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EMBEDDINGS OF ELLIPTIC CURVES IN FLAG VARIETIES AND POLYSTABLE VECTOR BUNDLES

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Abstract: Let C be an elliptic curve. Here we prove the stability of chains of polystable vector bundles obtained from suitable embedding of C in flag varieties.

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Fix integers $s > 0$ and $n > r_1 > \cdots > r_s > 0$. Let $Fl(n, r_1, \dots, r_s)$ denote the flag variety of all linear subspaces $V_s \subset \cdots \subset V_1 \subset \mathbb{K}^n$, $\mathbb{K}$ an algebraically closed feld with $\mathrm{char}(\mathbb{K}) = 0$, such that $\dim(V_i) = n - r_i$ for all i. Hence on $Fl(n, r_1, \dots, r_s)$ there are $s+1$ universal bundles Q_i and r surjective morphisms $h_i : Q_{i-1} \to Q_i$, $1 \le i \le s$, such that $Q_0 = \mathcal{O}^{\oplus}_{Fl(n,r_1,\dots,r_s)}$ and $\mathrm{rank}(Q_i) = r_i$ for $1 \le i \le s$. For the main definition of coherent systems and holomorphic triples, see respectively [2], [3] and references therein.

Theorem 1. *Let C be an elliptic curve. Fix integers $s > 0$, $n > r_1 > \cdots > r_s > 0$ and $d_1, \dots, d_r$ such that $d_1 > n$ and $d_i/r_i < d_{i+1}/r_{i+1}$ for all $1 \le i \le s-1$. Fix $\alpha_i \in \mathbb{R}$, $0 \le i \le s-1$, such that $\alpha_0 > 0$ and $d_{i+1}/r_{i+1} - d_i/r_i < \alpha_i < (1 + (r_i + r_{i+1})/(r_{i+1} - r_i))(d_{i+1}/r_{i+1} - d_i/r_i)$ for all $1 \le i \le s-1$. Fix s polystable vector bundles E_i on C such that $\deg(E_i) = d_i$, $\mathrm{rank}(E_i) = r_i$,*

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the indecomposable factors of each E_i are pairwise non-isomorphic and E_1 is general. Then $h^0(C, E_1) = d_1$. Let $V \subset H^0(C, E_1)$ be a general linear subspace. The induced evaluation map $f_0 : V \otimes \mathcal{O}_C \to E_1$ is surjective. Let $f_i : E_i \to E_{i+1}$, $1 \le i \le s-1$, be a general morphism. Each f_i is surjective. These data induces an embedding $\phi : C \to Fl(n, r_1, \dots, r_s)$ such that $E_i \cong \phi^*(Q_i)$ for $1 \le i \le s$, and $f_i = \phi^*(h_i)$ for $0 \le i \le s-1$. The coherent system (E_1, V) is α_0-stable and each holomorphic triple (E_{i+1}, E_i, f_i), $1 \le i \le s-1$ is α_i-stable.

For nice embeddings of smooth curves of genus $g \ge 2$ into $Fl(n, r_1, \dots, r_s)$, see [1].

We work over an algebraically closed feld $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

Proof of Theorem 1. The existence of the pair (E_1, V) is [2], Theorem 5.4. Since E_1 is semistable and $d_1 > 0$, $h^1(C, E_1) = 0$. Hence $h^0(C, E_1) = d_1$ (Riemann-Roch) For the existence of E_i, $2 \le i \le s$ such that for all $1 \le i \le s-1$ the triple (E_{i+1}, E_i, f_i) is α_i-stable, see the dual of [3], Theorem 5.4. Since $d_1 > r_1$ and E_1 is semistable, $h^0(C, E_1(-P)) = h^0(C, E_1) - r_1$ for all $P \in C$ (i.e. E_1 is spanned) and $h^0(C, E_1(-2P)) < h^0(C, E_1(-P))$. These two properties implies that the map $\psi : C \to Fl(d_1, r_1)$ (a Grassmannian) induced by $H^0(C, E_1)$ is an embedding. Since $\dim(C) = 1$, $\dim(V) > \mathrm{rank}(E_1)$ and V is general, even the map $\beta : C \to Fl(n, r_1)$ is an embedding. Hence the map ϕ is an embedding. □

Taking duals and applying again [3], Theorem 5.4, to all maps j_i, $1 \le i \le s-1$, we easily get the following result.

Theorem 2. *Let C be an elliptic curve. Fix integers $s > 0$, $n > r_1 > \cdots > r_s > 0$ and $d_1, \dots, d_r$ such that $d_1 > n$ and $d_i/r_i < d_{i+1}/r_{i+1}$ for all $1 \le i \le s-1$. Fix $\alpha_i \in \mathbb{R}$, $0 \le i \le s-1$, such that $\alpha_0 > 0$ and $d_{i+1}/r_{i+1} - d_i/r_i < \alpha_i < (1 + (r_i + r_{i+1})/(r_{i+1} - r_i))(d_{i+1}/r_{i+1} - d_i/r_i)$ for all $1 \le i \le s-1$. Fix s polystable vector bundles E_i on C such that $\deg(E_i) = d_i$, $\mathrm{rank}(E_i) = r_i$, the indecomposable factors of each E_i are pairwise non-isomorphic, E_1 and E_s are general. Then $h^0(C, E_1) = d_1$. Let $V \subset H^0(C, E_1)$ be a general linear subspace. The induced evaluation map $f_0 : V \otimes \mathcal{O}_C \to E_1$ is surjective. Let $f_i : E_i \to E_{i+1}$, $1 \le i \le s-1$, be a general morphism. Each f_i is surjective. These data induces an embedding $\phi : C \to Fl(n, r_1, \dots, r_s)$ such that $E_i \cong \phi^*(Q_i)$ for $1 \le i \le s$, and $f_i = \phi^*(h_i)$ for $0 \le i \le s-1$. The coherent system (E_1, V) is α_0-stable and each holomorphic triple (E_{i+1}, E_i, f_i), $1 \le i \le s-1$ is α_i-stable. Set $u_i := f_i \circ \cdots \circ f_0$, $0 \le i \le s-1$, and $F_i := \mathrm{Ker}(u_i)$. Hence $\mathrm{rank}(F_i) = n - r_{i+1}$ and $\deg(F_i) = -d_{i+1}$. Dualizing each f_i, $1 \le i \le s-1$, we get injective morphisms $j_i : F_i \to F_{i-1}$. Dualizing u_{s-1} we get an injection with locally free cokernel $F_s \to \mathcal{O}_C^{\oplus n}$) whose dual is a coherent system (F_1^*, V) which is α_0-stable.*

Each triple (F_{i-1}, F_i, j_i), $1 \le i \le s-1$, *is* α_i*-stable.*

Notice that in the statement of Theorem 2 we only required that each E_1, E_s is general among the polystable vector bundles on C with its degree and rank, not that the pair (E_1, E_s) is general among the pairs of polystable vector bundles with fixed ranks and degrees.

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