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EXPONENTIAL PERIODICITY OF THE DISCRETE-TIME RECURRENT NEURAL NETWORKS WITH VARIABLE DELAYS AND IMPULSES

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Abstract: In this paper, the problems on the existence and global exponential stability of periodic solution are investigated for the impulsive discrete-time recurrent neural network with periodic coefficients and time-varying delays. By using analytical methods, inequality technique and M-matrix theory, several sufficient conditions are obtained to ensure the existence, uniqueness and global exponential stability of periodic solution for the addressed neural network. Moreover, the exponential convergence rate index is estimated, which depends on the system parameters. The obtained results in this paper improve and extend some previously related results. An example with simulation is given to show the effectiveness of the obtained results.

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1. Introduction

In last few decades, recurrent neural networks, especially Hopfield neural net-

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works, cellular neural networks and the neural networks of bidirectional associative memory, have been widely studied due to their applications in various areas such as pattern recognition, signal processing, associative memory, parallel computation and optimization problems [6].

In the hardware implementation of a neural network using analog electronic circuits, time delay is inevitable and occurs in the signal transmission among the neurons, which may lead to some complex dynamic behaviors [3]. Therefore, the study of the stability for delayed neural networks is of both theoretical and practical importance. In recent years, some results on the stability of the recurrent neural networks with delays have been obtained, see [3], [13], [17], [2], [18], [21], [16] and the references therein.

However, besides delay effects, impulsive effects are also likely to exist in neural networks [8]. For instance, in the implementation of electronic networks, the state of the networks is subject to instantaneous perturbation and experiential abrupt change at certain instants, which may be caused by switching phenomenon, frequent change or other sudden noise, that does exhibit the impulsive effects. Therefore, it is necessary to take both impulsive effects and delay effects on the stability of neural networks into consideration. Some results on the impulsive effects have been gained for delayed neural networks, see [8], [1], [7], [5], [25], [22], [23] and the references therein.

Moreover, studies on neural dynamical systems not only involve a discussion of the stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior, bifurcation, and chaos. In theories and applications, global exponential periodicity is an important dynamic property of recurrent neural networks, and the global exponential stability at an equilibrium point can be viewed as a special case of global exponential periodicity since an equilibrium point can be viewed as a special periodic solution of the neural networks with any arbitrary period [20]. The analysis of periodic solution of neural networks may be considered to be more general than that of equilibrium point [24]. As to neural networks with constant delay or time-varying delays, periodic oscillatory solution has been studied, see [20], [24], [19] and the references therein.

On the other hand, in numerical simulation and practical implementation of the continuous-time neural networks, it is essential to formulate a discrete-time system, that is an analogue of the continuous-time system. Therefore, it is of both theoretical and practical importance to study the dynamics of discrete-time neural networks, see [15]. Some authors have studied the dynamics of some discrete-time neural networks, see [15], [12], [10], [4], [14], [14], [27], [9], [29], [11], [26] and the references therein. In [15], [12], [10], [4], [14], the global exponential stability and the robust stability were studied for the discrete-time neural networks with constant and time-varying delays. In [28], the authors investigated the global exponential stability for the discrete-time neural network with variable delays and impulses. In [27], [9], [29], [11], [26], the authors discussed the existence and the global exponential stability of periodic solution and almost periodic solution for discrete-time neural networks. To our knowledge, few authors have studied the existence and the stability of periodic solutions for the discrete-time recurrent neural networks with variable delays and impulses.

Motivated by the discussions above, the objective of this paper is to study the existence and the stability of periodic solution of the discrete-time recurrent neural network with variable delays and impulses. By using analytic methods, inequality technique and M-matrix theory, we obtain several sufficient conditions ensuring the existence, uniqueness and global exponential stability of periodic solution for the addressed neural network. Moreover, the exponential convergence rate index is estimated.

2. Model Description and Preliminaries

In this paper, we consider the existence and global exponential stability of periodic solutions of the following model

$$\begin{cases} u_i(m+1) = c_i(m)u_i(m) + \sum\limits_{j=1}^{n} a_{ij}(m)f_j(u_j(m)) \\ \quad + \sum\limits_{j=1}^{n} b_{ij}(m)f_j(u_j(m-\tau_{ij}(m))) + I_i(m), \quad m \neq m_k, \\ u_i(m) = p_{ik}(u_1(m^-), \cdots, u_n(m^-)) + q_{ik}(u_1((m-\tau_{i1}(m))^-), \cdots, \\ u_n((m-\tau_{in}(m))^-)) + J_{ik}(m), \quad m = m_k \end{cases} \tag{1}$$

for $i = 1, 2, \cdots, n$; $\quad k = 1, 2, \cdots$. Here n corresponds to the number of units in the neural network; $u_i(m)$ corresponds to the state of the i-th unit at time m; f_j is the activation function; $\tau_{ij}(m)$ corresponds to the transmission delay along the axon of the j-th unit from the i-th unit and satisfies $0 \leq \tau_{ij}(m) \leq \tau$ (τ is a nonnegative integer); $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively; $I_i(m)$ denotes the exogenous input of the i-th neurons at time m; $c_i(m) \in (0, 1)$ denotes the passive decay rate of the i-th neurons at time m. m_k is called impulsive moment and satisfies $0 \leq m_1 < m_2 < \cdots$, $\lim_{k\to+\infty} m_k = +\infty$; $p_{ik}(u_1(m^-), \cdots, u_n(m^-))$ represents impulsive perturbations of the i-th unit at time m_k; $q_{ik}(u_1((m-\tau_{i1}(m))^-),\cdots,u_1((m-\tau_{in}(m))^-))$ denotes impulsive perturbations of the i-th unit at time m_k which caused by the transmission delays; J_{ik} represents external impulsive input at time m_k.

For convenience, we introduce some notations. $u=(u_1,u_2,\cdots,u_n)^T\in R^n$ denotes a column vector; $|u|$ denotes the absolute-value vector given by $|u|=(|u_1|,|u_2|,\cdots,|u_n|)^T$; $\|u\|$ denotes a vector norm defined by $\|u\|=\Big(\sum\limits_{i=1}^{n}|u_i|^2\Big)^{\frac{1}{2}}$. For matrix $A=(a_{ij})_{n\times n}\in R^{n\times n}$, $\rho(A)$ denotes the spectral radius of A; $|A|$ denotes the absolute-value matrix given by $|A|=(|a_{ij}|)_{n\times n}$; $\|A\|$ denotes a matrix norm defined by $\|A\|=\Big(\lambda_{max}(A^TA)\Big)^{\frac{1}{2}}$. $\mathrm{diag}(b_1,b_2,\cdots,b_n)$ denotes the diagonal matrix with diagonal entries $b_1,b_2,\cdots,b_n$. E denotes an $n\times n$ unit matrix. For integers a, b, and $a<b$, $N[a,b]$ denotes the discrete interval given $N[a,b]=\{a,a+1,\cdots,b-1,b\}$. $C(N[-\tau,0],R^n)$ denotes the set of all functions φ: $N[-\tau,0]\to R^n$. For a discrete periodic function $g(m)$ with positive integer period ω, we denote $\overline{g}=\sup_{m\in N[0,\omega]}|g(m)|$.

Throughout this paper, we make the following assumptions:

(**H1**) $c_i(m)$, $a_{ij}(m)$, $b_{ij}(m)$, $I_i(m)$, $\tau_{ij}(m)$, $p_{ik}(m)$, $q_{ik}(m)$ and $J_{ik}(m)$ $(i,j=1,2,\cdots,n;k=1,2,\cdots)$ are periodic function with positive integer period ω, and there exists a positive integer l such that

$$p_{i,k+l}=p_{ik},\quad q_{i,k+l}=q_{ik},\quad J_{i,k+l}=J_{ik},\quad i=1,2,\cdots,n,\quad k=1,2,\cdots.$$

(**H2**) There exists a positive diagonal matrices $F=\mathrm{diag}(F_1,F_2,\cdots,F_n)$ such that

$$|f_i(u_1)-f_i(u_2)|\le F_i|u_1-u_2|$$

for all u_1, $u_2\in R$, $i=1,2,\cdots,n$.

(**H3**) There exist nonnegative matrices $P_k=(p_{ij}^{(k)})_{n\times n}$ and $Q_k=(q_{ij}^{(k)})_{n\times n}$ such that

$$|p_{ik}(u_1,\cdots,u_n)-p_{ik}(v_1,\cdots,v_n)|\le\sum_{j=1}^{n}p_{ij}^{(k)}|u_j-v_j|,$$

$$|q_{ik}(u_1,\cdots,u_n)-q_{ik}(v_1,\cdots,v_n)|\le\sum_{j=1}^{n}q_{ij}^{(k)}|u_j-v_j|,$$

for all $(u_1,\cdots u_n)^T\in R^n$, $(v_1,\cdots v_n)^T\in R^n$, $i=1,2,\cdots,n$, $k=1,2,\cdots$.

Definition 1. For any given $\phi\in C(N[-\tau,0],R^n)$, a real valued sequence $u(m)\in C(N[m_0-\tau,+\infty),R^n)$ is called a solution of model (1) through $(0,\phi)$, if $u(m)$ satisfies the initial conditions in the form

$$u(m_0 + s) = \phi(s), \quad s \in N[-\tau, 0],$$

and satisfies model (1) for all $m \geq 0$. Especially, a point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$ is called an equilibrium point of model (1), if $u(m) = u^*$ is a solution of model (1).

Definition 2. (see [19]) A real matrix $A = (a_{ij})_{n\times n}$ is said to be a nonsingular M-matrix if $a_{ij} \leq 0$ $(i, j = 1, 2, \cdots, n; i \neq j)$ and successive principle minors of A are positive.

To prove our results, the following lemmas are necessary.

Lemma 1. (see [19]) *Let Q be $n \times n$ matrix with non-positive off-diagonal elements, if Q is a nonsingular M-matrix, then:*
(i) The real parts of all eigenvalues of Q are positive.
(ii) There exists a vector $\xi > 0$ such that $\xi^T Q > 0$.
(iii) There exist a positive diagonal matrix D such that $DQ + Q^T D$ is positive definite matrix.

Lemma 2. (see [22]) *Let A be a nonnegative matrix, then $\rho(A)$ is a nonnegative eigenvalue of A and its corresponding eigenvectors have at least one be positive.*

When A is a nonsingular M-matrix, B is a nonnegative matrix, we denote

$$\Omega(A) = \{\xi \in R^n | A\xi > 0, \xi > 0\}, \quad \Gamma(B) = \{\xi \in R^n | B\xi = \rho(B)\xi, \xi > 0\}.$$

Obviously, $\Omega(A)$ and $\Gamma(B)$ are nonempty.

3. Main Results

Theorem 1. *Under assumptions (H1)-(H3), model (1) has exactly one ω-periodic solution, and all other solutions of model (1) converge exponentially to it as $m \to +\infty$ and the exponential convergence rate index equals $\varepsilon - \lambda$, if the following conditions are satisfied:*
(i) $\Xi = E - \overline{C} - (\overline{A} + \overline{B})F$ is a nonsingular $M-$ matrix, where $\overline{C} = \text{diag}(\overline{c_1}, \cdots, \overline{c_n})$, $\overline{A} = (\overline{a_{ij}})_{n\times n}$, $\overline{B} = (\overline{b_{ij}})_{n\times n}$.
(ii) $\Delta = \bigcap\limits_{k=1}^{\infty} \left[\Gamma(P_k) \bigcap \Gamma(Q_k)\right] \bigcap \Omega(\Xi)$ is non-empty.
(iii) There exists a constant λ such that

$$\frac{\ln \gamma_k}{m_k - m_{k-1}} \leq \lambda < \varepsilon, \quad k = 1, 2, \cdots, \tag{2}$$

where the scalar $\varepsilon > 0$ is determined by the following inequality

$$\xi_i(-1 + e^{\varepsilon}\overline{c_i}) + e^{\varepsilon}\sum_{j=1}^{n}\xi_j F_j(\overline{a_{ij}} + e^{\varepsilon\tau}\overline{b_{ij}}) \leq 0 \tag{3}$$

for a given $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T \in \Delta$, and

$$\gamma_k \geq \max\{1, \rho(P_k) + e^{\varepsilon\tau}\rho(Q_k)\}. \tag{4}$$

Proof. Let $\Upsilon = C(N[-\tau, 0], R^n)$, for $\phi \in \Upsilon$, define

$$\|\phi\| = \sup_{s\in N[-\tau,0]}\Big(\sum_{i=1}^{n}|\phi_i(s)|^2\Big)^{\frac{1}{2}},$$

then Υ is a Banach space with the norm above.

Given any $\phi, \psi \in \Upsilon$, let $u(m, \phi) = (u_1(m, \phi), \cdots, u_n(m, \phi))^T$ and $u(m, \psi) = (u_1(m, \psi), \cdots, u_n(m, \psi))^T$ be the solutions of model (1) starting from ϕ and ψ, respectively. Defined $u^m(\phi) = u(m + s, \phi)$, $s \in N[-\tau, 0]$, $m \in N[m_0, +\infty)$, then $u^m(\phi) \in \Upsilon$ for all $m \in N[m_0, +\infty)$. Thus, it follows from model (1) that

$$\begin{cases} u_i(m+1, \phi) - u_i(m+1, \psi) = c_i(m)\Big(u_i(m, \phi) - u_i(m, \psi)\Big) \\ +\sum\limits_{j=1}^{n} a_{ij}(m)\Big(f_j(u_j(m, \phi)) - f_j(u_j(m, \psi))\Big) \\ +\sum\limits_{j=1}^{n} b_{ij}(m)\Big(f_j(u_j(m - \tau_{ij}(m), \phi)) - f_j(u_j(m - \tau_{ij}(m), \psi))\Big), \\ m \neq m_k, \\ u_i(m, \phi) - u_i(m, \psi) \\ p_{ik}(u_1(m^-, \phi), \cdots, u_n(m^-, \phi)) - p_{ik}(u_1(m^-, \psi), \cdots, u_n(m^-, \psi)) \\ +q_{ik}(u_1((m - \tau_{i1}(m))^-, \phi), \cdots, u_n((m - \tau_{in}(m))^-, \phi)) \\ -q_{ik}(u_1((m - \tau_{i1}(m))^-, \psi), \cdots, u_n((m - \tau_{in}(m))^-, \psi)), \quad m = m_k \end{cases} \tag{5}$$

for $i = 1, 2, \cdots, n; \quad k = 1, 2, \cdots$.

Using the discrete part of (5) and assumption (H2), we can obtain that

$$\begin{aligned} &|u_i(m+1, \phi) - u_i(m+1, \psi)| \\ &\quad \leq c_i(m)|u_i(m, \phi) - u_i(m, \psi)| + \sum_{j=1}^{n}|a_{ij}(m)|F_j|u_j(m, \phi) - u_j(m, \psi)| \\ &\qquad + \sum_{j=1}^{n}|b_{ij}(m)|F_j|u_j(m - \tau_{ij}(m), \phi) - u_j(m - \tau_{ij}(m), \psi)|, \end{aligned} \tag{6}$$

for $m \in N[m_{k-1}, m_k)$, $k = 1, 2, \cdots$, $i = 1, 2, \cdots, n$.

From Ξ is an M-matrix and Δ is nonempty, we know that there exists a vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T \in \Delta \subseteq \Omega(\Xi)$ such that

$$-\xi_i(1 - \overline{c_i}) + \sum_{j=1}^{n} \xi_j F_j(\overline{a_{ij}} + \overline{b_{ij}}) < 0, \quad i = 1, 2, \cdots, n. \tag{7}$$

Consider the following functions

$$L_i(x) = \xi_i(-1 + e^x \overline{c_i}) + e^x \sum_{j=1}^{n} \xi_j F_j \Big(\overline{a_{ij}} + e^{x\tau} \overline{b_{ij}}\Big), \quad i = 1, 2, \cdots, n.$$

From (7), we know that $L_i(0) < 0$ and $L_i(x)$ is continuous. Since $\frac{dL_i(x)}{dx} > 0$, $L_i(x)$ is strictly monotone increasing, there exists $\varepsilon_i > 0$ such that $L_i(\varepsilon_i) = 0$ for $i = 1, 2, \cdots, n$. Choosing $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$, then $\varepsilon > 0$ and inequality (3) hold. Let

$$x_i(m) = e^{\varepsilon(m - m_0)} |u_i(m, \phi) - u_i(m, \psi)|, \qquad i = 1, 2, \cdots, n.$$

Then, from (6), we have

$$\begin{aligned} x_i(m+1) &= e^{\varepsilon(m+1-m_0)} |u_i(m+1, \phi) - u_i(m+1, \psi)| \\ &\leq e^{\varepsilon} \Big(\overline{c_i} x_i(m) + \sum_{j=1}^{n} \overline{a_{ij}} F_j x_j(m) + \sum_{j=1}^{n} e^{\varepsilon\tau} \overline{b_{ij}} F_j x_j(m - \tau_{ij}(m))\Big) \end{aligned} \tag{8}$$

for $m \in N[m_{k-1}, m_k)$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$.

For the initial condition $u(m_0 + s) = \phi(s) \in C(N[-\tau, 0], R^n)$, let $l_0 = \frac{\|\phi - \psi\|}{\min\limits_{1 \leq i \leq n} \{\xi_i\}}$, then

$$x_i(s) \leq |u_i(s, \phi) - u_i(s, \psi)| \leq \|\phi - \psi\| \leq \xi_i l_0 \tag{9}$$

for $s \in N[m_0 - \tau, m_0]$, $\quad i = 1, 2, \cdots, n$.

In the following, we prove that for any $i \in \{1, 2, \cdots, n\}$, the inequalities

$$x_i(m) \leq \xi_i l_0, \quad m \in N[m_0, m_1), \quad i = 1, 2, \cdots, n \tag{10}$$

hold.

In fact, if inequality (10) is not true, then there exist some r and $m^* \in N[m_0, m_1)$ such that

$$x_r(m^* + 1) > \xi_r l_0$$

and

$$x_j(m) \le \xi_j l_0, \quad m \in N[m_0 - \tau, m^*], \quad j = 1, 2, \cdots, n.$$

However, from (8) and (3), we have

$$\begin{aligned} &x_r(m^* + 1) \\ &\le e^{\varepsilon}\Big(\overline{c_r} x_r(m^*) + \sum_{j=1}^{n} \overline{a_{rj}} F_j x_j(m^*) + \sum_{j=1}^{n} e^{\varepsilon\tau} \overline{b_{rj}} F_j x_j(m^* - \tau_{rj}(m^*))\Big) \\ &\le e^{\varepsilon}\Big(\overline{c_r}\xi_r + \sum_{j=1}^{n} \overline{a_{rj}} F_j \xi_j + \sum_{j=1}^{n} e^{\varepsilon\tau} \overline{b_{rj}} F_j \xi_j\Big) l_0 \le \xi_r l_0, \quad (9) \end{aligned}$$

this is a contradiction. So inequality (10) is true. Thus,

$$|u_i(m, \phi) - u_i(m, \psi)| \le \xi_i l_0 e^{-\varepsilon(m - m_0)} \tag{11}$$

for $m \in N[m_0, m_1)$, $i = 1, 2, \cdots, n$.

In the following, we will use the mathematical induction to prove that

$$|u_i(m, \phi) - u_i(m, \psi)| \le \gamma_0 \gamma_1 \cdots \gamma_{k-1} \xi_i l_0 e^{-\varepsilon(m - m_0)} \tag{12}$$

for $m \in [m_{k-1}, m_k)$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$, where $\gamma_0 = 1$.

When $k = 1$, we know that inequality (12) hold from inequalities (11).

Suppose that the following inequality

$$|u_i(m, \phi) - u_i(m, \psi)| \le \gamma_0 \gamma_1 \cdots \gamma_{k-1} \xi_i l_0 e^{-\varepsilon(m - m_0)} \tag{13}$$

hold for $m \in N[m_{k-1}, m_k)$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots, h$.

From assumption (H3) and inequality (13), we know that for $i = 1, 2, \cdots$, n, the second equation of model (5) satisfies

$$\begin{aligned} &|u_i(m_h, \phi) - u_i(m_h, \psi)| \le \sum_{j=1}^{n} p_{ij}^{(h)} |u_i(m_h^-, \phi) - u_i(m_h^-, \psi)| \\ &\quad + \sum_{j=1}^{n} q_{ij}^{(h)} |u_i((m_h - \tau_{ij}(m_h))^-, \phi) - u_i((m_h - \tau_{ij}(m_h))^-, \psi)| \\ &\le \sum_{j=1}^{n} p_{ij}^{(h)} \gamma_0 \gamma_1 \cdots \gamma_{h-1} \xi_j l_0 e^{-\varepsilon(m_h - m_0)} + \sum_{j=1}^{n} q_{ij}^{(h)} \gamma_0 \gamma_1 \cdots \gamma_{h-1} \xi_j l_0 \\ &\times e^{-\varepsilon(m_h - \tau_{ij}(m_h) - m_0)} \le \sum_{j=1}^{n} (p_{ij}^{(h)} + e^{\varepsilon\tau} p_{ij}^{(h)}) \gamma_0 \gamma_1 \cdots \gamma_{h-1} \xi_j l_0 e^{-\varepsilon(m_h - m_0)}. \quad (14) \end{aligned}$$

From $\xi \in \Delta \subseteq \Gamma(P_h) \bigcap \Gamma(Q_h)$, we have

$$\sum_{j=1}^{n} p_{ij}^{(h)} \xi_j = \rho(P_h)\xi_i, \quad \sum_{j=1}^{n} q_{ij}^{(h)} \xi_j = \rho(Q_h)\xi_i, \quad i = 1, 2, \cdots, n. \tag{15}$$

It follows from (14), (15) and (4) that

$$|u_i(m_h, \phi) - u_i(m_h, \psi)| \leq \gamma_0\gamma_1 \cdots \gamma_{h-1}\gamma_h \xi_i l_0 e^{-\varepsilon(m_h - m_0)} \tag{16}$$

for $i = 1, 2, \cdots, n$.

So, from (9), (13) and (16), this leads to

$$|u_i(m_h, \phi) - u_i(m_h, \psi)| \leq \gamma_0\gamma_1 \cdots \gamma_{h-1}\gamma_h \xi_i l_0 e^{-\varepsilon(m - m_0)}$$

for $m \in N[m_0 - \tau, m_h], \quad i = 1, 2, \cdots, n$. Thus,

$$x_i(m) \leq \gamma_0\gamma_1 \cdots \gamma_{h-1}\gamma_h \xi_i l_0, \quad m \in N[m_0 - \tau, m_h], \quad i = 1, 2, \cdots, n. \tag{17}$$

In the following, we will prove that

$$x_i(m) \leq \gamma_0\gamma_1 \cdots \gamma_{h-1}\gamma_h \xi_i l_0 \tag{18}$$

hold for $m \in N[m_h, m_{h+1}), \quad i = 1, 2, \cdots, n$.

If inequality (18) is not true, then there exist some l and $m^{**} \in N[m_h, m_{h+1})$ such that

$$x_l(m^{**} + 1) > \xi_l l_0$$

and

$$x_j(m) \leq \xi_j l_0, \quad m \in N[m_0 - \tau, m^{**}], \quad j = 1, 2, \cdots, n\,.$$

However, from (8) and (3), we have

$$\begin{aligned} x_l(m^{**} + 1) & \\ \leq e^{\varepsilon}\Big(\overline{c_l} x_l(m^{**}) &+ \sum_{j=1}^{n} \overline{a_{lj}} F_j x_j(m^{**}) + \sum_{j=1}^{n} e^{\varepsilon\tau} \overline{b_{lj}} F_j x_j(m^{**} - \tau_{lj}(m^{**}))\Big) \\ & \leq e^{\varepsilon}\Big(\overline{c_l}\xi_l + \sum_{j=1}^{n} \overline{a_{lj}} F_j \xi_j + \sum_{j=1}^{n} e^{\varepsilon\tau} \overline{b_{lj}} F_j \xi_j\Big) l_0 \leq \xi_l l_0, \end{aligned}$$

this is a contradiction. So inequality (18) holds.

By the mathematical induction, we can conclude that inequality (12) holds.

From (2), (12) and the definition of l_0, we have

$$|u_i(m,\phi)-u_i(m,\psi)| \le e^{\lambda(m_1-m_0)}e^{\lambda(m_2-m_1)}\cdots e^{\lambda(m_{k-1}-m_{k-2})}\xi_i l_0 e^{-\varepsilon(m-m_0)} \le \frac{\xi_i}{\min\limits_{1\le i\le n}\{\xi_i\}}\|\phi-u^*\|e^{-(\varepsilon-\lambda)(m-m_0)}$$

for $m\in N[m_{k-1},m_k)$, $k=1,2,\cdots$, $i=1,2,\cdots,n$. So

$$\|u(m,\phi)-u(m,\psi)\| \le M\|\phi-\psi|e^{-(\varepsilon-\lambda)(m-m_0)}, \quad m\in N[m_0,+\infty), \tag{19}$$

where $M=(\sum\limits_{j=1}^{n}\xi_i^2)^{\frac{1}{2}}/\min\limits_{1\le i\le n}\{\xi_i\}\ge 1$.

We can choose a positive integer L such that

$$Me^{-(\varepsilon-\lambda)(L\omega-m_0)} \le \frac{1}{4}. \tag{20}$$

Define a Poincaré mapping $\theta:\Upsilon\to\Upsilon$ by $\theta(\phi)=u^{\omega}(\phi)$. Then we can derive from (19) and (20) that

$$\|\theta^L(\phi)-\theta^L(\psi)\| \le \frac{1}{4}\|\phi-\psi\|. \tag{21}$$

This implies that θ^L is a contraction mapping, hence there exists a unique fixed point $\phi^*\in\Upsilon$ such that

$$\theta^L(\phi^*)=\phi^*.$$

Note that

$$\theta^L(\theta(\phi^*))=\theta(\theta^L(\phi^*))=\theta(\phi^*).$$

This shows that $\theta(\phi^*)\in\Upsilon$ is also a fixed point of θ^L, so

$$\theta(\phi^*)=\phi^*,$$

this is,

$$u^{\omega}(\phi^*)=\phi^*.$$

Let $u(m,\phi^*)$ be the solution of model (1) through $(0,\phi^*)$. From assumption (H1), we know that $u(m+\omega,\phi^*)$ is also a solution of model (1). Note that

$$u^{m+\omega}(\phi^*)=u^m(u^{\omega}(\phi^*)=u^m(\phi^*)$$

for $m\in N[m_0,+\infty)$, hence

$$u^{m+\omega}(\phi^*)=u(m,\phi^*)$$

for $m\in N[m_0,+\infty)$. This shows that $u(m,\phi^*)$ is exactly one ω-periodic solution of model (1) and it is easy to see that all other solutions of model (1) converge exponentially to it as $m\to+\infty$. The proof is completed. □

Remark 1. We may properly choose matrices P_k and Q_k in assumption (H3) to guarantee Δ in Theorem 1 to be non-empty. Especially, when $P_k = p_k E$ and $Q_k = q_k E$ (p_k and q_k are nonnegative constants), Δ is certainly nonempty. So, from Theorem 1, we easily obtain the following corollary.

Corollary 2. *Under assumptions (H1)-(H3) with $P_k = p_k E$ and $Q_k = q_k E$, model (1) has exactly one ω-periodic solution, and all other solutions of model (1) converge exponentially to it as $m \to +\infty$ and the exponential convergence rate index equals $\varepsilon - \lambda$, if conditions (i) and (iii) in Theorem 1 are satisfied.*

Remark 2. If $p_{ik}(u_1, \cdots, u_n) = u_i$, $q_{ik}(u_1, \cdots, u_n) = 0$, $J_{ik} = 0$ ($i = 1, 2, \cdots, n$; $k = 1, 2, \cdots$), then model (1) turns to the following non-impulsive discrete-time recurrent neural network with variable delays

$$u_i(m+1) = c_i(m)u_i(m) + \sum_{j=1}^{n} a_{ij}(m) f_j(u_j(m)) + \sum_{j=1}^{n} b_{ij} f_j(u_j(m - \tau_{ij}(m))) + I_i(m) \quad (22)$$

for $m = 1, 2, 3, \cdots$.

For model (22), we can derive the following result.

Corollary 3. *Under assumptions (H1), (H2) and condition (i) in Theorem 1, model (22) has exactly one ω-periodic solution, and all other solutions of model (22) converge exponentially to it as $m \to +\infty$ and the exponential convergence rate index is ε.*

4. Example

Example 1. Consider a two-neuron neural network (1), where

$$C = \begin{pmatrix} \frac{1}{4}|\sin(\frac{\pi}{20}m)| & 0 \\ 0 & \frac{1}{5}|\cos(\frac{\pi}{20}m)| \end{pmatrix}, \ A = \begin{pmatrix} -\frac{1}{6}\sin(\frac{\pi}{10}m) & \frac{1}{8}\cos(\frac{\pi}{10}m) \\ -\frac{1}{8}\sin(\frac{\pi}{10}m) & \frac{1}{10}\sin(\frac{\pi}{10}m) \end{pmatrix},$$

$$B = \begin{pmatrix} -\frac{1}{6}\cos(\frac{\pi}{10}m) & \frac{3}{8}\sin(\frac{\pi}{10}m) \\ -\frac{5}{24}\sin(\frac{\pi}{10}m) & \frac{1}{5}\sin(\frac{\pi}{10}m) \end{pmatrix}, \ I = \begin{pmatrix} -\frac{19}{24}\sin(\frac{\pi}{10}m) \\ \frac{11}{12}\cos(\frac{\pi}{10}m) \end{pmatrix},$$

$$f_1(x) = f_2(x) = x, \quad \tau_{11}(m) = \tau_{12}(m) = \tau_{21}(m) = \tau_{22}(m) = 1 + \cos\frac{m\pi}{2},$$

$$p_{1k}(u_1,u_2)=0.08e^{0.05}|\sin\frac{k\pi}{10}\cos\frac{m\pi}{10}|u_1-0.028e^{0.05}|\sin\frac{k\pi}{10}|u_2,$$
$$q_{1k}(u_1,u_2)=0.2e^{0.05}|\sin\frac{k\pi}{10}|u_1,$$
$$p_{2k}(u_1,u_2)=0.05e^{0.05}|\sin\frac{k\pi}{10}|u_1+0.03e^{0.05}|\sin\frac{k\pi}{10}\sin\frac{m\pi}{10}|u_2,$$
$$q_{2k}(u_1,u_2)=0.2e^{0.05}|\sin\frac{k\pi}{10}\cos\frac{m\pi}{10}|u_2,$$
$$J_{1k}=1-e^{0.05}|\sin\frac{k\pi}{10}\sin\frac{m\pi}{10}|,\quad J_{2k}=-e^{0.05}|\sin\frac{k\pi}{10}\cos\frac{m\pi}{10}|,$$
$$m_1=2,\quad m_k=m_{k-1}+2,\quad k=2,3,\cdots.$$

It is easy to check that assumptions (H1)-(H3) are satisfied with $F=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $P_k=e^{0.05}\begin{pmatrix}0.08&0.028\\0.05&0.03\end{pmatrix}$, $Q_k=0.2e^{0.05}\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $l=10$, $\omega=20$.

It is easily computing that

$$\Xi=E-\overline{C}-(\overline{A}+\overline{B})F=\begin{pmatrix}\frac{5}{12}&-\frac{1}{2}\\-\frac{1}{3}&\frac{1}{2}\end{pmatrix}$$

is an M-matrix, and $\rho(P_k)=0.1e^{0.05}$, $\rho(Q_k)=0.2e^{0.05}$, $\Gamma(P_k)=\{(\xi_1,\xi_2)^T|\,\xi_1=1.4\xi_2,\xi_1>0,\xi_2>0\}$, $\Gamma(Q_k)=\{(\xi_1,\xi_2)^T|\xi_1>0,\xi_2>0\}$, $\Omega(\Xi)=\{(\xi_1,\xi_2)^T|\,1.2\xi_2<\xi_1<1.5\xi_2,\xi_1>0,\xi_2>0\}$. So $\Delta=\{(\xi_1,\xi_2)^T|\xi_1=1.4\xi_2,\xi_1>0,\xi_2>0\}$ is non-empty.

Take $\xi=(1.4,1)^T\in\Delta$, $\tau=2$, from the following inequalities

$$\xi_i(-1+e^{\varepsilon}\overline{c_i})+e^{\varepsilon}\sum_{j=1}^{n}\xi_jF_j(\overline{a_{ij}}+e^{\varepsilon\tau}\overline{b_{ij}})\le 0,\quad i=1,2,$$

we can get that a maximum value of ε is 0.0167.

Take $\gamma_k=e^x$, $\lambda=\frac{x}{2}$, then

$$\gamma_k\ge\max\{1,0.1e^{0.05}+e^{2\times 0.0167}\times 0.2e^{0.05}\},\quad k=1,2,\cdots,$$

and $\frac{\ln\gamma_k}{t_k-t_{k-1}}=\frac{\ln e^{0.05}}{2}=\lambda<\varepsilon$, $k=1,2,\cdots$.

Clearly, all conditions of Theorem 1 are satisfied. So this neural network has exactly one 20-periodic solution, and all other solutions of the neural network converge exponentially to it as $m\to+\infty$ and the exponential convergence rate index equals $0.0167-x$. Figure 1 shows the simulation result of the existence of periodic solution of the neural network with the initial condition $u_1(s)=2\cos(s)$, $u_2(s)=-3\sin(s)$, $s\in N[-2,0]$.

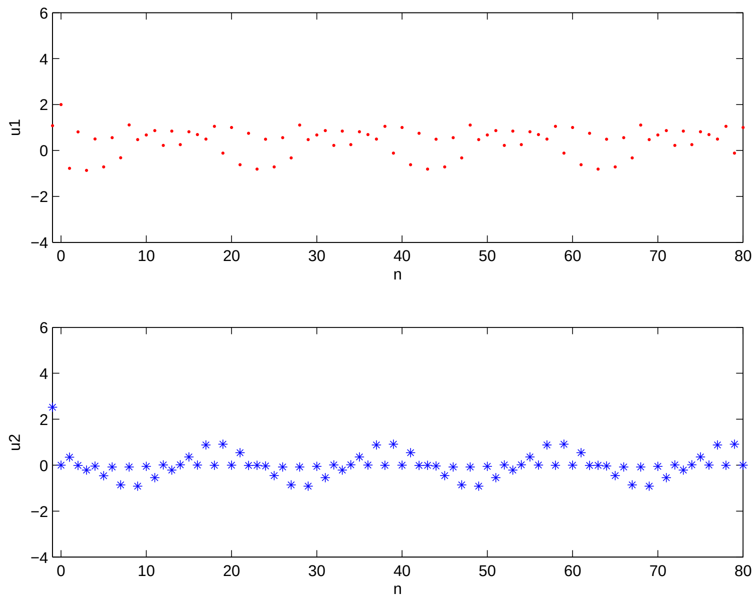

Figure 1: The existence of periodic solution for the discrete-time recurrent neural networks with variable delays and impulses, where the initial state $u_1(s) = 2\cos(s)$, $u_2(s) = -3\sin(s)$, $s \in N[-2, 0]$.

5. Conclusions

In this paper, by using analytic methods, inequality technique and M-matrix theory, several simple sufficient conditions checking the existence, uniqueness and global exponential stability of periodic solution have been obtained for the discrete-time recurrent neural network with variable delays and impulses. Moreover, the estimation for exponential convergence rate index was also proposed. An example with simulation has been given to show the effectiveness of the obtained results.

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