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SOME SOLUTIONS FOR A CLASS OF THIRD ORDER EQUATIONS AND THEIR ITERATES

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Abstract: We obtain solutions of radial type for a class of linear partial differential equations with singular coefficients and of third order. We also obtain similar type of solutions for the iterated forms of the same equations.

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1. Introduction

Several authors have studied second order partial differential equations of various types and their iterates [1-4]. In [1], radial type solutions for a class of singular equations are obtained. The essential operators there were second order elliptic or ultra-hyperbolic. In [2], Altın obtained a solution for the iterates of a class of partial differential equations.

In [4], the class of equations

$$Lu = \sum_{i=1}^{n} \left(\frac{r}{x_i}\right)^p \left[x_i^2 \frac{\partial^2 u}{\partial x_i^2} + \alpha_i x_i \frac{\partial u}{\partial x_i}\right] + \lambda u = 0 \tag{1.1}$$

are considered with $r^p = x_1^p + \cdots + x_n^p$ and some expansion formulas and Kelvin principle are obtained for these equations and their iterates.

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In this study, using a similar method given in [1], [2] we will obtain solutions of type r^m for the equations of the form

$$\left(\prod_{j=1}^{\nu} L_j^{q_j}\right) u = \left(L_1^{q_1} \dots L_\nu^{q_\nu}\right) u = 0\,, \tag{1.2}$$

where ν, $q_1, \dots, q_\nu$ are positive integers and

$$L_j = \sum_{i=1}^{n} \{ \left(\frac{r}{x_i}\right)^p \frac{1}{x_i^{p-3}} \frac{\partial^3}{\partial x_i^3} - p \left(\frac{r}{x_i}\right)^p \frac{1}{x_i^{p-2}} \frac{\partial^2}{\partial x_i^2}$$

$$+ \left[2\,(p-1) \left(\frac{r}{x_i}\right)^p + \gamma_i^j\right] \frac{1}{x_i^{p-1}} \frac{\partial}{\partial x_i} \} + \frac{\lambda_j}{r^p}\,. \tag{1.3}$$

The domain of the operator L_j is the set of all real-valued functions $u(x_1, \dots, x_n)$ of class $C^3(D)$ where D is a regularity domain of u in R^n. The iterated operators $L_j^{q_j}$ are defined by the relations

$$L_j^{k+1}(u) = L_j \left[L_j^k(u)\right]; k = 1, \dots, q_j - 1\,.$$

In (1.3) γ_i^j and λ_j $(i = 1, \dots, n, j = 1, \dots, \nu)$ are real parameters, $p(> 0)$ is a real constant and r is defined by

$$r^p = \sum_{i=1}^{n} x_i^p = x_1^p + \dots . + x_n^p. \tag{1.4}$$

2. Solutions of Type r^m

In this section, we will find solution of type r^m, which means solution depends upon r^m, where m is any real or complex parameter, for equation (1.2).

Now, we first establish the following lemma.

Lemma 2.1. *Let ν and $q_1, \dots, q_\nu$ be arbıtrary positive integers and m be a real or complex parameter. Then*

$$\left(\prod_{j=1}^{\nu} L_j^{q_j}\right)(r^m) = \prod_{j=1}^{\nu} \prod_{k=0}^{q_j-1} \Phi_j \left[m - p\left[Q\left(\nu\right) - Q\left(j\right)\right] - pk\right] r^{m-pQ(\nu)}\,, \tag{2.1}$$

where $Q(j) = q_1 + \cdots + q_j$, $1 \leq j \leq \nu$ *and* $\Phi_j(m)$ *is a thirth degree polynomial given by*

$$\Phi_j\,(m) = m^3 + (-3p + 2np - 3n)m^2 + (2p^2 - 2np^2 + 3np + \gamma_j^*)m + \lambda_j \quad (2.2)$$

with $\gamma_j^* = \sum\limits_{i=1}^{n} \gamma_i^j$.

Proof. From the definitions of L_j and r, for any real or complex parameter m, by direct calculation, it is easily seen that

$$L_j(r^m) = \Phi_j(m) r^{m-p}\,. \quad (2.3)$$

Applying the operator L_j repeatedly $q-1$ times on both sides of (2.3), we then obtain

$$L_j^q(r^m) = \left\{ \prod_{k=0}^{q-1} \Phi_j(m-kp) \right\} r^{m-qp}\,. \quad (2.4)$$

Replace q in (2.4) by q_j, then we have

$$L_j^{q_j}(r^m) = \left\{ \prod_{k=0}^{q_j-1} \Phi_j(m-kp) \right\} r^{m-q_j p}\,. \quad (2.5)$$

Now we will show the truth of (2.1) by induction on p. Hence for $j = 1$, (2.5) can be written as

$$L_1^{q_1}(r^m) = \left\{ \prod_{k=0}^{q_1-1} \Phi_j(m-kp) \right\} r^{m-pq_1} \quad (2.6)$$

considering the relation $Q(j) = q_1 + \cdots + q_j, 1 \leq j \leq \nu$, we can write (2.6) as

$$\left(\prod_{j=1}^{1} L_j^{q_j} \right) (r^m) = \prod_{j=1}^{1} \prod_{k=0}^{q_j-1} \Phi_j \left[m - p \left[Q\,(1) - Q\,(j) \right] - pk \right] r^{m-pQ(1)}\,.$$

Therefore, (2.1) is true for $\nu = 1$. Now assume that (2.1) is valid for $v-1$, that is,

$$\left(\prod_{j=1}^{\nu-1} L_j^{q_j} \right) (r^m)$$

$$= \prod_{j=1}^{\nu-1} \prod_{k=0}^{q_j-1} \Phi_j \left[m - p\left[Q\left(\nu-1\right) - Q\left(j\right)\right] - pk\right] r^{m-pQ(\nu-1)}. \quad (2.7)$$

We set $j = \nu$ in (2.5), we have

$$L_\nu^{q_\nu}\left(r^m\right) = \left\{ \prod_{k=0}^{q_\nu-1} \Phi_\nu(m-kp) \right\} r^{m-q_\nu p}. \quad (2.8)$$

Applying the linear operator $\prod_{j=1}^{\nu-1} L_j^{q_j}$ on both sides of (2.8), we then find

$$\left(\prod_{j=1}^{\nu-1} L_j^{q_j} \right) L_\upsilon^{q_\nu}\left(r^m\right) = \prod_{k=0}^{q_\nu-1} \Phi_\nu(m-kp) \prod_{j=1}^{\nu-1} L_j^{q_j}\left(r^{m-q_\nu p}\right). \quad (2.9)$$

If we replace m in (2.7) by $m - pq_\nu$, then (2.9) can be written as

$$\left(\prod_{j=1}^{\nu} L_j^{q_j} \right) \left(r^m\right) = \prod_{k=0}^{q_\nu-1} \Phi_\nu(m-kp)$$
$$\times \prod_{j=1}^{\nu-1} \prod_{k=0}^{q_j-1} \Phi_j \left[m - q_\nu p - p\left[Q\left(\nu-1\right) - Q\left(j\right)\right] - pk\right] r^{m-q_\nu p-pQ(\nu-1)}. \quad (2.10)$$

Since $pq_\nu + pQ(\nu - 1) = pQ(\nu)$, (2.10) gives formula (2.1). Thus, the lemma is proved. □

We now turn to formula (2.1) and write the algebraic polynomial equation

$$\prod_{j=1}^{\nu} \prod_{k=0}^{q_j-1} \Phi_j \left[m - p\left[Q\left(\nu\right) - Q\left(j\right)\right] - pk\right] = 0 \quad (2.11)$$

which is degree of $3Q(\nu)$. The number of real or complex roots of equation (2.11) is $3Q(\nu)$, $(1 \le j \le \nu)$.

Now using Lemma 1, we can prove the following theorem.

Theorem 2.2. *Let the algebraic polynomial equation (2.11) have distinct real roots $c_1, \ldots, c_M$, each having multiplicity $\xi_1, \ldots, \xi_M$, respectively, and distinct complex roots $\alpha_1 \pm i\beta_1, \ldots, \alpha_N \pm i\beta_N$, each having multiplicity $\eta_1, \ldots, \eta_N$, respectively. Then the solutions of type r^m of the equation (1.2) are given by formula*

$$u(r)=\sum_{w=1}^{M}\sum_{k_1=0}^{\xi_w-1}A_{wk_1}r^{c_w}(\ln r)^{k_1}+\sum_{s=1}^{N}\sum_{k_2=0}^{\eta_s-1}r^{\alpha_s}(\ln r)^{k_2}$$
$$\times\left[B_{sk_2}\cos(\beta_s\ln r)+C_{sk_2}\sin(\beta_s\ln r)\right], \quad (2.12)$$

where A_{wk_1}, B_{sk_2}, C_{sk_2} are arbitrary constants.

Proof. By the hypothesis concerning the real and complex roots of (2.11), we have the following two factors for this algebraic equation

$$\prod_{w=1}^{M}(m-c_w)^{\xi_w} \text{ and } \prod_{s=1}^{N}(m^2-2\alpha_s m+\alpha_s^2+\beta_s^2)^{\eta_s}.$$

Therefore, (2.1) can be written as

$$\left(\prod_{j=1}^{\nu}L_j^{q_j}\right)(r^m)=\left\{\prod_{w=1}^{M}(m-c_w)^{\xi_w}\prod_{s=1}^{N}(m^2-2\alpha_s m+\alpha_s^2+\beta_s^2)^{\eta_s}\right\}$$
$$\times r^{m-pQ(\nu)}, \quad (2.13)$$

where $\sum_{w=1}^{M}\xi_w+2\sum_{s=1}^{N}\eta_s=3Q(\nu)$ is the order of equation (1.2).

On the other hand, the following equalities are well known

$$\frac{\partial^l}{\partial m^l}\left(\prod_{j=1}^{\nu}L_j^{q_j}\right)(r^m)=\left(\prod_{j=1}^{\nu}L_j^{q_j}\right)\left(\frac{\partial^l r^m}{\partial m^l}\right)$$
$$=\left(\prod_{j=1}^{\nu}L_j^{q_j}\right)\left(r^m(\ln r)^l\right), \quad l\in N, \quad (2.14)$$

$$r^{\alpha_s\pm i\beta_s}=r^{\alpha_s}r^{\pm i\beta_s}=r^{\alpha_s}e^{\pm i\beta_s\ln r}=r^{\alpha_s}\left[\cos(\beta_s\ln r)\pm i\sin(\beta_s\ln r)\right]. \quad (2.15)$$

Now again consider (2.13). It is obvious that the right hand side of (2.13) has the factors $(m-c_w)^{\xi_w}$; $w=1,\dots,M$ which vanish for $m=c_w$; $w=1,\dots,M$ together with its derivatives with respect to m

$$\frac{d^{k_1}}{dm^{k_1}}(m-c_w)^{\xi_w}, k_1=1,\dots,\xi_w-1, w=1,\dots,M.$$

Thus, the function r^{c_w} and from (2.14), each of the functions

$$\left.\frac{d^{k_1}r^m}{dm^{k_1}}\right|_{m=c_w}=r^{c_w}(\ln r)^{k_1};\ k_1=1,\dots,\xi_w-1, w=1,\dots,M$$

satisfies equation (1.2). Since the given equation is linear, by the superposition principle the

$$\sum_{w=1}^{M}\sum_{k_1=0}^{\xi_w-1} A_{wk_1} r^{c_w} (\ln r)^{k_1} \tag{2.16}$$

also satisfies (1.2). Similarly the factors of (2.13)

$$\left(m^2 - 2\alpha_s m + \alpha_s^2 + \beta_s^2\right)^{\eta_s} = [m - (\alpha_s + i\beta_s)]^{\eta_s} [m - (\alpha_s - i\beta_s)]^{\eta_s} \ , \qquad s = 1, \dots, N$$

and the expressions

$$\frac{d^{k_2}}{dm^{k_2}} [m - (\alpha_s \pm i\beta_s)]^{\eta_s} \, ; \, k_2 = 1, \dots, \eta_s - 1, \, s = 1, \dots, N$$

are zero for $m = \alpha_s \pm i\beta_s$. Hence by (2.14) and (2.15) for $k_2 = 0, 1, \dots, \eta_s - 1$; $s = 1, \dots, N$ each of the functions

$$\left.\frac{d^{k_2} r^m}{dm^{k_2}}\right|_{m=\alpha_s \pm i\beta_s} = r^{\alpha_s \pm i\beta_s} (\ln r)^{k_2} = r^{\alpha_s} (\ln r)^{k_2} [\cos(\beta_s \ln r) \pm i \sin(\beta_s \ln r)]$$

and their superposition

$$\sum_{s=1}^{N}\sum_{k_2=0}^{\eta_s-1} r^{\alpha_s} (\ln r)^{k_2} [B_{sk_2} \cos(\beta_s \ln r) + C_{sk_2} \sin(\beta_s \ln r)] \tag{2.17}$$

satisfy (1.3). Therefore, the sum of (2.16) and (2.17) gives (2.12). Thus the theorem is proved. □

3. Solution of Type $u = u(r)$

In this section, we will show that all solutions which depend only on r for the equation (1.3) can be expressed by formula (2.12).

Lemma 3.1. *Let q be an arbitrary positive integer. Then for the function $u = u(r)$*

$$L_j^q u = e^{-pqt} \left\{ \prod_{k=0}^{q-1} \Phi_j(D - pk) \right\} u \, , \tag{3.1}$$

where $D = \frac{d}{dt}$, $r = e^t$ and Φ_j as given by (2.2).

Proof. We will prove this lemma by induction on q. Noticing definition of r given by (1.5), if we apply operator L_j to $u = u(r)$, then we find

$$\begin{aligned} L_j u(r) = \Bigg[& r^3 \frac{\partial^3 u}{\partial r^3} + [2np + 3(1-n-p)]\, r^2 \frac{\partial^2 u}{\partial r^2} \\ & + \left\{ (1-p)(2np - 3n + 1 - 2p) + \sum_{i=1}^{n} \gamma_i^j \right\} r \frac{\partial u}{\partial r} + \lambda_j u \Bigg] r^{-p} . \end{aligned} \tag{3.2}$$

It is easy to see that L_j becomes an Euler type operator. We let $r = e^t$, then we have

$$\frac{d}{dr} = e^{-t} D\,, \qquad \frac{d^2}{dr^2} = e^{-2t}(D^2 - D)\,,$$
$$\frac{d^3}{dr^3} = e^{-3t}(D^3 - 3D^2 + 2D)\,.$$

Thus, substituting into (3.2), we obtain

$$\begin{aligned} L_j u(r) = e^{-pt} \big[& D^3 - 3D^2 + 2D + (2np + 3(1-n-p))\,(D^2 - D) \\ & + \left\{ (1-p)(2np - 3n + 1 - 2p) + \sum_{i=1}^{n} \gamma_i^j \right\} D + \lambda_j \Bigg] u = e^{-pt} \\ \times & \left[D^3 + (2np + 3(-n-p))D^2 + (2p^2 - 2np^2 + 3np + \sum_{i=1}^{n} \gamma_i^j) D + \lambda_j \right] u\,, \end{aligned}$$

$$L_j u(r) = e^{-pt} \Phi_j(D) u\,. \tag{3.3}$$

Hence, (3.1) is true for $q = 1$. Now we suppose that (3.1) is true for $q - 1$, that is,

$$L_j^{q-1} u = e^{-p(q-1)t} \left\{ \prod_{k=0}^{q-2} \Phi_j(D - pk) \right\} u\,. \tag{3.4}$$

Applying the operator L_j on both sides of (3.4), we find

$$L_j^q u = L_j \left(e^{-p(q-1)t} \left\{ \prod_{k=0}^{q-2} \Phi_j(D - pk) \right\} u \right) .$$

We know from (3.3) that $L_j = e^{-pt}\Phi_j(D)$, therefore the right-hand side of the above equality can be written as

$$L_j^q u = e^{-pt} \Phi_j(D) \left(e^{-p(q-1)t} \left\{ \prod_{k=0}^{q-2} \Phi_j(D - pk) \right\} u \right) . \tag{3.5}$$

From ordinary differential equations, it is known that, for any two polynomials of the operator D with constant coefficients G and H and for any constant α, the following relation holds [1]

$$G(D)\left\{e^{-\alpha t}H(D)u\right\} = e^{-\alpha t}G(D-\alpha)H(D)u\,. \tag{3.6}$$

Using this property, we can write (3.5) as

$$\begin{aligned} L_j^q u &= e^{-pt}e^{-p(q-1)t}\Phi_j\left(D-p(q-1)\right)\left\{\prod_{k=0}^{q-2}\Phi_j\left(D-pk\right)\right\}u \\ &= e^{-pqt}\left\{\prod_{k=0}^{q-1}\Phi_j\left(D-pk\right)\right\}u. \end{aligned}$$

Thus, the proof is completed. □

Lemma 3.2. *Let ν and $q_1,\ldots,q_v$ be arbitrary positive integers. Then*

$$\left(\prod_{j=1}^{\nu}L_j^{q_j}\right)u = e^{-Q(\nu)pt}\prod_{j=1}^{\nu}\prod_{k=0}^{q_j-1}\Phi_j\left(D-p\left[Q(v)-Q(j)\right]-pk\right)u. \tag{3.7}$$

Proof. Using (3.1), it is easily proved in a similar manner of Lemma 2.

Now, we will set the following theorem.

Theorem 3.3. All solutions of type $u=u(r)$ for equation (1.2) can be expressed by formula (2.12).

Proof. Equating (3.7) to zero, we find the following ordinary differential equation with constant coefficients and of order $3Q(v)=3(q_1+\cdots+q_v)$.

$$\prod_{j=1}^{v}\prod_{k=0}^{q_j-1}\Phi_j\left(D-p\left[Q(v)-Q(j)\right]-pk\right)u = 0. \tag{3.8}$$

The characteristic equation of (3.8) is

$$\prod_{j=1}^{v}\prod_{k=0}^{q_j-1}\Phi_j\left(m-p\left[Q(v)-Q(j)\right]-pk\right)u = 0.$$

This was obtained in Lemma 1. Therefore, from Theorem 1, we know that this equation has the following factors

$$\prod_{w=1}^{M}(m-c_w)^{\xi_w} \text{ and } \prod_{s=1}^{N}(m^2-2\alpha_s m+\alpha_s^2+\beta_s^2)^{\eta_s}.$$

Hence solution of (3.8) is given by

$$u(t) = \sum_{w=1}^{M} \sum_{k_1=0}^{\xi_w - 1} A_{wk_1} t^{k_1} e^{c_w t} + \sum_{s=1}^{N} \sum_{k_2=0}^{\eta_s - 1} e^{\alpha_s t} t^{k_2} \left[B_{sk_2} \cos(\beta_s t) + C_{sk_2} \sin(\beta_s t) \right] \quad (3.9)$$

Since $e^t = r$, we set $t = \ln r$ in (3.9), we find formula (2.12). Thus the theorem is proved. □

References

[1] A. Altın, Solution of type r^m for a class of singular equations, *Int. J. Math. and Math. Sci.*, **5** (1982), 613-619.

[2] A. Altın, Radial type solutions for a class of third order equations and their iterates, *Mathematica Slovaca*, **49** (1999), 183-187.

[3] A.O. Celebi, On the generalized Tricomi's equation, *Comm. Fac. Sci. Univ. Ankara*, Ser. A, **17** (1986), 1-31.

[4] N. Özalp, A. Çetinkaya, Radial solutions of a class of iterated partial differential equations, *Czechoslovak Math. J.*, **55**, No. 2 (2005), 531-541.

[5] A. Weinstein, On a class of partial differential equations of even order, *Ann. Mat. Pura. Appl.*, **39**, No. 4 (1955), 245-254.