International Journal of Pure and Applied Mathematics
Volume 39 No. 1 2007, 1-16

ON THE q-ANALOGUE OF DUNKL OPERATOR AND ITS APPELL CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract: We introduce an operator $\mathcal{D}(\theta,q)$ giving a connection with H_q-semiclassical orthogonal polynomials and we investigate it to characterize a well known symmetric (MOPS) related to the Wall's one, us the unique $\mathcal{D}(\theta,q)$-Appell classical. This (MOPS) is also $H_{\sqrt{q}}$ -semiclassical of class one. The moments and integral representation of the corresponding linear form are given.

AMS Subject Classification: 42C05, 33C45
Key Words: q-semiclassical orthogonal polynomials, q-Dunkl operator, q-Appell orthogonal polynomials, moments and integral representation

1. Introduction and Preliminaries

The concept of O-semiclassical orthogonal polynomials was extensively studied by P. Maroni and coworkers for $O = D$ the derivative operator, $O = D_\omega$ or H_q the Hahn's operators through the following distributional equation satisfied by the regular linear form associated with a such sequence:

$$O(\Phi(x)u) + \Psi(x)u = 0\,, \tag{1.1}$$

where Φ is a monic polynomial and Ψ a polynomial with $\deg\Psi \geq 1$, see [1, 2, 15, 16, 17, 19]. For $O \in \left\{D, D_\omega, H_q\right\}$, O-semiclassical of class zero are

Received: March 8, 2007

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usually called O-classical and they are completely discribed in [1, 15, 19]. In particular, their corresponding O-Appell classical are well known. For other relevant researches in the subject from other point of view with perhaps other operators see [4, 14, 20].

In [6, 7], T.S. Chihara solved the following problem: Find all Brenke type polynomials which are also orthogonal. By solving a difference equation he found all (OPS) in this class. They are:

(a) The Laguerre (MOPS) (D-classical, see [19]).

(b) The generalized Hermite (MOPS) (symmetric D-semiclassical of class one, see [2]).

(c) The Al-Salam and Carlitz (MOPS) (H_q-Appell classical, see [15]).

(d) The generalized Stieltjes-Wigert (MOPS) (H_q-classical, see [15]).

(e) A symmetric (MOPS) related to the Stieltjes-Wigert (MOPS)'s.

(f) The Wall $\{W_n(.;b,q)\}_{n\geq 0}$ (MOPS)(H_q-classical, see [15]).

(g) A non-symmetric (MOPS) related to the Wall.

(h) A symmetric (MOPS) $\{Y_n(.;b,q)\}_{n\geq 0}$ related to the Wall (MOPS):

$$Y_{2n}(x;b,q) = W_n(x^2;b,q) \ \ , \ \ Y_{2n+1}(x;b,q) = xW_n(x^2;bq,q) \ , \ \ n \geq 0 \tag{1.2}$$

having the recurrence formula (see [5])

$$\begin{cases} Y_0(x;b,q) = 1 \ \ , \ \ Y_1(x;b,q) = x, \\ Y_{n+2}(x;b,q) = xY_{n+1}(x;b,q) - \widetilde{\gamma}_{n+1}Y_n(x;b,q), \ n \geq 0, \\ \widetilde{\gamma}_{2n} = b(1-q^n)q^n, \ n \geq 1 \ ; \ \widetilde{\gamma}_{2n+1} = (1-bq^n)q^{n+1}, \ n \geq 0, \\ b \neq 0, \ b \neq q^{-n}, \ n \geq 0. \end{cases} \tag{1.3}$$

So the aim of our contribution is to give another characterization of the (MOPS) $\{Y_n(.;b,q)\}_{n\geq 0}$ and it is associated regular linear form $\mathcal{Y}(b,q)$ based on the $\mathcal{D}_{(\theta,q)}$-Appell classical character, where $\mathcal{D}_{(\theta,q)} := H_q + \theta H_{-q}$ is the q-analogue of Dunkl operator, see [8]. This first section contains preliminary results and notations used in the sequel. The second section is devoted to the study of the connection between the $\mathcal{D}_{(\theta,q)}$-Appell classical and the H_q-semiclassical characters. In the third section we determine all symmetric $\mathcal{D}_{(\theta,q)}$-Appell classical linear forms; there is a unique solution, up to affine transformations, it is the regular linear form $\mathcal{Y}(b,q^2)$. Also the relationship between $\mathcal{Y}(b,q)$ and the H_q-classical Wall one is highlited. Consequently, moments and integral representation of $\mathcal{Y}(b,q)$ are obtained.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$ the moments of u. Moreover, a linear form u is called symmetric if $(u)_{2n+1} = 0,\ n \geq 0$.

Let us introduce some useful operations in $\mathcal{P}'$. For any linear form u, any polynomial g and any $(a, c) \in \mathbb{C} - \{0\} \times \mathbb{C}$, we let $H_q u$, gu, $h_a u$, $(x-c)^{-1}u$ and δ_c, be the linear forms defined by duality

$$\langle H_q u, f \rangle := -\langle u, H_q f \rangle \ , \ \langle gu, f \rangle := \langle u, gf \rangle \ , \langle h_a u, f \rangle := \langle u, h_a f \rangle,\ f \in \mathcal{P},$$

$$\langle (x-c)^{-1} u, f \rangle := \langle u, \theta_c f \rangle \ , \ \langle \delta_c, f \rangle := f(c) \ , f \in \mathcal{P},$$

where $(H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}$ [12], $(h_a f)(x) = f(ax)$ and $(\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}$ [17, 19]. It is easy to see that, see [15]

$$H_q(fu) = (h_{q^{-1}} f) H_q u + q^{-1} (H_{q^{-1}} f) u, f \in \mathcal{P},\ u \in \mathcal{P}'. \tag{1.4}$$

Let $\{P_n\}_{n \geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n \geq 0$ (polynomial sequence: PS) and let $\{u_n\}_{n \geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by $\langle u_n, P_m \rangle := \delta_{n,m}$, $n, m \geq 0$.

Lemma 1. (see [19]) *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent:*

i) $\langle u, P_{m-1} \rangle \neq 0$, $\langle u, P_n \rangle = 0$, $n \geq m$,

ii) $\exists \lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq m-1$, $\lambda_{m-1} \neq 0$ *such that*

$$u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$$

The linear form u is called *regular* if we can associate with it a sequence of polynomials $\{P_n\}_{n \geq 0}$ such that $\langle u, P_m P_n \rangle = r_n \delta_{n,m}$, $n, m \geq 0$; $r_n \neq 0$, $n \geq 0$. The sequence $\{P_n\}_{n \geq 0}$ is then said orthogonal with respect to u. Necessarily, $u = \lambda u_0$, $\lambda \neq 0$ and $\{P_n\}_{n \geq 0}$ is an (OPS) such that any polynomial can be supposed monic (MOPS). In this case, we have $u_n = r_n^{-1} P_n u_0$, $n \geq 0$ and conversely. Also, the (MOPS)$\{P_n\}_{n \geq 0}$ fulfils the recurrence relation

$$\begin{cases} P_0(x) = 1 \ , \ P_1(x) = x - \beta_0 \ , \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x) \ , \ \gamma_{n+1} \neq 0 \ , \ n \geq 0. \end{cases} \tag{1.5}$$

When u is regular, let ϕ be a polynomial such that $\phi u = 0$. Then $\phi = 0$. Also, when u is regular, u is symmetric if and only if $\beta_n = 0$, $n \geq 0$.

Finally, we introduce an operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ defined by $(\sigma f)(x) := f(x^2)$ for all $f \in \mathcal{P}$. Consequently, we define σu by duality (see [18])

$$\langle \sigma u, f \rangle = \langle u, \sigma f \rangle \ , f \in \mathcal{P} \ , u \in \mathcal{P}'.$$

We have the well known formula, see [18]

$$f(x)\sigma u = \sigma(f(x^2)u), \tag{1.6}$$

and it is easy to prove

$$\sigma(H_q u) = (q+1)H_{q^2}(\sigma(xu)). \tag{1.7}$$

A linear form u is called H_q-semiclassical when it is regular and there exist two polynomials Φ and Ψ, Φ monic, $\deg\Phi = t \geq 0$, $\deg\Psi = p \geq 1$ such that

$$H_q(\Phi u) + \Psi u = 0\,. \tag{1.8}$$

The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called H_q-semiclassical, see [10, 15, 16]. The H_q-semiclassical character is kept by shifting. In fact, let $\{\widetilde{P}_n := a^{-n}(h_a P_n)\}_{n\geq 0}$, $a \neq 0$; when u_0 satisfies (1.8), then $\widetilde{u}_0 = h_{a^{-1}}u_0$ fulfils the equation

$$H_q\Big(a^{-t}\Phi(ax)\widetilde{u}_0\Big) + a^{1-t}\Psi(ax)\widetilde{u}_0 = 0 \tag{1.9}$$

and the recurrence elements $\widetilde{\beta}_n,\ \widetilde{\gamma}_{n+1},\ n \geq 0$ of the sequence $\{\widetilde{P}_n\}_{n\geq 0}$ are

$$\widetilde{\beta}_n = \frac{\beta_n}{a}\ ,\ \widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2},\ n \geq 0. \tag{1.10}$$

Also, the H_q-semiclassical linear form u_0 is said to be of class $s = \max(p-1, t-2) \geq 0$ if and only if (see [16])

$$\prod_{c\in\mathcal{Z}_\Phi}\Big\{\Big|q(h_q\Psi)(c) + (H_q\Phi)(c)\Big| + \Big|\langle u_0, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi)\rangle\Big|\Big\} > 0, \tag{1.11}$$

where $\mathcal{Z}_\Phi$ is the set of zeroes of Φ. In particular, when $s = 0$, the linear form u_0 is usually called H_q-*classical* (Al-Salam Carlitz, Big q-Laguerre, q-Meixner, Wall, etc., see [15]).

Lastly, let us recall the following result useful for our work (see [2])

Lemma 2. *Let $\{P_n\}_{n\geq 0}$ be a (MOPS) and $M(x,n)$, $N(x,n)$ two polynomials such that*

$$M(x,n)P_{n+1}(x) = N(x,n)P_n(x),\ n \geq 0.$$

Then, for any index n for which $\deg N(x,n) \leq n$, we have

$$N(x,n) = 0 \ \text{ and } \ M(x,n) = 0.$$

2. Connection between $\mathcal{D}_{(\theta,q)}$-Appell Classical and H_q-Semiclassical Characters

Let us introduce the q-Dunkl operator in $\mathcal{P}$ by

$$\mathcal{D}_{(\theta,q)} = H_q + \theta H_{-q}\ ,\ \theta \neq 0\ ,\ q \in \widetilde{\mathbb{C}}$$

where $\widetilde{\mathbb{C}} := \Big\{ q \in \mathbb{C},\ \ q \neq 0,\ q^n \neq 1,\ n \geq 1 \Big\}$ and H_q is the q-derivative operator. It is obvious that $\mathcal{D}_{(\theta,q)}$ tends to $\mathcal{D}_{(\theta,1)} = D + \theta H_{-1} := \mathcal{D}_\theta$ the Dunkl operator when q tends to 1. We have $\mathcal{D}^{\mathrm{T}}_{(\theta,q)} = -H_q - \theta H_{-q}$ [15], where $\mathcal{D}^{\mathrm{T}}_{(\theta,q)}$ denotes the transposed of $\mathcal{D}_{(\theta,q)}$. We can define $\mathcal{D}_{(\theta,q)}$ from $\mathcal{P}'$ to $\mathcal{P}'$ by $\mathcal{D}_{(\theta,q)} = -\mathcal{D}^{\mathrm{T}}_{(\theta,q)}$ so that

$$\langle \mathcal{D}_{(\theta,q)} u, f\rangle = -\langle u, \mathcal{D}_{(\theta,q)} f\rangle\ ,\ u \in \mathcal{P}'\ ,\ f \in \mathcal{P}.$$

In particular this yields

$$(\mathcal{D}_{(\theta,q)} u)_n = -[[n]]_{(\theta,q)} (u)_{n-1}\ ,\ n \geq 0,$$

where $(u)_{-1} := 0$ and

$$[[n]]_{(\theta,q)} = [n]_q + \theta [n]_{-q}\ ,\ n \geq 0\,, \tag{2.1}$$

with

$$[n]_p := \frac{p^n - 1}{p - 1},\ n \geq 0. \tag{2.2}$$

Lemma 3. *For any $f, g \in \mathcal{P}$, $u \in \mathcal{P}'$ and $a \in \mathbb{C} - \{0\}$ we have*

$$\mathcal{D}_{(\theta,q)} \circ h_a = a h_a \circ \mathcal{D}_{(\theta,q)} \ \text{ in }\ \mathcal{P}\ ;\ \mathcal{D}_{(\theta,q)} \circ h_a = a^{-1} h_a \circ \mathcal{D}_{(\theta,q)} \ \text{ in }\ \mathcal{P}', \tag{2.3}$$

$$\mathcal{D}_{(\theta,q)}(fg) = (h_{-q} f)(\mathcal{D}_{(\theta,q)} g) + g(\mathcal{D}_{(\theta,q)} f) + (h_q f - h_{-q} f)(H_q g), \tag{2.4}$$

$$\mathcal{D}_{(\theta,q)}((h_{-q} f) u) = f(\mathcal{D}_{(\theta,q)} u) + (\mathcal{D}_{(\theta,q)} f) u + H_q((h_{-q} f - h_q f) u). \tag{2.5}$$

Proof. It is easy to establish (2.3)-(2.4) from the following well known results (see [15])

$$H_p \circ h_a = a h_a \circ H_p \ \text{ in }\ \mathcal{P},$$
$$H_p \circ h_a = a^{-1} h_a \circ H_p \ \text{ in }\ \mathcal{P}'$$

and (1.4). For (2.5) we have $\forall P \in \mathcal{P}$

$$\begin{aligned}\langle \mathcal{D}_{(\theta,q)}((h_{-q} f) u), P\rangle &= -\langle u, (h_{-q} f)(\mathcal{D}_{(\theta,q)} P)\rangle \\ &= -\langle u, \mathcal{D}_{(\theta,q)}(fP) - P(\mathcal{D}_{(\theta,q)} f) - (h_q f - h_{-q} f)(H_q P)\rangle \\ &\qquad\qquad\qquad\qquad \text{(by (2.4))} \\ &= \langle f(\mathcal{D}_{(\theta,q)} u) + (\mathcal{D}_{(\theta,q)} f) u + H_q((h_{-q} f - h_q f) u), P\rangle.\end{aligned}$$

□

Now consider a (PS) $\{P_n\}_{n\geq 0}$ as above and let

$$P_n^{[1]}(x;\theta,q) = \frac{1}{[[n+1]]_{(\theta,q)}}(\mathcal{D}_{(\theta,q)}P_{n+1})(x),\ n\geq 0$$

with $\theta \neq \frac{q+1}{q-1}$, $\theta \neq \frac{q+1}{q-1}\ \frac{1-q^{2n+1}}{1+q^{2n+1}}$, $n\geq 0$.

Denoting by $\{u_n^{[1]}(\theta,q)\}_{n\geq 0}$ the dual sequence of $\{P_n^{[1]}(.;\theta,q)\}_{n\geq 0}$, we have the following result.

Lemma 4. *The following equalities hold*

$$\mathcal{D}_{(\theta,q)}(u_n^{[1]}(\theta,q)) = -[[n+1]]_{(\theta,q)}u_{n+1}\ ,\ n\geq 0. \tag{2.6}$$

Proof. Indeed, from the definition:

$$\langle u_n^{[1]}(\theta,q), P_m^{[1]}(.;\theta,q)\rangle = \delta_{n,m},\ n,m\geq 0,$$

we have $-\langle \mathcal{D}_{(\theta,q)}(u_n^{[1]}(\theta,q)), P_{m+1}(.;\theta,q)\rangle = [[m+1]]_{(\theta,q)}\delta_{n,m}$, therefore

$$\begin{cases} \langle \mathcal{D}_{(\theta,q)}(u_n^{[1]}(\theta,q)), P_{n+1}(.;\theta,q)\rangle = -[[n+1]]_{(\theta,q)}, \qquad n\geq 0, \\ \langle \mathcal{D}_{(\theta,q)}(u_n^{[1]}(\theta,q)), P_m(.;\theta,q)\rangle = 0,\ m\geq n+2,\ n\geq 0. \end{cases}$$

By virtue of Lemma 1, we get

$$\mathcal{D}_{(\theta,q)}(u_n^{[1]}(\theta,q)) = \sum_{\nu=0}^{n+1}\lambda_{n,\nu}u_\nu,\ n\geq 0.$$

But

$$\langle \mathcal{D}_{(\theta,q)}(u_n^{[1]}(\theta,q)), P_\nu(.;\theta,q)\rangle = \lambda_{n,\nu},\ 0\leq \nu\leq n+1$$

and

$$\lambda_{n,\nu} = 0,\quad 0\leq\nu\leq n,\qquad \lambda_{n,n+1} = -[[n+1]]_{(\theta,q)},\ \ n\geq 0.$$

Hence (2.6) follows. □

Definition 5. The sequence $\{P_n\}_{n\geq 0}$ is called $\mathcal{D}_{(\theta,q)}$-Appell classical if $P_n^{[1]}(.;\theta,q) = P_n,\ n\geq 0$ and $\{P_n\}_{n\geq 0}$ is orthogonal. Then u_0 is called a $\mathcal{D}_{(\theta,q)}$-Appell classical linear form.

Now, let $\{P_n\}_{n\geq 0}$ be a $\mathcal{D}_{(\theta,q)}$-Appell classical sequence and u_0 its canonical linear form. Then $P_n^{[1]}(.;\theta,q) = P_n,\ n\geq 0$ and equivalently $u_n^{[1]}(\theta,q) = u_n,\ n\geq 0$. Consequently relation (2.6) becomes

$$\mathcal{D}_{(\theta,q)}(P_nu_0) = -r_{n+1}P_{n+1}u_0,\ n\geq 0, \tag{2.7}$$

where

$$r_{n+1} = \frac{[[n+1]]_{(\theta,q)}}{\gamma_{n+1}},\ n \geq 0. \tag{2.8}$$

In particular, taking $n=0$ in (2.7) we get

$$\mathcal{D}_{(\theta,q)}(u_0) = -r_1 P_1 u_0. \tag{2.9}$$

By virtue of (2.5) formula (2.7) gives for $n \geq 0$

$$(h_{-q^{-1}}P_n)(\mathcal{D}_{(\theta,q)}u_0) + (\mathcal{D}_{(\theta,q)} \circ h_{-q^{-1}}P_n)u_0 + H_q\big((P_n - h_{-1}P_n)u_0\big) = -r_{n+1}P_{n+1}u_0.$$

From (2.9) and (1.7) the above formula becomes

$$(h_{-q^{-1}}P_n - h_{q^{-1}}P_n)(H_q u_0) = \Big\{r_{n+1}P_{n+1} - r_1 P_1 (h_{-q^{-1}}P_n) + (\mathcal{D}_{(\theta,q)} \circ h_{-q^{-1}}P_n) + q^{-1}H_{q^{-1}}\big(P_n - h_{-1}P_n\big)\Big\}u_0, \quad n \geq 0\,. \tag{2.10}$$

Taking $n=1$ in (2.10), on account of (2.2)-(2.3), (2.8) and by virtue of (1.7)-(1.8) we obtain

$$H_q(xu_0) + \frac{1}{2}\Big\{\Big(\frac{q+1+\theta(1-q)}{\gamma_2} + \frac{1+\theta}{\gamma_1}q^{-1}\Big)x^2 + \Big((1-q^{-1})\beta_0\frac{1+\theta}{\gamma_1} - \frac{q+1+\theta(1-q)}{\gamma_2}(\beta_0+\beta_1)\Big)x + \frac{q+1+\theta(1-q)}{\gamma_2}(\beta_0\beta_1 - \gamma_1) - \frac{1+\theta}{\gamma_1}\beta_0^2 - q^{-1}(1+\theta)\Big\}u_0 = 0.$$

Consequently, we get the following result.

Proposition 6. *Let $\{P_n\}_{n\geq 0}$ be a $\mathcal{D}_{(\theta,q)}$-Appell classical sequence and u_0 its canonical linear form. Then u_0 is H_q-semiclassical of class $s \leq 1$ satisfying the functional equation*

$$H_q(\Phi(x)u_0) + \Psi(x)u_0 = 0\,, \tag{2.11}$$

with

$$\begin{cases} \Phi(x) = x \\ \Psi(x) = \frac{1}{2}\Big\{\Big(\frac{q+1+\theta(1-q)}{\gamma_2} + \frac{1+\theta}{\gamma_1}q^{-1}\Big)x^2 \\ \qquad + \Big((1-q^{-1})\beta_0\frac{1+\theta}{\gamma_1} - \frac{q+1+\theta(1-q)}{\gamma_2}(\beta_0+\beta_1)\Big)x \\ \qquad\qquad + \frac{q+1+\theta(1-q)}{\gamma_2}(\beta_0\beta_1-\gamma_1) - \frac{1+\theta}{\gamma_1}\beta_0^2 - q^{-1}(1+\theta)\Big\}. \end{cases} \tag{2.12}$$

3. Determination of All Symmetric $\mathcal{D}_{(\theta,q)}$-Appell Classical Orthogonal Polynomials

3.1. A New Characterization of $\mathcal{Y}(b,q)$

Lemma 7. *Let $\{P_n\}_{n\geq 0}$ be a symmetric $\mathcal{D}_{(\theta,q)}$-Appell classical sequence. The following formulas hold*

$$\frac{2q}{1+q}\theta(h_{-q}P_{n+1})(x)=\Big\{[[n+2]]_{(\theta,q)}-\frac{\gamma_{n+1}}{\gamma_n}[[n]]_{(\theta,q)}-1-\theta\frac{1-q}{1+q}\Big\}P_{n+1}(x)+\Big(\frac{\gamma_{n+1}}{\gamma_n}[[n]]_{(\theta,q)}-q[[n+1]]_{(\theta,q)}\Big)xP_n(x),\quad n\geq 1,\quad (3.1)$$

$$\gamma_2=q\frac{q+1+\theta(q-1)}{1+\theta}\gamma_1,\ \theta\neq\frac{1+q}{1-q}.\qquad (3.2)$$

Proof. From (1.5) and the fact that $\{P_n\}_{n\geq 0}$ is symmetric we have

$$P_{n+2}(x)=xP_{n+1}(x)-\gamma_{n+1}P_n(x),\ \ n\geq 0.\qquad (3.3)$$

Applying the operator $\mathcal{D}_{(\theta,q)}$ in (3.3), using (2.4) and in accordance of the $\mathcal{D}_{(\theta,q)}$-Appell classical character we obtain

$$[[n+2]]_{(\theta,q)}P_{n+1}(x)=-q[[n+1]]_{(\theta,q)}xP_n(x)+(1+\theta)P_{n+1}(x)+2qx(H_qP_{n+1})(x)-\gamma_{n+1}[[n]]_{(\theta,q)}P_{n-1}(x),\ n\geq 1.\quad (3.4)$$

From definition of the operator $\mathcal{D}_{(\theta,q)}$ and the recurrence relation in (1.5), formula (3.4) becomes

$$[[n+2]]_{(\theta,q)}P_{n+1}(x)=q[[n+1]]_{(\theta,q)}xP_n(x)+\big(1+\theta\frac{1-q}{1+q}\big)P_{n+1}(x)+\frac{2q}{1+q}\theta(h_{-q}P_{n+1})(x)-\frac{\gamma_{n+1}}{\gamma_n}[[n]]_{(\theta,q)}\big(xP_n-P_{n+1}(x)\big),\ n\geq 1.$$

Consequently (3.1) is proved.

On the other hand, taking $n=1$ in (3.1) and on account of $P_1(x)=x$ and $P_2(x)=x^2-\gamma_1$, we get (3.2) after identification. □

Now, we are able to give the system satisfied by γ_{n+1}, $n\geq 0$ written in terms of r_{n+1}, $n\geq 0$, where r_{n+1} is given by (2.8).

Proposition 8. *The sequence* $\{r_{n+1}\}_{n\geq 0}$ *fulfils the following system*

$$q^2 r_{n+1} = r_{n-1}, \qquad n \geq 2, \tag{3.5}$$

$$\frac{r_{n+1}}{r_{n+2}}[[n+2]]_{(\theta,q)} - \frac{r_{n-1}}{r_n}[[n]]_{(\theta,q)} = [[n+3]]_{(\theta,q)} - [[n+1]]_{(\theta,q)}, \quad n \geq 2, \tag{3.6}$$

$$\frac{r_1}{r_2} = q\frac{q+1+\theta(q-1)}{q+1-\theta(q-1)}. \tag{3.7}$$

Proof. Applying the dilatation h_{-q} for (3.3) and multiplying by $\frac{2q}{1+q}\theta$, according to (3.1), we get successively

$$(h_{-q}P_{n+2})(x) = -qx(h_{-q}P_{n+1})(x) - \gamma_{n+1}(h_{-q}P_n)(x), \;\; n \geq 0,$$

$$\frac{2q}{1+q}\theta(h_{-q}P_{n+2})(x) = -qx\frac{2q}{1+q}\theta(h_{-1}P_{n+1})(x) - \gamma_{n+1}\frac{2q}{1+q}\theta(h_{-1}P_n)(x), \quad n \geq 0,$$

$$\begin{aligned}&\Big([[n+3]]_{(\theta,q)} - \tfrac{\gamma_{n+2}}{\gamma_{n+1}}[[n+1]]_{(\theta,q)} - 1 - \theta\tfrac{1-q}{1+q}\Big)P_{n+2}(x) \\ &\qquad + \Big(\tfrac{\gamma_{n+2}}{\gamma_{n+1}}[[n+1]]_{(\theta,q)} - q[[n+2]]_{(\theta,q)}\Big)xP_{n+1}(x) \\ &= -qx\Big\{\Big([[n+2]]_{(\theta,q)} - \tfrac{\gamma_{n+1}}{\gamma_n}[[n]]_{(\theta,q)} - 1 - \theta\tfrac{1-q}{1+q}\Big)P_{n+1}(x) \\ &\qquad + \Big(\tfrac{\gamma_{n+1}}{\gamma_n}[[n]]_{(\theta,q)} - q[[n+1]]_{(\theta,q)}\Big)xP_n(x)\Big\} \\ &-\gamma_{n+1}\Big\{\Big([[n+1]]_{(\theta,q)} - \tfrac{\gamma_n}{\gamma_{n-1}}[[n-1]]_{(\theta,q)} - 1 - \theta\tfrac{1-q}{1+q}\Big)P_n(x) \\ &\qquad + \Big(\tfrac{\gamma_n}{\gamma_{n-1}}[[n-1]]_{(\theta,q)} - q[[n]]_{(\theta,q)}\Big)xP_{n-1}(x)\Big\}, \;\; n \geq 2.\end{aligned}$$

But from (3.3) another time we obtain

$$M(x,n)P_{n+1}(x) = N(x,n)P_n(x), \quad n \geq 2\,, \tag{3.8}$$

where for $n \geq 2$

$$M(x,n) = \Big([[n+3]]_{(\theta,q)} - \gamma_{n+1}\frac{[[n-1]]_{(\theta,q)}}{\gamma_{n-1}} - (1+q) + \theta(q-1)\Big)x,$$

$$\begin{aligned}N(x,n) &= \Big(q^2[[n+1]]_{(\theta,q)} - \gamma_{n+1}\tfrac{[[n-1]]_{(\theta,q)}}{\gamma_{n-1}}\Big)x^2 \\ &+\gamma_{n+1}\big([[n+3]]_{(\theta,q)} - [[n+1]]_{(\theta,q)}\big) - \gamma_{n+2}[[n+1]]_{(\theta,q)} + \gamma_{n+1}\gamma_n\tfrac{[[n-1]]_{(\theta,q)}}{\gamma_{n-1}}.\end{aligned}$$

Next, according to Lemma 2, for $n \geq 2$, $M(x,n)=0$, $N(x,n)=0$, that is to say

$$q^2[[n+1]]_{(\theta,q)} - \gamma_{n+1}\frac{[[n-1]]_{(\theta,q)}}{\gamma_{n-1}} = 0, \quad n \geq 2\,, \tag{3.9}$$

$$\gamma_{n+1}\big([[n+3]]_{(\theta,q)} - [[n+1]]_{(\theta,q)}\big) - \gamma_{n+2}[[n+1]]_{(\theta,q)} + \gamma_{n+1}\gamma_n\frac{[[n-1]]_{(\theta,q)}}{\gamma_{n-1}} = 0, \quad n \geq 2. \tag{3.10}$$

According to (2.8) relations (3.9)-(3.10) give the desired results (3.5)-(3.6). Also, from (2.8) and (2.2) we get

$$r_1 = \frac{1+\theta}{\gamma_1} \quad , \quad r_2 = \frac{q+1-\theta(q-1)}{\gamma_2}.$$

Therefore, taking into account (3.2) we obtain (3.7). □

Now we are going to solve the system (3.5)-(3.7).

By virtue of (3.5) and (2.1), (3.6) becomes

$$\frac{r_{n-1}}{r_n} = q\ \frac{q+1+\theta(-1)^n(q-1)}{q+1-\theta(-1)^n(q-1)}, \quad n \geq 2.$$

Consequently

$$r_{n+1} = q^{-n}\ \frac{q+1+\theta(-1)^n(q-1)}{q+1+\theta(q-1)}\ r_1, \quad n \geq 0 \tag{3.11}$$

and (3.5), (3.7) are valid.

From (2.8) and (2.1), (3.11) gives

$$\gamma_{n+1} = [[n+1]]_{(\theta,q)}\ q^n\ \frac{q+1+\theta(q-1)}{q+1+\theta(-1)^n(q-1)}\ \frac{\gamma_1}{1+\theta}, \quad n \geq 0. \tag{3.12}$$

When $q=1$ in (3.12) we recover again that the unique sequence of orthogonal polynomials which is symmetric $\mathcal{D}_\theta$-Appell classical is the generalized Hermite one, see [11].

When $q \in \widetilde{\mathbb{C}}$, from (3.12) and after some calculations we obtain

$$\begin{cases} \gamma_{2n} = \frac{q+1+\theta(q-1)}{1-q^2}\ \frac{\gamma_1}{1+\theta}\ (1-q^{2n})q^{2n-1}, \quad n \geq 1, \\[2ex] \gamma_{2n+1} = \frac{q+1-\theta(q-1)}{1-q^2}\ \frac{\gamma_1}{1+\theta}\ \left(1 - \frac{q+1+\theta(q-1)}{q+1-\theta(q-1)}\ q^{2n+1}\right)\ q^{2n}, \quad n \geq 0, \end{cases} \tag{3.13}$$

with the regularity conditions

$$\theta \neq 0 \ , \ \theta \neq \pm\frac{q+1}{q-1} \ , \ \theta \neq \frac{q+1}{q-1} \ \frac{1-q^{2n+1}}{1+q^{2n+1}}, \ n \geq 0. \tag{3.14}$$

Corollary 9. *The unique symmetric $\mathcal{D}_{(\theta,q)}$-Appell classical linear form, up to affine transformations, is the regular linear form $\mathcal{Y}(b,q^2)$ related to the Wall one $\Big(b \neq 0, \ b \neq q^{-2n}, \ n \geq 0\Big)$.*

Proof. Let $\{P_n\}_{n\geq 0}$ be a symmetric $\mathcal{D}_{(\theta,q)}$-Appell classical sequence and u_0 its canonical linear form. By virtue of (3.13),(3.2) and (2.11)-(2.12) we get

$$\begin{cases} \beta_n = 0, \ \ n \geq 0, \\ \gamma_{2n} = \frac{q+1+\theta(q-1)}{1-q^2} \ \frac{\gamma_1}{1+\theta} \ (1-q^{2n})q^{2n-1}, \ \ n \geq 1, \\ \gamma_{2n+1} = \frac{q+1-\theta(q-1)}{1-q^2} \ \frac{\gamma_1}{1+\theta} \ \Big(1 - \frac{q+1+\theta(q-1)}{q+1-\theta(q-1)} \ q^{2n+1}\Big) \ q^{2n}, \ \ n \geq 0, \\ \\ H_q(\Phi(x)u_0) + \Psi(x)u_0 = 0, \\ \Phi(x) = x \ , \ \Psi(x) = \frac{(1+\theta)(1+q^{-1})}{q+1+\theta(q-1)}\Big\{\frac{1}{\gamma_1}x^2 - 1\Big\}. \end{cases} \tag{3.15}$$

We have $q(h_q\Psi)(0) + (H_q\Phi)(0) = \frac{-2\theta}{q+1+\theta(q-1)} \neq 0$. According to (1.11) and (3.15), this allows us to conclude that u_0 is a symmetric H_q-semiclassical linear form of class one since (3.14).

On the other hand, with the choice $a = \sqrt{\frac{(q+1-\theta(q-1))q^{-2}\gamma_1}{(1-q^2)(1+\theta)}}$, in (1.9)-(1.10), and putting $b := q \ \frac{q+1+\theta(q-1)}{q+1-\theta(q-1)}$ we are led to the following canonical case

$$\begin{cases} \widetilde{\beta}_n = 0, \ \ n \geq 0, \\ \widetilde{\gamma}_{2n} = b(1-(q^2)^n)(q^2)^n, \ \ n \geq 1, \\ \widetilde{\gamma}_{2n+1} = (1-b(q^2)^n)(q^2)^{n+1}, \ \ n \geq 0, \\ H_q(\widetilde{\Phi}(x)\widetilde{u}_0) + \widetilde{\Psi}(x)\widetilde{u}_0 = 0, \\ \widetilde{\Phi}(x) = x \ , \ \widetilde{\Psi}(x) = -b^{-1}(q-1)^{-1}\Big\{q^{-2}x^2 + b - 1\Big\}. \end{cases} \tag{3.16}$$

Thus (see (1.3))

$$\widetilde{u}_0 = \mathcal{Y}(b,q^2), \ \ b \neq 0, \ \ b \neq q^{-2n}, \ \ n \geq 0. \qquad \square$$

Remark 10. It is easy to conclude that the linear form $\mathcal{Y}(b,q)$ is $H_{q^{\frac{1}{2}}}$-semiclassical of class one for $b \neq 0,\ \ b \neq q^{-n},\ \ n \geq 0$ satisfying the q-Pearson equation

$$H_{q^{\frac{1}{2}}}(x\mathcal{Y}(b,q)) - b^{-1}(q^{\frac{1}{2}}-1)^{-1}\Big\{q^{-1}x^2+b-1\Big\}\mathcal{Y}(b,q) = 0. \tag{3.17}$$

The linear form $\mathcal{Y}(b,q)$ is positive definite when

$$0<q<1\ \ ,\ \ 0<b<1\ \ ;\ \ q>1\ \ ,\ \ b<0.$$

Remark 11. The (MOPS) $\{\mathcal{F}_n(x|q)\}_{n\geq 0}$ cited in [13] is in fact orthogonal by respect to $h_{(1-q^{-1})^{\frac{1}{2}}}\mathcal{Y}(q^{-\alpha-1},q^{-1})$ with $\alpha \neq -n-1,\ \ n \geq 0$.

Remark 12. The (MOPS) $\{\widetilde{\mathcal{H}}_n(x;q)\}_{n\geq 0}$ obtained after normalization of $\{\mathcal{H}_n(x;q)\}_{n\geq 0}$ in [3] is associated to the regular linear form

$$h_{-iq^{-\frac{1}{2}}}\mathcal{Y}(q^{-\mu-\frac{1}{2}},q^{-1})$$

for $\mu \neq -n-\frac{1}{2},\ \ n \geq 0$.

Let now $\mathcal{W}(b,q),\ b \neq 0, b \neq q^{-n}\ n \geq 0$ be the Wall H_q-classical linear form. We have (see [15]):

$$H_q(x\mathcal{W}(b,q)) - b^{-1}(q-1)^{-1}\Big\{q^{-1}x+b-1\Big\}\mathcal{W}(b,q) = 0. \tag{3.18}$$

In the following proposition we are going to establish the relationship between $\mathcal{Y}(b,q)$ and $\mathcal{W}(b,q)$.

Proposition 13. *We have*

$$\sigma\mathcal{Y}(b,q) = \mathcal{W}(b,q). \tag{3.19}$$

Proof. Applying the operator σ to the both sides of (3.17) and in accordance of (1.6)-(1.7) we get

$$H_q\Big(x\big(\sigma\mathcal{Y}(b,q)\big)\Big) - b^{-1}(q-1)^{-1}\Big\{q^{-1}x+b-1\Big\}\big(\sigma\mathcal{Y}(b,q)\big) = 0. \tag{3.20}$$

Moreover, the linear form $\mathcal{Y}(b,q)$ is symmetric and regular then $\sigma\mathcal{Y}(b,q)$ is regular, see [5, 19]. Taking into account (3.20) the linear form $\sigma\mathcal{Y}(b,q)$ is H_q-classical. Lastly, the comparison with (3.18) yields the desired result (3.19). ☐

3.2. Moments and Integral Representation of $\mathcal{Y}(b,q)$

Let us first recall the following standard notations, see [9]

$$(a;q)_0 := 1\,; \qquad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1})\,, \qquad n \geq 1 \tag{3.21}$$

$$(a;q)_\infty := \prod_{k=1}^{+\infty}(1-aq^{k})\,, \qquad |q|<1. \tag{3.22}$$

We now give a general result on moments and integral representation needed to the sequel.

Lemma 14. *Let v be a positive definite linear form with Supp $\subset \mathbb{R}_+$ and u a symmetric regular linear form such that*

$$\sigma u = v. \tag{3.23}$$

(i) The moments of u are

$$(u)_{2n} = (v)_n \quad , \qquad (u)_{2n+1} = 0\,, \qquad n \geq 0. \tag{3.24}$$

(ii) If v possesses the integral representation

$$\langle v, f\rangle = \int_0^{+\infty} V(x)f(x)dx\,,\ f\in\mathcal{P}\,,\ \int_0^{+\infty} V(x)dx = 1\,, \tag{3.25}$$

then a possible integral representation of u is

$$\langle u, f\rangle = \int_{-\infty}^{+\infty} |x|V(x^2)f(x)dx\,,\ \ f\in\mathcal{P}. \tag{3.26}$$

(iii) If v possesses the discrete representation

$$v = \sum_{k=0}^{+\infty}\rho_k\delta_{\tau_k}\,,\ \sum_{k=0}^{+\infty}\rho_k = 1\,, \tag{3.27}$$

then a possible discrete measure of u is

$$u = \sum_{k=0}^{+\infty}\rho_k\frac{\delta_{(\tau_k)^{\frac{1}{2}}}+\delta_{-(\tau_k)^{\frac{1}{2}}}}{2}. \tag{3.28}$$

Proof. The result in (i) is a consequence of (3.23) and the definition of the operator σ.

For (ii)-(iii) consider $f \in \mathcal{P}$ and let us split up the polynomial f according to its even and odd parts

$$f(x) = f^e(x^2) + x f^o(x^2). \tag{3.29}$$

Therefore

$$\langle u, f(x)\rangle = \langle u, f^e(x^2)\rangle = \langle \sigma u, f^e(x)\rangle = \langle v, f^e(x)\rangle \tag{3.30}$$

since u is symmetric and (3.23). Also from (3.29) we get

$$f^e(x) = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}, \quad x \in \mathbb{R}_+. \tag{3.31}$$

In view of (3.25) and (3.27) and by the fact of (3.30)-(3.31) we recover representations in (3.26) and (3.28) after an easy calculation. □

Finally, the above result allows us to give moments and integral representation related to the linear form $\mathcal{Y}(b,q)$ since the relationship (3.19) and the well known components of the Wall H_q-classical linear form; see pp. 91-92 in [15].

Corollary 15. *For $\mathcal{Y}(b,q)$ the following q-identities hold*

$$(\mathcal{Y}(b,q))_{2n} = q^n (b;q)_n\,, \quad (\mathcal{Y}(b,q))_{2n+1} = 0\,,\ n \geq 0. \tag{3.32}$$

$$\langle \mathcal{Y}(b,q), f\rangle = K \int_{-1}^{1} |x|^{2\tau-1}(x^2;q)_\infty f(x)dx\,,\ f \in \mathcal{P}\,,\ 0<q<1,\ 0<b<1, \tag{3.33}$$

with $b := q^\tau, \tau > 0$ and $K^{-1} = \int_0^1 t^{\tau-1}(t;q)_\infty dt$.

$$\mathcal{Y}(b,q) = (b;q)_\infty \sum_{k=0}^{+\infty} \frac{b^k}{(q;q)_k} \frac{\delta_{q^{\frac{1+k}{2}}} + \delta_{-q^{\frac{1+k}{2}}}}{2}\,,\ 0<q<1,\ |b|<1, \tag{3.34}$$

$$\mathcal{Y}(b,q) = \frac{1}{(bq^{-1};q^{-1})_\infty} \sum_{k=0}^{+\infty} \frac{q^{-\frac{1}{2}k(k+1)}(-b)^k}{(q^{-1};q^{-1})_k} \frac{\delta_{q^{\frac{1+k}{2}}} + \delta_{-q^{\frac{1+k}{2}}}}{2}\,,\ q>1, \tag{3.35}$$

for $b<0$ or $0<b<1$.

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