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D'ALEMBERT'S SUPPLEMENTED PRINCIPLE AND NEWTON'S FIVE SUPPLEMENTED LAWS

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Abstract: In this paper the relationship between Newton's third law and D'Alembert's supplemented principle is shown, where the latter is upgraded to the rank of a new law of mechanics, referred to as d'Alembert's law, and which under its concept comprises six equations absolutely independent from one another. D'Alembert's law, in being generalised to cover any physical medium corrects the fundamentals of the mechanics of such media. Newton's five laws are also supplemented.

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1. D'Alembert's Supplemented Principle

There appears the question: How does the original law of physics look like, whose generalisation Euler's equations are?

$$\begin{cases} m\vec{a} = \vec{P} \;\Rightarrow\; m\ddot{x}_C = P_x\,, \quad m\ddot{y}_C = P_y\,, \qquad m\ddot{z}_C = P_z\,, \\ \qquad\qquad\qquad I_\xi\dot{\omega}_\xi - (I_\eta - I_\zeta)\,\omega_\eta\omega_\zeta = M_\xi\,, \\ \quad ? \quad\;\; \Rightarrow \quad I_\eta\dot{\omega}_\eta - (I_\zeta - I_\xi)\,\omega_\zeta\omega_\xi = M_\eta\,, \\ \qquad\qquad\qquad I_\zeta\dot{\omega}_\zeta - (I_\xi - I_\eta)\,\omega_\xi\omega_\eta = M_\zeta \end{cases}$$

According to the derivations found in the literature Newton's second law is also the aforesaid law:

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$$\begin{cases} m\vec{a} = \vec{P} & \Rightarrow & \begin{aligned} m\ddot{x}_C &= P_x\,, \\ m\ddot{y}_C &= P_y\,, \\ m\ddot{z}_C &= P_z\,, \end{aligned} \\ \left(m\vec{a} = \vec{P}\right)' & \Rightarrow & \begin{aligned} I_\xi\dot{\omega}_\xi - \left(I_\eta - I_\zeta\right)\omega_\eta\omega_\zeta &= M_\xi\,, \\ I_\eta\dot{\omega}_\eta - \left(I_\zeta - I_\xi\right)\omega_\zeta\omega_\xi &= M_\eta\,, \\ I_\zeta\dot{\omega}_\zeta - \left(I_\xi - I_\eta\right)\omega_\xi\omega_\eta &= M_\zeta\,, \end{aligned} \end{cases}$$

The explanation of that issue is related with the so-called complete definition of force. So a complete description of the physical phenomenon as the interaction between two objects which are in contact at a certain point, requires two surface forces and the relationship between the latter to be determined. This relationship is expressed by Newton's third supplemented law and D'Alembert's supplemented principle, in the cases of static equilibrium and dynamic equilibrium, respectively.

From the definition: For every action, there is an equal and opposite reaction, which is in equilibrium with the former, the law of static equilibrium, i.e. Newton's third supplemented law, arises:

$$U\left[\vec{R}_{BA}, \vec{R}_{AB}; l_{BA}, l_{AB}\right] \equiv U\left[\vec{R}_{BA}, \vec{R}_{AB}; \vec{r}_C\right] = 0\,. \tag{1a}$$

Necessary Condition: $\sum\left(\vec{R}_{BA}, \vec{R}_{AB}\right) = \vec{R}_{BA} + \vec{R}_{AB} = \vec{0}$.

Sufficient Condition:

$$\begin{cases} a) & l_{BA} = l_{AB}\,, \\ b) & \sum \vec{M}\left(\vec{R}_{BA}, \vec{R}_{AB}\right) = \vec{r}_C \times \vec{R}_{BA} + \vec{r}_C \times \vec{R}_{AB} \\ & = \vec{r}_C \times \left(\vec{R}_{BA} + \vec{R}_{AB}\right) = \vec{r}_C \times \vec{0} \equiv \vec{0}\,, \end{cases}$$

as well as the law of dynamic equilibrium does, i.e. D'Alembert's supplemented principle referred to as D'Alembert's law:

$$U\left[\vec{P}, \vec{A}; \vec{r}\,(t)\right] = 0, \quad \vec{A} = -m\vec{a}, \quad \text{for} \quad t = t_0, \quad \vec{r} = \vec{r}_0, \quad \vec{v} = \vec{v}_0. \tag{1b}$$

Necessary Condition:

$$\begin{cases} \sum\left(\vec{P},\vec{A}\right)=\vec{P}+\vec{A}=\vec{0}: \\ a) \qquad m\vec{a}=\frac{d}{dt}\left(m\vec{v}\right)=\vec{P}, \quad \text{for} \quad t=t_0, \quad \vec{r}=\vec{r}_0, \quad \vec{v}=\vec{v}_0\,, \\ b) \qquad m\vec{a}=\vec{P}, \quad \vec{r}=\vec{r}\left(t\right): \\ \text{Cartesian product,} \\ \text{Newtons multi-dimensiona law for a rigid object} \\ 1\cdot\vec{a}=\vec{Q}+\vec{R}_h/\cdot\underset{\alpha}{\vec{r}}, \\ \vec{r}=\vec{r}\left(u^{\alpha}\right), \quad \underset{\alpha}{\vec{r}}=\frac{\partial\vec{r}}{\partial u^{\alpha}}, \quad \alpha=1,2,\ldots,6\,, \\ \text{for} \qquad t=t_0, \qquad \vec{r}=\vec{r}_0, \qquad \vec{v}=\vec{v}_0\,, \\ \text{Newtons multi-dimensiona law for a non-holonomic system}\,, \\ 1\cdot\vec{a}=\vec{Q}+\vec{R}_h+\vec{R}_{nh}/\cdot\underset{\nu}{\vec{R}}, \\ \vec{r}=\vec{r}\left(u^{\alpha},t\right), \quad \underset{\alpha}{\vec{r}}=\frac{\partial\vec{r}}{\partial u^{\alpha}}, \quad \alpha=1,2,\ldots,N\,, \\ \vec{v}=\vec{v}\left(\dot{e}^{\nu},t\right), \quad \underset{\nu}{\vec{R}}=\frac{\partial\vec{a}}{\partial\ddot{e}_{\nu}}, \quad \nu=1,2,\ldots,N'\,, \\ \text{for} \qquad t=t_0, \qquad \vec{r}=\vec{r}_0, \qquad \vec{v}=\vec{v}_0\,, \\ \vec{a}\cdot\underset{\nu}{\vec{R}}=a_{\nu}=\frac{\partial S}{\partial\ddot{e}_{\nu}}=\vec{Q}\cdot\underset{\nu}{\vec{R}}+\vec{R}_n\cdot\underset{\nu}{\vec{R}}+\vec{R}_{nh}\cdot\underset{\nu}{\vec{R}}=\Phi_{\nu}, \quad S=\sum_{i=1}^{n}\frac{m_i a_i^2}{2}\,. \end{cases}$$

Sufficient Condition:

$$\begin{cases} a) \quad \sum\vec{M}_O\left(\vec{P},\vec{A}\right)=\vec{r}\times\vec{P}+\vec{r}\times\vec{A}=\vec{r}\times\left(\vec{P}+\vec{A}\right)=\vec{r}\times\vec{0}\overset{!}{\equiv}\vec{0}\,, \\ b) \quad -\vec{r}\times\vec{A}=\vec{r}\times m\vec{a}=\frac{d}{dt}\left(\vec{r}\times m\vec{v}\right)=\frac{d\vec{K}}{dt}\equiv\vec{r}\times\vec{P}=\vec{M}, \;\; \vec{K}=\vec{r}\times m\vec{v}\,. \end{cases}$$

D'Alembert's law is generalised to cover mass forces:

$$U\left[\vec{F}_{AB},\vec{F}_{BA};\vec{r}_A,\vec{r}_B\right]=U\left[\vec{F}_{AB},\vec{F}_{BA};\vec{r}_A,\vec{r}_B\left(t\right)\right] \equiv U\left[\vec{A},\vec{F}_{BA};\vec{r}_B\left(t\right)\right]=0, \quad (2)$$

$$\vec{F}_{AB}\equiv\vec{A}=-m_B\vec{a}_B, \qquad \mid\vec{F}_{BA}\mid=\mid\vec{F}_{AB}\mid=\frac{\kappa m_A m_B}{\mid\vec{r}_{AB}\mid^2}\;\;;$$

any non-inertial systems, where a single force acts on a material point:

$$U\left[\vec{P},\vec{A}_b;\vec{r}\left(t\right)\right]=U\left[\vec{P},\vec{A}_w+\vec{A}_u+\vec{A}_c;\vec{r}\left(t\right)\right]$$

$$\equiv U\left[\vec{A}_w, \vec{P}, \vec{A}_u, \vec{A}_c; \vec{\rho}\,(t)\right] = 0, \quad (3)$$

$$\vec{A}_b = -m\vec{a}_b,\ \ \vec{A}_w = -m\vec{a}_w,\ \ \vec{A}_u = -m\vec{a}_u = -m\left(\vec{a}_O + \vec{\varepsilon}\times\vec{\rho} + \vec{\omega}\times\vec{\omega}\times\vec{\rho}\right),$$
$$\vec{A}_c = -m\vec{a}_c = -2m\vec{\omega}\times\vec{v}_w;$$

the case, in which by employing the principle of the independence of forces, many forces of any types act on a single point:

$$\sum_{i=1}^{n} U_i\left[\vec{Q}_i, \vec{A}_i; \vec{r}\,(t)\right] = U\left[\vec{Q}_i, \vec{A}_i; \vec{r}\,(t)\right] = U\left[\vec{Q}, \vec{A}; \vec{r}\,(t)\right] = 0, \quad (4)$$

$$\vec{A} = \sum_{i=1}^{n}\vec{A}_i = -m\sum_{i=1}^{n}\vec{a}_i = -m\vec{a}, \qquad \vec{Q} = \sum_{i=1}^{n}\vec{Q}_i,$$
$$\vec{Q}_i \equiv \vec{P}_1, \vec{P}_2, \ldots, \vec{P}_{n'},\ \ \vec{F}_{n'+1} \equiv \vec{A}_u,\ \ \vec{F}_{n'+2} \equiv \vec{A}_c,\ \ \vec{F}_{n'+3}, \ldots, \vec{F}_n;$$

or any non-inertial systems, where resultant forces of mass forces and surface forces of any types act on a single material point:

$$U\left[\vec{A}, \vec{F}, \vec{P}; \vec{r}\,(t)\right] = 0, \quad (5)$$

$$\vec{F} = m\left[\vec{g} - \left(\vec{a}_O + \vec{\varepsilon}\times\vec{r} + \vec{\omega}\times\vec{\omega}\times\vec{r} + 2\vec{\omega}\times\vec{v}\right)\right] + \vec{E}, \quad \vec{W} = \vec{P} + \vec{F}.$$

D'Alembert's principle is generalised to cover a single point and an entire free system of material points:

$$U_i\left[\vec{A}_i, \vec{W}_i, \vec{W}_i'; \vec{r}_i\,(t)\right] = 0, \qquad i = 1, 2, \ldots, n, \quad (6)$$

$$\sum_{i=1}^{n} U_i = U\left[\vec{A}_i, \vec{W}_i, \vec{W}_i'; \vec{r}_i\,(t)\right] = U\left[\vec{A}_i, \vec{W}_i; \vec{r}_i\,(t)\right]$$
$$= U\left[\vec{A}_i, \vec{F}_i\vec{P}_i; \vec{r}_i\,(t)\right] = 0, \qquad \frac{d\vec{K}_O}{dt} \overset{!}{\equiv} \vec{M}_O; \quad (7)$$

as well as liquids and single-atomic gases:

$$U\left[\delta\vec{A}, \delta\vec{F}, \delta\vec{P}; \vec{r}\,(t)\right] = 0, \quad (8)$$

$$\int\limits_A \delta\vec{A} + \int\limits_F \delta\vec{F} + \int\limits_P \delta\vec{P} = \vec{0}\,, \quad \rho\frac{d\vec{V}}{dt} = \rho\vec{f} + \mathrm{Div}\,\mathbf{P},$$

$$\int\limits_A \vec{r} \times \delta\vec{A} + \int\limits_F \vec{r} \times \delta\vec{F} + \int\limits_P \vec{r} \times \delta\vec{P} \overset{!}{\equiv} \vec{0}, \quad \varepsilon_{ijk} p_{jk} \overset{!}{\equiv} 0, \qquad i,j,k = 1,2,3.$$

D'Alembert's principle is generalised to cover a rigid system of material points and on a perfectly rigid object:

$$\mid \vec{r}_{ik} \mid = \text{const.}, \qquad i,k = 1,2,\ldots,n,$$

$$U\left[\vec{A}_i, \vec{W}_i, ; \vec{r}_i(t)\right] = U\left[\vec{A}_i, \vec{F}_i, \vec{P}_i; \vec{r}_i(t)\right] = 0, \qquad i = 1,2,\ldots,n, \tag{9}$$

$$U\left[\delta\vec{A}, \delta\vec{F}, \delta\vec{P}; \vec{r}(t)\right] = U\left[\delta\vec{A}, \delta\vec{F}, \vec{P}_j; \vec{r}(t), \vec{r}(t), \vec{r}_j(t)\right] = 0, \tag{10}$$

$$U\left[\delta\vec{P}; \vec{r}(t)\right] = U\left[\vec{P}_j; \vec{r}_j(t)\right], \qquad j = 1,2,\ldots,n',$$

$$\int\limits_A \delta\vec{A} + \int\limits_F \delta\vec{F} + \sum_{j=1}^{n'} \vec{P}_j = \vec{0}, \quad m\vec{a}_C = \vec{W},$$

$$\int\limits_A \vec{r} \times \delta\vec{A} + \int\limits_F \vec{r} \times \delta\vec{F} + \sum_{j=1}^{n'} \vec{r}_j \times \vec{P}_j \overset{!}{=} \vec{0}, \quad \frac{d\vec{K}_O}{dt} \overset{!}{=} \vec{M}_O;$$

and a rigid object being in translation motion, rotation, plane rotation about a fixed point and any motion:

$$U\left[\delta\vec{A}, \delta\vec{F}, \vec{P}_i, \vec{M}_{ux}, \vec{M}_{uy}, \vec{M}_{uz}; \vec{\rho}(t), \vec{\rho}(t), \vec{\rho}_i(t)\right] = 0, \tag{11}$$

$$U\left[\delta\vec{A}, \delta\vec{F}, \vec{P}_i, \vec{R}_A, \vec{R}_B; \vec{r}(t), \vec{r}(t), \vec{r}_i(t), A, B\right] = 0, \tag{12}$$

$$U\left[\delta\vec{A}, \delta\vec{F}, \vec{P}_i, \vec{R}_A, \vec{R}_B, \vec{R}_C; \vec{\rho}(t), \vec{\rho}(t), \vec{\rho}_i(t), A, B, C\right] = 0, \tag{13}$$

$$U\left[\delta\vec{A}, \delta\vec{F}, \vec{P}_i, \vec{R}_O; \vec{r}(t), \vec{r}(t), \vec{r}_i(t), O\right] = 0, \tag{14}$$

$$U\left[\delta\vec{A}, \delta\vec{F}, \vec{P}_i; \vec{\rho}(t), \vec{\rho}(t), \vec{\rho}_i(t)\right] = 0\,. \tag{15}$$

D'Alembert's principle is generalised to cover a single rigid object and a system of rigid objects, to which the general equation of analytical dynamics may be applied:

$$U\left[\delta\vec{A}, \delta\vec{F}, \vec{P}_j; \vec{r}(t), \vec{r}(t), \vec{r}_j(t)\right] \overset{!}{\equiv} U\left[\vec{A}_i, \vec{W}_i; \vec{r}_i(t)\right] = 0, \tag{16}$$

$$j = 1,2,\ldots,n', \qquad i = 1,2,\ldots,n,$$

$$\int_A \delta\vec{A} + \int_F \delta\vec{F} + \sum_{j=1}^{n'} \vec{P}_j = \vec{0} \Rightarrow \sum_{i=1}^{n} \left(\vec{A}_i + \vec{W}_i\right) = \vec{0} \qquad / \cdot \delta\vec{r}_C$$

$$\int_A \vec{\rho} \times \delta\vec{A} + \int_F \vec{\rho} \times \delta\vec{F} + \sum_{j=1}^{n'} \vec{\rho}_j \times \vec{P}_j \Rightarrow \sum_{i=1}^{n} \vec{\rho}_i \times \left(\vec{A}_i + \vec{W}_i\right) = \vec{0} \qquad / \cdot \delta\vec{\varphi}$$

$$\delta\vec{r}_i = \delta\vec{r}_C + \delta\vec{\varphi} \times \vec{\rho}_i, \qquad \sum_{i=1}^{n} \left(\vec{W}_i - m_i \ddot{\vec{r}}_i\right) \cdot \delta\vec{r}_i = 0.$$

D'Alembert's law is generalised to cover any system of rigid objects being in a translation motion, rotation, plane rotation about a fixed point and any motion:

$$\begin{aligned} U \Big[& \vec{A}_{C\alpha}, \vec{W}_\alpha, \vec{0}_{C\alpha}, \vec{M}_{C\alpha}, \vec{A}_{C\beta}, \vec{W}_\beta, \vec{K}_{O\beta}, \vec{M}_{O\beta}, \vec{A}_{C\gamma}, \vec{W}_\gamma, \vec{K}_{C\gamma}\ , \\ & \vec{M}_{C\gamma}, \vec{A}_{C\nu}, \vec{W}_\nu, \vec{K}_{O\nu}, \vec{M}_{O\nu}, \vec{A}_{C\sigma}, \vec{W}_\sigma, \vec{K}_{C\sigma}, \vec{M}_{C\sigma}; \\ & C_\alpha, C_\alpha, O_\beta, O_\beta, C_\gamma, C_\gamma, O_\nu, O_\nu, C_\sigma, C_\sigma \Big] = 0, \end{aligned} \tag{17}$$

whereby we have got:

$$\begin{aligned} & \sum_{\alpha=1}^{N_1} \left(\vec{A}_{C\alpha} + \vec{W}_\alpha\right) + \sum_{\beta=1}^{N_2} \left(\vec{A}_{C\beta} + \vec{W}_\beta\right) + \sum_{\gamma=1}^{N_3} \left(\vec{A}_{C\gamma} + \vec{W}_\gamma\right) \\ & \qquad + \sum_{\nu=1}^{N_4} \left(\vec{A}_{C\nu} + \vec{W}_\nu\right) + \sum_{\sigma=1}^{N_5} \left(\vec{A}_{C\sigma} + \vec{W}_\sigma\right) = \vec{0}, \end{aligned}$$

$$\begin{aligned} \sum \vec{M}_O \Big[& \vec{A}_{C\alpha}, \vec{W}_\alpha, \vec{0}_{C\alpha}, \vec{M}_{C\alpha}, \vec{A}_{C\beta}, \vec{W}_\beta, \vec{K}_{O\beta}, \vec{M}_{O\beta}, \vec{A}_{C\gamma}, \vec{W}_\gamma, \\ & \vec{K}_{C\gamma}, \vec{M}_{C\gamma}, \vec{A}_{C\nu}, \vec{W}_\nu, \vec{K}_{O\nu}, \vec{M}_{O\nu}, \vec{A}_{C\sigma}, \vec{W}_\sigma, \vec{K}_{C\sigma}, \vec{M}_{C\sigma} \Big] = \vec{0}; \end{aligned}$$

as well as the so-called working form of the general equation of analytical dynamics for any material system comprising perfectly rigid objects, being in a translation motion, rotation, plane rotation about a fixed point and any motion:

D'Alembert's law is generalised to cover molecules of multi-atomic gas:

$$U_i \left[\vec{A}_i, \vec{W}_i, \vec{W}_i', \vec{K}_i, \vec{M}_i\left(\vec{P}_k\right), \vec{M}_i'\left(\vec{S}_{ik}\right); \vec{r}_i(t)\right] = 0, \tag{18}$$

$$\vec{K}_i = -\frac{d}{dt}\left(\mathbf{B}_i'\vec{\omega}_i\right),$$

$$\sum_{i=1}^{n} U_i = U\left[\vec{A}_i, \vec{W}_i, \vec{W}_i', \vec{K}_i, \vec{M}_i\left(\vec{P}_k\right), \vec{M}_i'\left(\vec{S}_{ik}\right); \vec{r}_i\left(t\right)\right]$$

$$= U\left[\vec{A}_i, \vec{W}_i, \vec{K}_i, \vec{M}_i\left(\vec{P}_k\right); \vec{r}_i\left(t\right)\right] = 0\,, \tag{19}$$

$$U\left[\delta\vec{A}, \delta\vec{F}, \delta\vec{P}, \delta\vec{K}, \delta\vec{M}, \delta\vec{Y}; \vec{r}\left(t\right)\right] = 0\,. \tag{20}$$

D'Alembert's supplemented and generalised principle corrects the original fundamentals of dynamics of those media and leads to determination of a formula on the power of the moments from external forces:

$$\rho N_m^w = m \operatorname{div} \vec{\omega},$$

to supplementing the first law of thermodynamics:

$$\rho \frac{dq}{dt} = \rho \frac{de}{dt} + \frac{d\left(1/\rho\right)}{dt} + m \operatorname{div} \vec{\omega} = \rho \frac{de}{dt} + p \operatorname{div} \vec{V} + m \operatorname{div} \vec{\omega},$$

and to development of 10 parameters' dynamics of multi-atomic gases:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \vec{V} = 0, \qquad \rho \frac{d\vec{V}}{dt} = \rho \vec{f} + \operatorname{Div} \mathbf{P},$$

$$\rho \frac{d\left(i^2 \vec{\omega}\right)}{dt} = \rho \vec{Y} + \operatorname{Div} \mathbf{M},$$

$$p = \rho R T, \qquad m = \rho R_1 T, \qquad \mathbf{M} = -m\varepsilon,$$

$$\rho \frac{d}{dt}\left(\frac{V^2}{2} + \frac{i^2 \omega^2}{2} + e\right) = \rho \vec{f} \cdot \vec{V} + \operatorname{div}\left(\mathbf{P}\vec{V}\right)$$

$$+\rho \vec{Y} \cdot \vec{\omega} + \operatorname{div}\left(\mathbf{M}\vec{\omega}\right) + \rho \dot{q} - \operatorname{div} \vec{j}\,.$$

2. Principle of Kineto-Statics

Application of D'Alembert's supplemented and generalised principle to a system of real objects, which may be replaced with models of rigid objects.

In any reference system true objects being in motion, each of which – in given conditions – may be replaced by a model of a perfectly rigid object, those true objects being imaginably cut out of a system of true objects and then – each of them separately – being replaced by a model of a perfectly rigid object, remain in a dynamic equilibrium, i.e. active forces, mass forces and superficial forces, external reactions and forces of inertia constitute a system equal to zero, i.e.

$$U\left[\delta\vec{A}_{\alpha},\delta\vec{F}_{\alpha},\delta\vec{P}_{\alpha};\vec{r}_{\alpha}\left(t\right)\right]=0, \tag{21}$$

as well as

$$U\left[\delta\vec{A}_{\alpha},\delta\vec{F}_{\alpha},\vec{P}_{\alpha i};\vec{r}_{\alpha}\left(t\right),\vec{r}_{\alpha}\left(t\right),\vec{r}_{\alpha i}\left(t\right)\right]=0, \tag{22}$$

$$\text{if}\quad U\left[\delta\vec{P}_{\alpha};\vec{r}_{\alpha}\left(t\right)\right]=U\left[\vec{P}_{\alpha i};\vec{r}_{\alpha i}\left(t\right)\right].$$

Bearing in mind a practical use of this principle of kineto – statics we note that the above representation of this principle indicates that the so-called "cut-out" a system of objects out of a system of objects is firstly applied to an entire system of objects pertaining to a specific issue in order to replace such a system with a system of perfectly rigid objects and for the latter six dynamical equation are formulated, which arise out of the sum of forces equal to zero and sum of the moments of such forces equal to zero, and then we employ such a "cut-out" again to selected objects of the system so may times that an additional dynamical equation of motion may be developed.

3. Newton's First Law

There are real reference systems, referred to as the inertial systems, in which real objects arbitrarily imaginably cut out of a cluster of such real objects remain at rest or keep moving straight and uniformly, if there are not any external forces acting externally upon the objects, or those forces make up a system equal to zero, which may be expressed by the following formulae:

$$U\left[\vec{P}_{i};A_{i}\right]=U\left[\vec{P}_{i};l_{i}\right]=U\left[\vec{P}_{i};\vec{r}_{i}\right]=0.\qquad i=1,2,\ldots,n,$$

where

$$\sum_{i=1}^{n}\vec{P}_{i}=\vec{0},$$

and

$$\sum_{i=1}^{n}\vec{M}_{O}\left(\vec{P}_{i}\right)=\sum_{i=1}^{n}\vec{r}_{i}\times\vec{P}_{i}=\vec{0},$$

as well as:

$$1)\quad \sum_{i=1}^{n} P_{ix} = 0, \qquad 2)\quad \sum_{i=1}^{n} P_{iy} = 0, \qquad 3)\quad \sum_{i=1}^{n} P_{iz} = 0,$$

$$4)\quad \sum_{i=1}^{n} (y_i P_{iz} - z_i P_{iy}) = 0,$$

$$5)\quad \sum_{i=1}^{n} (z_i P_{ix} - x_i P_{iz}) = 0,$$

$$6)\quad \sum_{i=1}^{n} (x_i P_{iy} - y_i P_{ix}) = 0\,.$$

Equation of the co-ordinates, which result from the above conditions of equilibrium, specify – each separately – the so-called directional equilibrium of uniform translation motion (equations (1), (2), (3)) and directional equilibrium of uniform rotation (equations (4), (5), (6)).

4. Principle of the Independence of Forces

Supplemented by:

if $$m\vec{a} = \sum_{i=1}^{n} \vec{P}_i,$$

then $$m\vec{a}_i = \vec{P}_i,$$

for $$i = 1, 2, \ldots, n,$$

where $$\vec{a} = \sum_{i=1}^{n} \vec{a}_i.$$

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