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COHERENT SYSTEMS ON $\mathbf{P}^n$ OBTAINED FROM THE TANGENT AND THE COTANGENT BUNDLE

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: We study the α-stability of the coherent systems on $\mathbf{P}^n$ with as bundles $T\mathbf{P}^n)(t)$. $\Omega^1_{\mathbf{P}^n}(t)$ and $T\mathbf{P}^n \otimes \Omega^1_{\mathbf{P}^n}(t)$.

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For the definition of coherent systems, see [3], [6] and references therein. Here we will use [6] (and a few other computations on elliptic curves) to prove the following results.

Theorem 1. *Fix integers n, t_i, $i = 1, 2, 3$, such that $n \geq 3$, $t_1 \geq 2$, $t_2 \geq 3$ and $t_3 \geq 3$. Set $E_1 := T\mathbf{P}^n(t_1)$, $E_2 := \Omega^1_{\mathbf{P}^n}(t_2)$, $E_3 := T\mathbf{P}^n \otimes \Omega^1_{\mathbf{P}^n}(t_3)$, $n_1 := n_2 := n$, $n_3 := n^2$, $d_1 := (n+1)^2 + n(n+1)t_1$, $d_2 := -(n+1)^2 + n(n+1)t_2$ and $d_3 := n^2(n+1)t_3$. Fix integers k_i, $1 \leq i \leq 3$, such that $0 < k_i \leq d_i$ and $\alpha_i in \mathbb{R}$, $1 \leq i \leq 3$, $\alpha_i > 0$. If $k_i < n_i$, then assume $\alpha_i < d_i/(n_i - k_i)$. We have $h^0(\mathbf{P}^n, E_i) \geq d_i$ for all i. Let V_i be a general k_i-dimensional linear subspace of $H^0(\mathbf{P}^n, E_i)$. Then for all i the coherent system (E_i, V_i) is α_i-stable.*

Notice that $T\mathbf{P}^n \otimes \Omega^1_{\mathbf{P}^n}(t)$ is semistable, but not stable. Hence for $i = 3$ in the statement of Theorem 1 we need the restriction $\alpha_3 > 0$, while for $i = 1, 2$ the case $\alpha_i = 0$ is trivially true.

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

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The following remark is well known (see [4] and [2] for much more).

Remark 1. Fix an integer $n \geq 1$, an elliptic curve X and a rank r semistable vector bundle E on X with rank n and degree $n+1$. Since $n+1$ and n are coprime, E is stable. Notice that $h^1(X,E) = 0$, $h^0(X,E) = n+1$, E is spanned, $h^0(X, \det(E))$ and $E^* \cong \mathrm{Ker}(\epsilon)$, where $\epsilon : H^0(X, \det(E)) \otimes \mathcal{O}_X \to \det(E)$ is the evaluation map. Since $\det(E)$ is very ample, we see that the pair $(E, H^0(C,E)$ determines an embedding $j : X \to \mathbf{P}^n$ such that $E \cong j^*(T\mathbf{P}^n(-1))$. The degree $n+1$ curve $j(X)$ is a linearly normal elliptic curve. Hence there is a linearly normal curve elliptic curve $Y := j(X) \subset \mathbf{P}^n$ such that $T\mathbf{P}^n(t)|Y$ and $\Omega^1_{\mathbf{P}^n}(t)$ are stable for all $t \in \mathbb{Z}$.

Proof of Theorem 1. Look at the twists of the Euler's sequence of $T\mathbf{P}^n$ and of its dual:

$$0 \to \mathcal{O}_{\mathbf{P}^n}(t) \to \mathcal{O}_{\mathbf{P}^n}(t+1)^{\oplus(n+1)} \to T\mathbf{P}^n(t) \to 0\,, \tag{1}$$

$$0 \to \Omega^1_{\mathbf{P}^n}(t) \to \mathcal{O}_{\mathbf{P}^n}(t-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n}(t) \to 0\,. \tag{2}$$

Let $X \subset \mathbf{P}^n$ be a linearly normal elliptic curve such that $T\mathbf{P}^n|X$ is stable (Remark 1). Set $F_i := E_i(-t_i)|X$, $1 \leq i \leq 3$. Hence F_1 and F_2 are stable. By [1], Lemma 22, F_3 is the direct sum of all n-roots of $\mathcal{O}_X$. Hence F_3 is the direct sum of n^2 pairwise non-isomorphic line bundles with the same degree. Tensor (1) by $\mathcal{I}_X$ and use that $h^1(\mathbf{P}^n, \mathcal{I}_X(t)) = 0$ for all $t \geq 2$ and that $h^2(\mathbf{P}^n, \mathcal{I}_X(t)) = 0$ for all $t \geq 2$. We get $h^1(\mathbf{P}^n, \mathcal{I}_X \otimes T\mathbf{P}^n(t)) = 0$ and the surjectivity of the restriction map $H^0(\mathbf{P}^n, T\mathbf{P}^n(t)) \to H^0(C, T\mathbf{P}^n(t)|X)$ for all $t \geq 1$. Tensor (2) by $\mathcal{I}_X$ and use that the homogeneous ideal of X is generated in degree 2 if $n \geq 3$ 2, i.e. that the map $H^0(\mathbf{P}^n, \mathcal{I}_X(t-1)^{\oplus(n+1)}) \to H^0(\mathbf{P}^n, \mathcal{I}_X(t))$ is surjective for all $t \geq 3$ when $n \geq 3$, and in degree 3 for $n = 2$. We get the surjectivity of the restriction map $H^0(\mathbf{P}^n, \Omega^1_{\mathbf{P}^n}(t)(t)) \to H^0(X, \Omega^1_{\mathbf{P}^n}(t)|X)$ for all $t \geq 3$ when $n \geq 3$, all $t \geq 4$, when $n = 2$. Hence for $i = 1, 2$ we get $h^0(\mathbf{P}^n, E_i) \geq d_i$ and that the restriction map $\rho_i : H^0(\mathbf{P}^n, E_i) \to H^0(X, E_i|X)$ is surjective. Since $\dim(V_i) \leq d_i = h^0(X, E_i|X)$ and V_i is general, $\rho_i(V_i)$ is a general k_i-dimensional linear subspace of $H^0(X, E_i|X)$. Hence the coherent system $(E_i|X, \rho_i(V_i))$ is α_i-stable. Now we will do the case $i = 3$. Since $\dim(X) = 1$, we have $h^2\mathbf{P}^n, \mathcal{I}_X \otimes \Omega^1_{\mathbf{P}^n}(z)) = h^0(\mathbf{P}^n, \Omega^1_{\mathbf{P}^n}(z))$ for all z. Hence $h^2\mathbf{P}^n, \mathcal{I}_X \otimes \Omega^1_{\mathbf{P}^n}(z)) = 0$ for all z. We new that $h^1\mathbf{P}^n, \mathcal{I}_X \otimes \Omega^1_{\mathbf{P}^n}(z)) = 0$ for all $z \geq 3$. Since $t_3 \geq ++$, tensor(1) with $\mathcal{I}_X \otimes \Omega^1_{\mathbf{P}^n}$, we get the surjectivity of the restriction map $H^0(\mathbf{P}^3, E_3) \to H^0(X, E_3|X)$. Now we apply again [6], except that now F_3 cannot be considered a general polystable vector bundle with its degree and rank. However, since it is so simple (a direct sum of pairwise non-isomorphic line bundles) everything works. Assume that (E_i, V_i) is not α_i-stable and take an α_i-destabilizing coherent subsystem (G_i, W_i) with G_i torsion free. Hence G_i is locally free outside a closed subset $T \subset \mathbf{P}^n$ such that $\dim(T) \leq n-2$. By a transversality result due to Kleima (see [5]), there is $g \in \mathrm{Aut}(\mathbf{P}^n)$ such that $g(X) \cap T = \emptyset$. Hence $G_i|X$ is locally free. Since V_i is general and $\dim(V_i) \leq d_i$, the restriction map $j : V_i \to H^0(g(X), E_i|g(X))$ is injective. Hence the restriction map $j_{|W_i} : W_i \to H^0(g(X), G_i|g(X))$ is injective. Hence the coherent subsystem $(G_i|X, j(W_i))$ α_i-destabilizes the coherent system $(E_i|g(X), j(V_i))$, contradiction. □

Remark 2. In the proof of Theorem 1 we checked explicitly the case $n = 2$ of Theorem 1.

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