, $\bar w \in C([-h,0],E)$;

(ii) There exist functions $\bar r$, $\tilde r\in C(I_a,I_a)$, $\bar r(t)$, $\tilde r(t)\in[0,t]$, such that

$$|\Psi_i(t,u,v,)|\le \bar r(t), \quad |\Theta_i(t,u,v)|\le \tilde r(t), \quad i=1,\dots,p,$$

where $\|u\|_0\le\|\varphi(0)\|+\bar r(a)\bar g(\bar r(a))$, $\|v\|_0\le \bar g(\tilde r(a))$, $\|y\|_0\le\|\varphi(0)\|+a\bar g(a)$, $\|w\|_0\le\bar g(a)$;

(iii) $\varphi\in C^1([-h,0,E])$, and $\|\varphi'(\tau)\|\le\bar g(0)$, $\tau\in[-h,0]$ where $\bar g$ is defined in Lemma 1.

The consequence of the point (i) of Assumption H_1 is the following estimation

$$\|f(t,u_1,\dots,u_p,v_1,\dots,v_p)\|\le\sum_{i=1}^{p}[\bar k_i(t)\|u_i\|_0+\bar l_i(t)\|v_i\|_0]+\bar\omega(t),$$

where $\bar\omega(t)=\sup_{s\in I_t}\|f(s,\mathbb{O},\dots,\mathbb{O})\|$, $t\in I_a$, and $\mathbb{O}$ means zero in the space $C([-h,0],E)$.

We define the operator $\mathcal F$ in the following way

$$\mathcal F[z](t)=f(t,(Vz)^\star_{\Psi(t,(Vz)_{\alpha(t)},z_{\beta(t)})},z^\star_{\Theta(t,(Vz)_{\gamma(t)},z_{\delta(t)})}), \quad t\in I_a,$$
$$\mathcal F[z](t)=\varphi'(t), \quad t\in[-h,0],$$

where the operator V is given by (3), and

$$(Vz)^\star_{\Psi(t,(Vz)_{\alpha(t)},z_{\beta(t)})} = ((Vz)_{\Psi_1(t,(Vz)_{\alpha_1(t)},z_{\beta_1(t)})},\dots,(Vz)_{\Psi_p(t,(Vz)_{\alpha_p(t)},z_{\beta_p(t)})}), \quad t\in I_a,$$
$$z^\star_{\Theta(t,(Vz)_{\gamma(t)},z_{\delta(t)})}=(z_{\Theta_1(t,(Vz)_{\gamma_1(t)},z_{\delta_1(t)})},\dots,z_{\Theta_p(t,(Vz)_{\gamma_p(t)},z_{\delta_p(t)})}), \quad t\in I_a.$$

Lemma 2. *Suppose that Assumption H_1 and assumptions of Lemma 1 are satisfied with $k_i(t)=\bar k_i(t)$, $l_i(t)=\bar l_i(t)$, $\sigma_i(t)=\bar r(t)$, $\zeta_i(t)=\tilde r(t)$, $i=1,\dots,p$, there exist functions $\bar\alpha$, $\bar\beta$, $\bar\gamma$, $\bar\delta\in C(I_a,I_a^p)$, $\bar\alpha=(\bar\alpha_1,\dots,\bar\alpha_p)$, $\bar\beta=(\bar\beta_1,\dots,\bar\beta_p)$, $\bar\gamma=(\bar\gamma_1,\dots,\bar\gamma_p)$, $\bar\delta=(\bar\delta_1,\dots,\bar\delta_p)$, such that $\bar\alpha_i(t)$, $\bar\beta_i(t)$, $\bar\gamma_i(t)$, $\bar\delta_i(t)\in[0,t]$, $t\in I_a$, $i=1,\dots,p$, and $H(t)=\sum_{i=1}^{p}\bar k_i(t)\|\varphi(0)\|+\bar\omega(t)$, and let $\bar g$ be the corresponding solution of (7) then*

$$\mathcal F:\Omega([-r,a],E,\bar g)\to\Omega([-r,a],E,\bar g),$$

Proof. Let $z \in \Omega([-r,0],E,\bar{g})$, and $w(t) = \mathcal{F}[z](t)$. Then we have

$$\|w(t)\| = \|\mathcal{F}[z](t)\|$$
$$\leq \sum_{i=1}^{p} [\bar{k}_i(t)\|(Vz)_{\Psi_i(t,(Vz)_{\alpha_i(t)},z_{\beta_i(t)})}\|_0 + \bar{l}(t)\|\Theta_i(t,(Vz)_{\gamma_i(t)},z_{\delta_i(t)})\|_0 + \bar{\omega}(t)$$
$$\leq \sum_{i=1}^{p} \bar{k}_i(t)\|\varphi(0)\| + \bar{\omega}(t) + \sum_{i=1}^{p}[\bar{k}_i(t) \int\limits_0^{\sigma_i(t)} \bar{g}(s)ds + \bar{l}_i(t)\bar{g}(\zeta_i(t))] = \bar{g}(t).$$

for $t \in I_a$. Therefore $\|w(t)\| \leq \bar{g}(t)$ for $t \in I_a$. Hence it follows that $w \in \Omega([-h,a],E,\bar{g})$. The lemma is proved. □

Assumption H_2. Suppose that:

(i) There exist $\bar{m} \in \mathbb{R}_+$, and b, d, $\bar{s}$, $\tilde{s}$, $\bar{q}$, $\tilde{q} \in \mathbb{R}_+^p$, such that for $t \in I_a$, and $i = 1,\dots,p$, we put

$$\|f(t,u,v) - f(\bar{t},u,v)\| \leq \bar{m}|t-\bar{t}|,$$
$$\|\Psi_i(t,u,v) - \Psi_i(\bar{t},u,v)\| \leq b_i|t-\bar{t}|, \quad \|\Theta_i(t,u,v) - \Theta_i(\bar{t},u,v)\| \leq d_i|t-\bar{t}|,$$
$$|\alpha_i(t) - \alpha_i(\bar{t})| \leq \bar{s}_i|t-\bar{t}|, \quad |\beta_i(t) - \beta_i(\bar{t})| \leq \tilde{s}_i|t-\bar{t}|,$$
$$|\gamma_i(t) - \gamma_i(\bar{t})| \leq \bar{q}_i|t-\bar{t}|, \quad |\delta_i(t) - \delta_i(\bar{t})| \leq \tilde{q}_i|t-\bar{t}|;$$

(ii) The following compatibility condition holds

$$\varphi'(0_-) = f(t,\varphi,\dots,\varphi,\varphi',\dots,\varphi'),$$

where $\varphi'(0_-)$ means the left hand derivative of the function φ at the point $t = 0$.

Let

$$A(t) = \sum_{i=1}^{p} \bar{l}_i(t)\rho_i(t)\tilde{q}_i,$$
$$B(t) = \bar{g}(t)\sum_{i=1}^{p} \bar{k}_i(t)\mu_i(t)\tilde{s}_i + \sum_{i=1}^{p} \bar{l}_i(t)[d_i + \bar{g}(t)\bar{q}_i\chi_i(t)], \tag{11}$$
$$C(t) = \bar{m} + \bar{g}(t)\sum_{i=1}^{p} \bar{k}_i(t)[b_i + \nu_i(t)s_i\bar{g}(t)],$$

where $t \in I_a$, A, B, $C \in C(I_a,\mathbb{R}_+)$ are nondecreasing functions.

Suppose that $B(a) < 1$ and $[B(a)-1]^2 - 4A(a)C(a) > 0$. Let λ_1, λ_2 be two different positive roots of the equation $A(a)\lambda^2 + [B(a)-1]\lambda + C(a) = 0$. Now we define the following class of functions

$$\Delta([-h,a],E,\lambda) = \{y \in \Omega([-h,a],E,\bar{g}) : \|y(t) - y(\bar{t})\| \leq \lambda|t-\bar{t}|, t,\bar{t} \in I_a\},$$

where $\lambda \in [\lambda_1, \lambda_2]$, if $A(a) \neq 0$, and $\lambda \geq C(a)[1 - B(a)]^{-1}$, if $A(a) = 0$.

Lemma 3. *Suppose that Assumption H_2, and assumptions of Lemma 2 are satisfied, $B(a) < 1$, and $[B(a) - 1]^2 - 4A(a)C(a) > 0$, where A, B, C are defined by (11).*

Then the operator $\mathcal{F}$ maps $\Delta([-h, a], E, \lambda)$ into itself.

Proof. Let $z \in D([-r, a], \bar{g}, \lambda)$. It follows from Lemma 2, that $\mathcal{F}[z] \in \Omega([-h, a], E, \bar{g})$. For $t, \bar{t} \in I_a$ we have

$$\|\mathcal{F}[z](t) - \mathcal{F}[z](\bar{t})\| \leq m|t - \bar{t}|$$
$$+ \sum_{i=1}^{p} \bar{k}_i(t)\bar{g}(t)[b_i|t - \bar{t}| + \nu_i(t)\|(Vz)_{\alpha_i(t)} - (Vz)_{\alpha_i(\bar{t})}\|_0 + \mu_i(t)\|z_{\beta_i(t)} - z_{\beta_i(\bar{t})}\|_0]$$
$$+ \sum_{i=1}^{p} \bar{l}_i(t)\lambda[d_i|t - \bar{t}| + \chi_i(t)\|(Vz)_{\gamma_i(t)} - (Vz)_{\gamma_i(\bar{t})}\|_0 + \rho_i(t)\|z_{\delta_i(t)} - z_{\delta_i(\bar{t})}\|_0]$$
$$\leq \left\{\bar{m} + \sum_{i=1}^{p} \bar{k}_i(t)\bar{g}(t)[b_i + \bar{s}_i\nu_i(t)\bar{g}(t) + \tilde{s}_i\lambda\mu_i(t)]\right.$$
$$\left. + \sum_{i=1}^{p} \bar{l}_i(t)\lambda[d_i + \bar{q}_i\chi_i(t)\bar{g}(t) + \lambda\rho_i(t)\bar{g}(t)]\right\} |t - \bar{t}|$$
$$\leq [A(t)\lambda^2 + B(t)\lambda + C(t)]|t - \bar{t}| \leq \lambda|t - \bar{t}|.$$

Therefore $\mathcal{F}[z] \in \Delta([-h, a], E, \lambda)$. The lemma is proved. □

Remark 3. If $E = \mathbb{R}^n$, and the Assumptions of Lemma 3 are satisfied, then the problem (4), (5) has at least one solution $\bar{z} \in \Delta([-h, 0], E, \lambda)$.

Indeed, we see at once that the continuous operator $\mathcal{F}$ maps the bounded, closed, and convex set $\Delta([-h, 0], E, \lambda)$ into its compact subset $\mathcal{F}[\Delta([-h, a], E, \lambda)]$. Hence, and from the Schauder Fixed Point Theorem it follows that $\mathcal{F}$ has at least one fixed point.

For an arbitrary Banach space we have the following result.

Remark 4. If assumptions of Lemma 3 are satisfied, and $Q < 1$, where

$$Q = \sum_{i=1}^{p} \bar{k}_i(a)[a + \bar{g}(a)(a + \nu_i(a) + \mu_i(a))] + \sum_{i=1}^{p} \bar{l}_i(a)[1 + \lambda(a\chi_i(a) + \rho_i(a))],$$

then the problem (4), (5) has a unique solution in $\Delta([-h, a], E, \lambda)$.

It is obvious that under these assumptions the operator $\mathcal{F}$ is a contraction in the space $\Delta([-h, a], E, \lambda)$. The assertion of this remark follows from the Banach Fixed Point Theorem.

We shall relax this restrictive condition.

3. The Main Theorem

For the function $v \in C([-h,0],E)$ we define the function $Tv : [-r,a] \to E$ by

$$(Tv)(t) = v(t), \quad t \in [-h,0],$$

$$(Tv)(t) = v(0), \quad t \in I_a.$$

We define the sequence $\{z_n\}$ in the following way:
(i) $z_0 \in \Omega([-h,a],\bar{g})$ and $z_0(t) = (T\varphi')(t)$ for $t \in [-h,a]$,
(ii) if $z_n : [-h,a] \to E$ is given, then

$$z_{n+1}(t) = \mathcal{F}[z_n](t) \quad \text{for} \quad t \in I_a,$$

$$z_{n+1}(t) = \varphi'(t) \quad \text{for} \quad t \in [-h,0].$$

To prove the convergence of the sequence $\{z_n\}$ we define the sequence $\{g_n\}$ as follows

$$g_{n+1} = Kg_n + Lg_n, \quad n = 0,1,\dots,$$

$$g_0 = \bar{g},$$

where $\bar{g}$ is a solution of equation (8) with functions $k_i,\ l_i \in C([-h,a],\mathbb{R}_+)$, $\sigma_i,\ \zeta_i \in C([-h,a],I_a)$, $\sigma_i(t),\ \zeta_i(t) \in [0,t]$, where $i = 1,\dots,p$, and $H \in C([-h,a],\mathbb{R}_+)$ given by

$$k_i(t) = \bar{k}_i(t)[1 + \bar{g}(t)\nu_i(t)] + \lambda\bar{l}_i(t)\chi_i(t),$$

$$l_i(t) = \bar{k}_i(t)\bar{g}(t)\mu_i(t) + \bar{l}_i[\lambda\rho_i(t) + 1],$$

$$H(t) = \max\left\{\|\mathcal{F}[z_0] - z_0\|, \sum_{i=1}^{p}\bar{k}_i(t)\|\phi(0)\| + \bar{\omega}(t)\right\}, \tag{12}$$

$$\sigma_i(t) = \max\left\{\bar{r}_i(t), \alpha_i(t), \gamma_i(t)\right\},$$

$$\zeta_i(t) = \max\left\{\tilde{r}_i(t), \beta_i(t), \delta_i(t)\right\},$$

where $t \in I_a$.

By induction, we can prove the following (see [16]).

Lemma 4. *Suppose that assumptions of Lemma 1 are satisfied with functions k_i, l_i, H, σ_i, ζ_i given by relations (12). Then*

$$0 \le g_{n+1} \le g_n \le \bar{g}, \quad n = 0, 1 \ldots,$$

and

$$\lim_{n\to+\infty} g_n(t) = 0 \quad \text{uniformly on} \quad I_a.$$

Theorem 1. *If Assumptions H_1, H_2, and assumptions of Lemma 1 are satisfied for functions k_i, l_i, H, σ_i, ζ_i given by relations (12) then there exists the only one solution $\bar{z} \in \Delta([-h, a], E, \lambda)$ of the problem (4), (5), the sequence $\{z_n\}$ is convergent to $\bar{z}$ uniformly on I_a, and the following estimations*

$$\|\bar{z}(t) - z_n(t)\| \le g_n(t), \qquad n = 0, 1, \ldots, \qquad t \in I_a, \tag{13}$$

hold.

Proof. First we note that from assumptions of this theorem it follows that the assumptions of Lemmas 2 and 3 are satisfied. Hence $z_n \in \Delta([-h, a], E, \lambda)$. Now we prove the estimations

$$\|z_n(t) - z_0(t)\| \le \bar{g}(t), \quad n = 0, 1, \ldots, \quad t \in I_a, \tag{14}$$

and

$$\|z_{n+k}(t) - z_n(t)\| \le g_n(t), \quad n, k = 0, 1, \ldots, \quad t \in I_a. \tag{15}$$

The estimate (14) is obvious for $n = 0$. Assume that the estimate (14) holds for a certain $n > 0$. Then for $n + 1$ we have

$$\begin{aligned}
&\|z_{n+1}(t) - z_0(t)\| \le \|\mathcal{F}[z_n](t) - \mathcal{F}[z_0](t)\| + \|\mathcal{F}[z_0](t) - z_0(t)\| \\
&\quad \le \sum_{i=1}^{p} \bar{k}_i(t)[\int_0^{\bar{r}_i(t)} \|(z_n - z_0)(s)\| ds + \bar{g}(t)\|\Psi_i(t, (Vz_n)_{\alpha_i(t)}, (z_n)_{\beta_i(t)}) \\
&\qquad\qquad - \Psi_i(t, (Vz_0)_{\alpha_i(t)}, (z_0)_{\beta_i(t)})\|] \\
&+ \sum_{i=1}^{p} \bar{l}_i(t)[\|z_n - z_0\|_{\tilde{r}_i(t)} \lambda \|\Theta_i(t, (Vz_n)_{\gamma_i(t)}, (z_n)_{\delta_i(t)}) - \Theta_i(t, (Vz_0)_{\gamma_i(t)}, (z_0)_{\delta_i(t)})\|] \\
&\quad + H(t) \le \sum_{i=1}^{p} \left\{\bar{k}_i(t)[1 + \bar{g}(t)\nu_i(t)] + \lambda \bar{l}_i(t)\chi_i(t)\right\} \int_0^{\max\{\bar{r}_i(t), \alpha_i(t), \gamma_i(t)\}} \bar{g}(s) ds \\
&\quad + \sum_{i=1}^{p} \left\{\bar{k}_i(t)\bar{g}(t)\mu_i(t) + \bar{l}_i(t)[\lambda\rho_i(t) + 1]\right\} \bar{g}(\max\{\tilde{r}_i(t), \beta_i(t), \delta_i(t)\}) + H(t)
\end{aligned}$$

$$\leq \sum_{i=1}^{p} k_i(t) \int_{0}^{\sigma_i(t)} \bar{g}(s)ds + \sum_{i=1}^{p} l_i(t)\bar{g}_i(\zeta_i(t)) + H(t) = \bar{g}(t).$$

So the estimate (14) holds for $n = 0, 1, \dots,$ $t \in I_a$. In the same manner we can prove the estimate (15). It follows from Lemma 4, that the sequence $\{z_n\}$ is convergent to the solution $\bar{z}$ of the problem (4), (5). It is obvious, that $\bar{z} \in \Delta([-h, a], E, \lambda)$. Letting $k \to +\infty$ in the estimate (15) we get the estimate (13) holds.

To prove uniqueness we assume that $\tilde{z} \in \Delta([-h, a], E, \lambda)$ is another solution of the problem (4), (5). Let

$$w(t) = \max_{s\in[0,t]} \|\tilde{z}(s) - \bar{z}(s)\|, \quad t \in I_a$$

and

$$w(t) \leq \max_{s\in I_t} \sum_{i=1}^{p} \left\{\bar{k}_i(s)[1 + \bar{g}(s)\nu_i(s)] + \lambda\bar{l}_i(s)\chi_i(s)\right\}$$

$$\leq \max_{s\in I_t} \sum_{i=1}^{p} \left\{\bar{k}_i(s)\bar{g}(s)\mu_i(s) + \bar{l}_(s)[1 + \lambda\rho_i(s)]\right\} \bar{g}(\zeta_i(s))$$

$$\leq (Kw)(t) + (Lw)(t), \quad t \in I_a.$$

Therefore

$$w(t) \leq (Kw)(t) + (Lw)(t) \quad \text{for} \quad t \in I_a.$$

Thus w is a solution of the inequality (9), so $w(t) = 0$. Therefore $\tilde{z}(t) = \bar{z}(t)$. The proof is finished. □

4. Some Examples

Let

$$l^\star = (l_1^\star, \dots, l_p^\star), \quad k^\star = (k_1^\star, \dots, k_p^\star) \in \mathbb{R}_+^p,$$

$$\sigma^\star = (\sigma_1^\star, \dots, \sigma_p^\star), \quad \zeta^\star = (\zeta_1^\star, \dots, \zeta_p^\star) \in I_1^p,$$

and Assumptions H_1 and H_2 are satisfied with functions k_i, l_i, σ_i, ζ_i, and H given by relation (12).

Example 1. Suppose that

$$l_i(t) \le l_i^\star, \quad \zeta_i(t) \le \zeta_i^\star,$$

and

$$k_i(t) \le k_i^\star t, \quad \sigma_i(t) \le \sigma_i^\star, \qquad \text{or} \qquad k_i(t) \le k_i^\star, \quad \sigma_i(t) \le \sigma_i^\star t, t \in I_a.$$

Then the conditions (6) in Lemma 1 can be replaced by

$$\sum_0^{+\infty} \sum_{i_1=1}^{p} \cdots \sum_{i_n=1}^{p} \prod_{j=1}^{n} l_{i_j}^\star H(\prod_{j=1}^{n} \zeta_{i_j}^\star) < +\infty,$$

and

$$\sum_{j=1}^{p} k_j^\star \sigma_j^\star < +\infty, \qquad \sum_0^{+\infty} \sum_{i_1=1}^{p} \cdots \sum_{i_n=1}^{p} \prod_{j=1}^{n} l_{i_j}^\star \zeta_{i_j}^\star < +\infty.$$

Suppose also that for some S, $G \in \mathbb{R}_+$ we have $H(t) \le St^G$ for $t \in I_a$. Then the assertion of Theorem 3 holds, if $B(a) < 1$, and $[B(a)-1]^2 - 4A(a)C(a) > 0$.

Example 2. Suppose that

$$l_i(t) \le l_i^\star, \quad \zeta_i(t) \le \zeta_i^\star, \quad k_i(t) \le k_i^\star t, \quad \sigma_i(t) \le \sigma_i^\star t, \quad t \in I_a.$$

Then conditions (6) take the form

$$\sum_0^{+\infty} \sum_{i_1=1}^{p} \cdots \sum_{i_n=1}^{p} \prod_{j=1}^{n} l_{i_j}^\star H(\prod_{j=1}^{n} \zeta_{i_j}^\star) < +\infty,$$

and

$$\sum_{j=1}^{p} k_j^\star \sigma_j^\star < +\infty, \qquad \sum_0^{+\infty} \sum_{i_1=1}^{p} \cdots \sum_{i_n=1}^{p} \prod_{j=1}^{n} l_{i_j}^\star (\zeta_{i_j}^\star)^2 < +\infty.$$

We can also come to a similar conclusion (concerning the assertion of Theorem 3) as in the Example 1.

Remark 5. Note that, if

$$f(t, u_1, \ldots, u_p, v_1, \ldots, v_p) = \tilde{f}(t, u_1(0), \ldots, u_p(0), v_1(0), \ldots, v_p(0)),$$

$$\Psi_i(t, y, w) = \tilde{\Psi}_i(t, y(0), w(0)), \quad \Theta_i(t, y, w) = \tilde{\Theta}_i(t, y(0), w(0)),$$

where $\tilde{f} : I_a \times C^{2p}([-h,0],E) \to E$, $\tilde{\Psi}_i$, $\tilde{\Theta}_i : I_a \times C([-h,0],E) \times C([-h,0],E) \to I_a^p$, $\tilde{\Psi}_i(t)$, $\tilde{\Theta}_i(t) \in [0,t]$ and u_i, v_i, y_i, $w_i \in C([-h,0],\mathbb{R})$, $i = 1,\dots,p$, then the equation (1) reduces to the following

$$x'(t) = \tilde{f}(t, x^\star(\tilde{\Psi}(t, x(\alpha(t)), x'(\beta(t)))), (x')^\star(\tilde{\Theta}(t, x(\gamma(t)), x'(\delta(t))))), \quad t \in I_a,$$

and such case of equation (1) for $p = 1$ is considered in [18].

For

$$\Psi_i(t,y,w) = \tilde{\alpha}_i(t), \quad \Theta_i(t,y,w) = \tilde{\beta}_i(t), \quad i = 1,\dots,p,$$

$$f(t,u_1,\dots,u_p,v_1,\dots,v_p) = \tilde{f}(t,u_1(0),\dots,u_p(0),v_1(0),\dots,v_p(0)),$$

where y, $w \in C([-h,0],E) \to E$, $\tilde{f} : I_a \times C^{2p}([-h,0],E) \to E$, $\tilde{\alpha}_i$, $\tilde{\beta}_i : I_a \to I_a$, and $\tilde{\alpha}_i(t)$, $\tilde{\beta}_i(t) \in [0,t]$, $i = 1,\dots,p$, then equation (1) reduces to

$$x'(t) = \tilde{f}(t, x^\star(\tilde{\alpha}(t)), (x')^\star(\tilde{\beta})),$$

and such equation we can find in [15].

If we suppose that $p = 1$ and

$$\Psi(t,y,w) = \tilde{\alpha}(t), \quad \Theta(t,y,w) = \tilde{\beta}(t),$$

where y, $w \in C([-h,0],E) \to E$, $\tilde{\alpha}$, $\tilde{\beta} : I_a \to I_a$, and $\tilde{\alpha}(t)$, $\tilde{\beta}(t) \in [0,t]$, then equation (1) reduces to the following one

$$x'(t) = f(t, x_{\tilde{\alpha}(t)}, x'_{\tilde{\beta}(t)}),$$

and such case of equation (1) with an unbounded initial function is analised in [16].

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