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WEIGHTED INCREASING TREES AND EXPONENTIAL GENERATING FUNCTIONS

Wen-Jin Woan
Department of Mathematics
Howard University
2103 Opal Ridge, Vista, CA 92081, USA
e-mail: wjwoan@sbcglobal.net

Abstract: We study the counting of various trees with labeled nodes and weighted edges and its exponential generating functions.

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1. Introduction

In this paper we study the counting of various trees with labeled nodes and weighted edges and its exponential generating functions. In Section 4 we construct labeled weighted trees structure to count certain sequences.

Definition 1. An *increasing tree* is a tree with nodes labeled with $[n] = \{0, 1, 2, 3, 4, .., n\}$ which satisfies the following two conditions:

(1) Increasing on any branch going up.

(2) Increasing from left to right of the immediate successors of a node.

The following properties of increasing trees are listed in Proposition 1.3.16 of [4].

1. The number of increasing trees on n edges is $n!$.

2. The number of such trees for which the root has k successors is the signless Stirling number $S(n, k)$ of the first kind.

3. The number of such trees with k endpoints is the Eulerian number $E(n, k)$.

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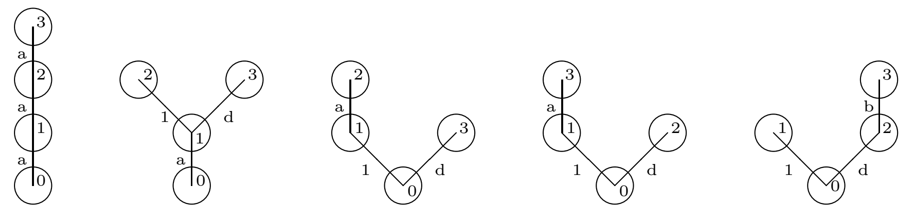

Figure 1:

Definition 2. A tree is said to be $1-2$ tree, if the outdegree of each node is at most two. Here we consider only $1-2$ trees. Let t_n be the counting of increasing $1-2$ trees, then $\{t_n\}$ is the sequence of zig-zag permutations: $1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521\ldots$ $(see[3], A000111)$.

Example 3. For $n=3$, the third tree has two choices (2 or 3) for position 2 and the count is $t_3 = 1+1+2+1 = 5$ (see Figure 1).

For $n=4$, $t_4 = 1+1+2+1+3+3+1+3+1 = 16$ (see Figure 2).

2. Main Results

Remark 4. There is a bijection between $1-2$ trees and Motzkin paths. Here we use the idea in weighted Motzkin paths by assigning weights to the edges. For *weighted increasing 1-2 trees* (WIT(b, c)), the node with outdegree 1 we assign weight b for that single edge and for a node with outdegree 2 we assign the left edge weight 1 and the right edge weight c. The weight of a weighted increasing tree is the product of the weights of the edges.

Let w_n be the sum of the weights of all weighted increasing trees in $WIT(b,c)$ of order n and $y = f(x) = \sum w_n x^{n+1}$. If we partition the tree by the first node (the one above the root) of the leftmost branch, then we have the recurrence relation

$$w_n = bw_{n-1} + c\sum \binom{n-1}{k} w_k w_{n-2-k}.$$

For $n = 3,\ 4$, refer to Example 3 and replace a by b and d by c,
$w_0 = 1$, $w_1 = b$, $w_2 = b^2 + c$,
$w_3 = 1(b^3) + 1(bc) + 2(bc) + 1(bc) = b^3 + 4bc$,
$w_4 = 1(b^4) + 1(b^2c) + 2(b^2c) + 1(b^2c)\ + 3(b^2c) + 3(b^2c) + 1(b^2c) + 3(c^2) + 1(c^2)$,

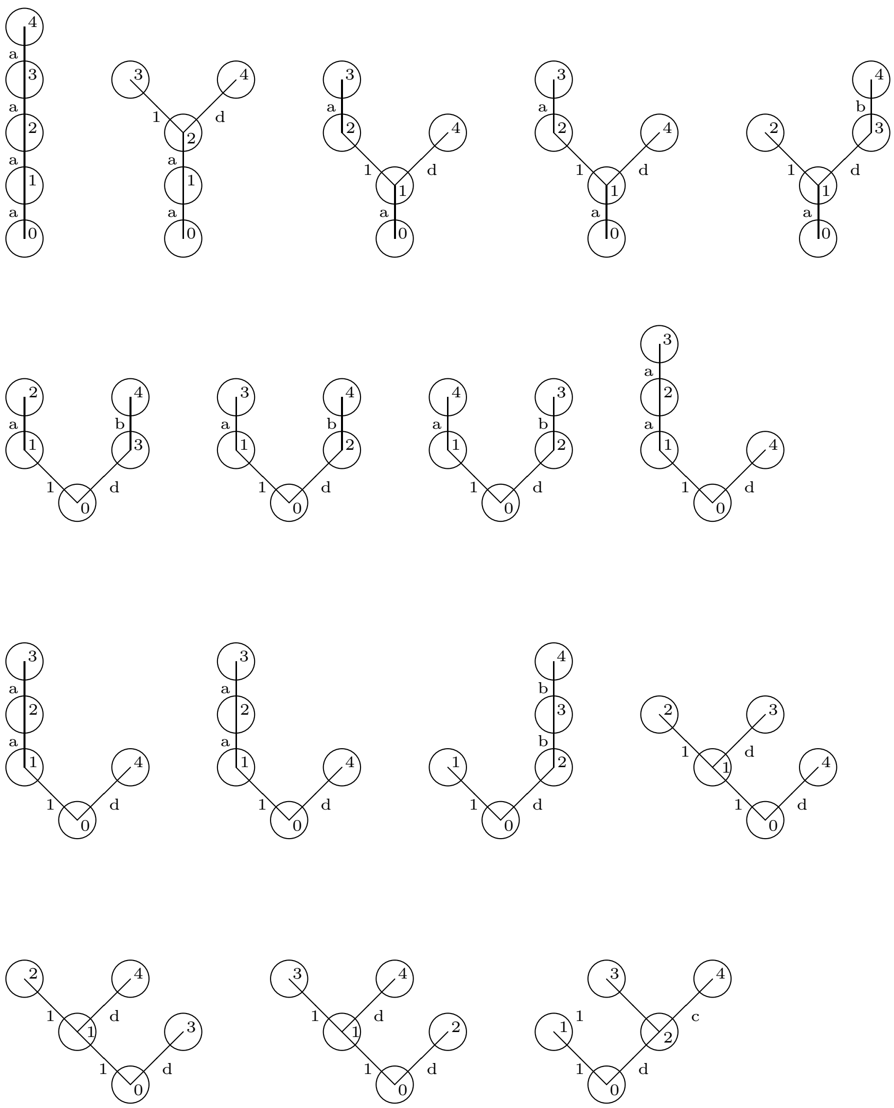

Figure 2:

$$= b^4 + 11(b^2c) + 4c^2,$$

$$w_5 = b^5 + 4b^3c + 9b^2bc + 9bb^2c + 9bcc + 3bc^2 + 4b^3c + 16bc^2 + 6bc^2,$$

$$= b^5 + 26b^3c + 34bc^2 \ ,$$

$y^{'}=\frac{dy}{dx}=1+by+\frac{c}{2!}y^2=\frac{c}{2}(y^2+\frac{2b}{c}y+\frac{2}{c})=\frac{c}{2}((y+\frac{b}{c})^2+\frac{2c-b^2}{c^2})$.

Solving the separable differential equation we have

Case 1. $2c-b^2=0$. $y=\frac{-b}{c}-\frac{2b}{c(bx-2)}=\frac{-bbx+2b-2b}{c(bx-2)}=\frac{b^2x}{c(2-bx)}=\frac{b^2x}{2c(1-\frac{b}{2}x)}=\frac{x}{1-\frac{b}{2}x}=x\sum(\frac{bx}{2})^n$.

Case 2. $2c-b^2>0$. $y=\frac{-b}{c}+\frac{\sqrt{2c-b^2}}{c}\tan(\frac{\sqrt{2c-b^2}x}{2}+\arctan\frac{b}{\sqrt{2c-b^2}})$.

Case 3. $2c-b^2<0$. $y=\frac{(\exp(x\sqrt{b^2-2c})-1)}{r_2-r_1\exp(x\sqrt{b^2-2c})}$ $r_2=\frac{b+\sqrt{b^2-2c}}{2}$, $r_1=\frac{b-\sqrt{b^2-2c}}{2}$.

Remark 5. For a *generalized weighted increasing trees* ($GWIT(a,b,c,\ d)$) we assign weights a,d instead of b,c for those edges with initial nodes on the left most branch. Let h_n be the sum of the weights of all trees in $GWIT(a,b,c,d)$ of order n. If we partition the trees by the first node of the leftmost branch. Then we have the recurrence relation

$$h_n=ah_{n-1}+d\sum\binom{n-1}{k}h_kw_{n-2-k}.$$

For $n=3,\ 4$, refer to Example 3:

$h_0=1,\ h_1=a,\ h_2=a^2+d,\ h_3=a^3+3ad+bd,$

$h_4=a^4+6a^2d+4abd+b^2d+cd+3d^2,$

$h_5=a^5+10a^3d+10a^2bd+5ab^2d+15ad^2+5adc+10bd^2+4bcd+b^3d.$

Let $h=h(x)=\sum\frac{h_n}{n!}x^n$ be the exponential generating function, by the recurrence relation above $h^{'}=\frac{dh}{dx}=ah+dhy$, where y is defined as in Remark 4.

Solve the separable differential equation we have

$$h=h(x)=\exp(\int(a+dy)dx).$$

3. Some Examples

Example 6. $b=1,\ c=2$, the sequence is $w_0=1,\ 1,\ 3,\ 9,\ 39,\ \ldots$ (see [3], A080635) and it counts the number of permutations on n letters without double falls and without initial falls.

$y^{'}=1+y+y^2,$

$\frac{2}{\sqrt{3}}\arctan\frac{2(y+\frac{1}{2})}{\sqrt{3}}=x+C=x+\frac{\pi}{3\sqrt{3}},$

$y=\frac{\sqrt{3}}{2}\ \tan(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})-\frac{1}{2}\ =x+\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{3}{8}x^4+\frac{13}{40}x^5+\frac{21}{80}x^6+\frac{123}{560}x^7+O\left(x^8\right).$

Let $a=d=1$,

$\int(1+\frac{\sqrt{3}}{2}\ \tan(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})-\frac{1}{2}\ )=\frac{x}{2}+\ln\sec\frac{\sqrt{3}}{2}(x+\frac{\pi}{3\sqrt{3}})-\ln\frac{2}{\sqrt{3}},$

$h(x) = \exp(\frac{x}{2} + \ln\sec\frac{\sqrt{3}}{2}(x+\frac{\pi}{3\sqrt{3}}) - \ln\frac{2}{\sqrt{3}}) = \frac{\sqrt{3}}{2}\exp(\frac{x}{2})(\sec(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}))$
$= 1 + x + x^2 + \frac{5}{6}x^3 + \frac{17}{24}x^4 + \frac{7}{12}x^5 + O\left(x^6\right).$

The sequence is $h_0 = 1, 1, 2, 5, 17, 70, \ldots$ (see [3], A049774) and it counts the number of permutations on n letters without double falls.

Example 7. $b = 3$, $c = 4$, the sequence is $w_0 = 1, 3, 13, 75, 541, \ldots$ (see [3], A000670) and it counts the number of ways n competitors can rank in a competition, allowing for the possibility of ties.

$y' = 1 + 3y + \frac{4}{2!}y^2,$

$y = \frac{\exp(x)-1}{2-\exp(x)} = x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + \frac{25}{8}x^4 + \frac{541}{120}x^5 + \frac{1561}{240}x^6 + \frac{47293}{5040}x^7 + O\left(x^8\right).$

Let $a = d = 1$,

$\ln\ g = \int(1 + \frac{\exp(x)-1}{2-\exp(x)})dx = \int\frac{1}{2-e^x}dx = -\frac{1}{2}\ln\left(-2+e^x\right) + \frac{1}{2}\ln\left(e^x\right),$

$g = h(x) = \exp(-\frac{1}{2}\ln\left(-2+e^x\right) + \frac{1}{2}\ln\left(e^x\right)) = \exp(\frac{x}{2})(2-e^x)^{-\frac{1}{2}} = 1 + x + x^2 + \frac{7}{6}x^3 + \frac{35}{24}x^4 + \frac{113}{60}x^5 + \frac{1787}{720}x^6 + \frac{16717}{5040}x^7 + O\left(x^8\right).$ The sequence is $h_0 = 1, 1, 2, 7, 35, 226, 1787, \ldots$ (see [3], A014307).

Example 8. $b = c = 2$, the sequence is $w_0 = 1, 2, 6, 24, 120, \ldots$ (see [3], A000142).

$y' = 1 + 2y + \frac{2}{2!}y^2,$

$y = \frac{4x}{2(2-bx)} = \frac{x}{1-x} = \sum x^n = \sum \frac{1}{n!}n!x^n.$

Let $a = 1$, $d = 3$,

$\int(1 + \frac{3x}{1-x})dx = -2x - 3\ln\left(x-1\right),$

$h(x) = \exp(-2x - 3\ln\left(1-x\right)) = \frac{e^{-2x}}{(1-x)^3} = 1 + x + 2x^2 + \frac{8}{3}x^3 + \frac{11}{3}x^4 + \frac{71}{15}x^5 + \frac{268}{45}x^6 + O\left(x^7\right).$

The sequence is $h_0 = 1, 1, 4, 16, 88, 568, 4288, 36832, 354688, \ldots$ (see [3], A052124).

4. Hankel Matrices Stieltjes Matrices and Weighted Increasing Trees

Given a sequence we want to find out wether there is a representation by weighted increasing trees. The Hankel matrix generated by the sequence $a_0, a_1, a_2, \ldots,$ is given by the infinite matrix

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & . \\ a_1 & a_2 & a_3 & a_4 & a_5 & . \\ a_2 & a_3 & a_4 & a_5 & a_6 & . \\ a_3 & a_4 & a_5 & a_6 & a_7 & . \\ a_4 & a_5 & a_6 & a_7 & a_8 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Without loss of generality we take $a_0 = 1$. Our basic assumption is that the Hankel matrix generated by the sequence has an LDU factorization, where L is a lower triangular matrix with all diagonal elements equal to 1, $U = L^T$, and D is a diagonal matrix with all diagonal elements nonzero. A necessary and sufficient condition for H to have an LDU factorization is that H be positive definite.

Let H be the Hankel matrix generated by the sequence $1, a_1, a_2, \ldots$ and let $H = LDL^T$. Then the Stieltjes matrix $S_L = L^{-1}\overline{L}$ ($\overline{L}$ is obtained from L by removing the first row) has the form

$$S_L = \begin{bmatrix} a & 1 & 0 & 0 & 0 & . \\ d & \lambda_1 & 1 & 0 & 0 & . \\ 0 & \mu_1 & \lambda_2 & 1 & 0 & . \\ 0 & 0 & \mu_2 & \lambda_3 & 1 & . \\ 0 & 0 & 0 & \mu_3 & \lambda_4 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Example 9. The Delannoy sequence: $1, 3, 13, 63, 321, 1683, \ldots$ (see [3], A001850).

$$H = \begin{bmatrix} 1 & 3 & 13 & 63 & 321 & . \\ 3 & 13 & 63 & 321 & 1683 & . \\ 13 & 63 & 321 & 1683 & 8989 & . \\ 63 & 321 & 1683 & 8989 & 48639 & . \\ 321 & 1683 & 8989 & 48639 & 265729 & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 3 & 1 & 0 & 0 & 0 & . \\ 13 & 6 & 1 & 0 & 0 & . \\ 63 & 33 & 9 & 1 & 0 & . \\ 321 & 180 & 62 & 12 & 1 & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & 4 & 0 & 0 & 0 & . \\ 0 & 0 & 8 & 0 & 0 & . \\ 0 & 0 & 0 & 16 & 0 & . \\ 0 & 0 & 0 & 0 & 32 & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 3 & 13 & 63 & 321 & . \\ 0 & 1 & 6 & 33 & 180 & . \\ 0 & 0 & 1 & 9 & 62 & . \\ 0 & 0 & 0 & 1 & 12 & . \\ 0 & 0 & 0 & 0 & 1 & . \\ . & . & . & . & . & . \end{bmatrix} = LDU.$$

The Stieltjes matrix S_L associated with L is

$$S_L = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & . \\ 4 & 3 & 1 & 0 & 0 & . \\ 0 & 2 & 3 & 1 & 0 & . \\ 0 & 0 & 2 & 3 & 1 & . \\ 0 & 0 & 0 & 2 & 3 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

For Riordan matrices refer to [1].

Definition 10. A *Riordan matrix* is a lower triangular matrix for which the generating function for the k^{th} column, $k \geq 0$, is given by $g(x)[f(x)]^k$, where

$$g(x) = 1 + g_1 x + g_2 x^2 + \cdots, \qquad f(x) = x + f_2 x^2 + f_3 x^3 + \cdots .$$

A *Riordan matrix with exponential generating functions* is a lower triangular matrix for which the generating function for the k-th column, $k \geq 0$, is given by $\frac{1}{k!} g(x)[f(x)]^k$, where

$$g(x) = 1 + g_1 x + g_2 \frac{x^2}{2!} + g_3 \frac{x^3}{3!} + \cdots, \qquad f(x) = x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \cdots .$$

For the following two theorems please refer to [2].

Theorem 11. *Let H be the Hankel matrix generated by the sequence $1, a_1, a_2, \ldots$, and let $H = LDL^T$. Then L is a Riordan matrix if and only if the Stieltjes matrix S_L has the form*

$$S_L = \begin{bmatrix} a & 1 & 0 & 0 & 0 & . \\ d & \lambda & 1 & 0 & 0 & . \\ 0 & \mu & \lambda & 1 & 0 & . \\ 0 & 0 & \mu & \lambda & 1 & . \\ 0 & 0 & 0 & \mu & \lambda & . \\ . & . & . & . & . & . \end{bmatrix}$$

and the ordinary generating function $g(x) = \sum a_n x^n$ of the sequence $1, a_1, a_2, \ldots$ is given by

$$g(x) = \frac{1}{1 - ax - dxf},$$

where

$$f = x(1 + \lambda f + \mu f^2), \quad f(0) = 0.$$

Theorem 12. *Let H be the Hankel matrix generated by the sequence $1, a_1, a_2, \ldots$ and let $H = LDL^T$. Then L is a Riordan matrix with exponential generating functions if and only if the Stieltjes matrix S_L has the form*

$$S_L = \begin{bmatrix} a & 1 & 0 & 0 & 0 & . \\ d & \lambda_1 & 1 & 0 & 0 & . \\ 0 & \mu_1 & \lambda_2 & 1 & 0 & . \\ 0 & 0 & \mu_2 & \lambda_3 & 1 & . \\ 0 & 0 & 0 & \mu_3 & \lambda_4 & . \\ . & . & . & . & . & . \end{bmatrix},$$

where $\{\lambda_i\}_{i\geq 0}$ is an arithmetic sequence with common difference λ and $\{\frac{\mu_i}{i+1}\}_{i\geq 0}$ an arithmetic sequence with common difference μ, and the exponential generating function $g(x) = \sum a_n \frac{x^n}{n!}$ for the sequence $1, a_1, a_2, \ldots$ is given by

$$\ln(g) = \int (a + df)dx\ , \qquad g(0) = 1,$$

where f is given by $f' = 1 + \lambda f + \mu f^2\ , \quad f(0) = 0\,.$

Remark 13. By comparing the above Theorem with Remark 4 we find our GWIT$(a, \lambda, 2\mu, d)$. For the rest of the section we work on truncated matrices only.

Remark 14. The algorithm of constructing GWIT$(a, \lambda, 2\mu, d)$ for a sequence is as follows:

1. Form the Hankel matrix for the sequence.
2. Gaussian elimination to produce DU.
3. Divide each row by the diagonal entry to produce $U = L^T$.
4. Find the inverse L^{-1}.
5. Find $S_L = L^{-1}\overline{L}$.

Remark 15. Let us start with the sequence 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, ... (see [3], A000111). It counts the number of zig-zag permutations on n letters.

$$H = \begin{bmatrix} 1 & 1 & 2 & 5 & 16 & 61 \\ 1 & 2 & 5 & 16 & 61 & 272 \\ 2 & 5 & 16 & 61 & 272 & 1385 \\ 5 & 16 & 61 & 272 & 1385 & 7936 \\ 16 & 61 & 272 & 1385 & 7936 & 50521 \\ 61 & 272 & 1385 & 7936 & 50521 & 353792 \end{bmatrix}, \text{Gaussian elimination :}$$

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 16 & 61 \\ 0 & 1 & 3 & 11 & 45 & 211 \\ 0 & 0 & 3 & 18 & 105 & 630 \\ 0 & 0 & 0 & 18 & 180 & 1530 \\ 0 & 0 & 0 & 0 & 180 & 2700 \\ 0 & 0 & 0 & 0 & 0 & 2700 \end{bmatrix} = DU$$

$$U = L^T = \begin{bmatrix} 1 & 1 & 2 & 5 & 16 & 61 \\ 0 & 1 & 3 & 11 & 45 & 211 \\ 0 & 0 & 1 & 6 & 35 & 210 \\ 0 & 0 & 0 & 1 & 10 & 85 \\ 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 5 & 11 & 6 & 1 & 0 & 0 \\ 16 & 45 & 35 & 10 & 1 & 0 \\ 61 & 211 & 210 & 85 & 15 & 1 \end{bmatrix} = L$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & 1 & 0 & 0 \\ -6 & -10 & 25 & -10 & 1 & 0 \\ 30 & -26 & -75 & 65 & -15 & 1 \end{bmatrix} = L^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 7 & -6 & 1 & 0 \\ -6 & -10 & 25 & -10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 5 & 11 & 6 & 1 & 0 & 0 \\ 16 & 45 & 35 & 10 & 1 & 0 \\ 61 & 211 & 210 & 85 & 15 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 6 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 5 & 1 \end{bmatrix} = S_L.$$

$\lambda = 1 = b,\ \mu = \frac{1}{2} = \frac{c}{2},\ a = 1,\ d = 1,$
$f(x) = \sec x + \tan x - 1 = \ x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{2}{15}x^5 + \frac{61}{720}x^6 + O\left(x^7\right),$
$g(x) \ = \ (\sec x + \tan x)(\sec x) \ \ = 1 + x + x^2 + \frac{5}{6}x^3 + \frac{2}{3}x^4 + \frac{61}{120}x^5 + \frac{17}{45}x^6$
$+O\left(x^7\right).$

Example 16. (The Derangements) The number of permutations of n

elements with no fixed points 1,0,1,2,9,44,265,1854,14833,. . . . $H = LDL^T$ and

$$S_L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & . \\ 1 & 2 & 1 & 0 & 0 & . \\ 0 & 4 & 4 & 1 & 0 & . \\ 0 & 0 & 9 & 6 & 1 & . \\ 0 & 0 & 0 & 16 & 8 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

So $\lambda = 2,\ \mu = 1,\ a = 0$ and $d = 1$. So $f' = 1 + 2f + f^2$ with $f(0) = 0$. That gives $f = \frac{x}{1-x}$ and $\ln(g) = \int f dx$, $g(0) = 1$.

$$\begin{aligned} g(x) &= \frac{e^{-x}}{1-x} \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{11}{30}x^5 + \frac{53}{144}x^6 + \frac{103}{280}x^7 + \frac{2119}{5760}x^8 + O\left(x^9\right). \end{aligned}$$

Example 17. Here we start with a Stieltjes matrix having the form in Theorem 12 with $\lambda = 3,\ \mu = 2,\ a = 3$ and $d = 1$.

$$S_L = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & . \\ 1 & 6 & 1 & 0 & 0 & . \\ 0 & 6 & 9 & 1 & 0 & . \\ 0 & 0 & 15 & 12 & 1 & . \\ 0 & 0 & 0 & 28 & 15 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

The exponential generating function for the sequence in the leftmost column of L is given by

$\ln(g) = \int(3 + f)dx$, $\quad g(0) = 1$ where $f' = 1 + 3f + 2f^2$, $\quad f(0) = 0$. We have

$$f = y = \frac{e^x - 1}{2 - e^x} \quad = x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + \frac{25}{8}x^4 + \frac{541}{120}x^5 + \frac{1561}{240}x^6 + O\left(x^7\right),$$

$$\int (3 + \frac{e^x - 1}{2 - e^x})dx = 3x - \frac{1}{2}\ln\left(\left(-2 + e^x\right)e^x\right),$$

$$\begin{aligned} \ln g &= \exp(3x - \frac{1}{2}\ln\left(\left(-2 + e^x\right)e^x\right)) = \exp\left(3x - \frac{1}{2}\ln\left(\left(-2 + e^x\right)e^x\right)\right) \\ &= e^{3x}\sqrt{\frac{1}{e^x(2 - e^x)}} = 1 + 3x + 5x^2 + \frac{13}{2}x^3 + \frac{187}{24}x^4 + \frac{47}{5}x^5 + \frac{1691}{144}x^6 \end{aligned}$$

$$+\frac{25453}{1680}x^7+O\left(x^8\right).$$

The associated sequence is $1, 3, 10, 39, 187, 1128, 8455\ldots$ ([3], $A(054912)$).

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