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γ-CONNECTED SPACES

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Abstract: We define and discuss γ-connected spaces, γ-components in a space X and γ-locally connected spaces.

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1. Introduction

S. Kasahara [6] defined an operation α on topological spaces and studied a α-closed graphs of a function in 1979. D.S. Jankovic [5] defined α-closed set and further worked on functions with α-closed graphs in 1983. H. Ogata [7] introduced separation axioms γ-T_i, $i = 0, 1/2, 1, 2$; and studied some of their properties in 1991. F.U. Rehman and B. Ahmad [8] (or [1]) defined and discussed several properties of γ-interior, γ-closure,γ-exterior and γ-boundary and (γ,β)-closed (open) mappings in topological spaces (resp. in product spaces [1]), and further investigated the characterizations of (γ,β)- continuous and (γ,β)-closed (open) mappings in 1992 (1993). In 2003 (2005), B. Ahmad and S. Hussain [2] (or [4]) continued studying the properties of γ-operations (γ-regular and

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γ-normal spaces) in topological spaces.

In this paper, we define γ-connected spaces and study their properties in topological spaces. Hereafter we shall write space in place of topological space.

We recall some definitions and results used in this paper to make it self contained.

Definition. (see [7]) Let (X,τ) be a space. An operation γ: $\tau \rightarrow$ P(X) is a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each V $\in \tau$, where V^γ denotes the value of γ at V.

The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = int cl(G) are examples of operation γ.

Definition. (see [8]) Let A $\subseteq$ X. A point a $\in$ A is said to be γ-interior point of A iff there exists an open nbd N of a such that $N^\gamma \subseteq$ A and we denote the set of all such points by int$_\gamma$(A).

Thus

$$\operatorname{int}_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\}.$$

Note that A is γ-open [7] iff A =int$_\gamma$(A).

A set A is called γ-closed [7] iff X-A is γ-open.

Definition. (see [8]) A point x $\in$ X is called a γ-closure point of A $\subseteq$ X, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A, is called γ-closure of A and is denoted by cl$_\gamma$(A). A subset A of X is called γ-closed, if cl$_\gamma(A) \subseteq$ A.

Note that cl$_\gamma$(A) is contained in every γ-closed superset of A.

Definition. (see [7]) An operation γ on τ is said be regular, if for any open nbds U, V of x $\in$ X, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition. (see [7]) An operation γ on τ is said to be open if for every nbd U of x $\in$ X, there exists a γ-open set B such that $x \in B$ and $U^\gamma \supseteq B$.

2. γ-Connected Spaces

Definition 1. A topological space X is said to be γ-connected if there does not exist a pair A, B of nonempty disjoint γ-open subset of X such that X= A $\cup$ B, otherwise X is called γ-disconnected. In this case, the pair (A, B) is called a γ-disconnection of X. A subset A of a space X is γ-connected if it is γ-connected as a subspace.

Example 1. Let X = $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}$. For b $\in$ X, define an operation $\gamma : \tau \rightarrow P(X)$ such that

$$\gamma(A) = \begin{cases} \text{cl}(A), & \text{if } b \in A\,, \\ \text{clint}\,(A), & \text{if } b \notin A\,. \end{cases}$$

Calculations give that ϕ, X, $\{a, c\}$ are the only γ-open sets. Clearly X is γ-connected but not connected.

Example 2. Any infinite space with co-finite γ-topology is γ-connected.

Example 3. Every γ-indiscrete space is γ-connected.

We give the characterization of γ-connected space, the proof of which is straight forward.

Theorem 1. *A topological space X is γ-disconnected (resp. γ-connected) iff there exists (resp. does not exist) non empty subset A of X which is both γ-open and γ-closed in X.*

Definition. (see [7]) A mapping f: $(X, \tau_1) \to (Y, \tau_2)$ is said to be (γ, β)-continuous if for each $x \in X$ and each open set V containing f (x), there exists an open set U such that $x \in U$ and $f(U^\gamma) \subseteq V^\beta$, where $\gamma : \tau_1 \to P(X); \beta : \tau_2 \to P(Y)$ are operations on τ_1 and τ_2 respectively.

A (γ, β)-continuous mapping [8] has been characterized as:

If $f : (X, \tau_1) \to (Y, \tau_2)$ is a mapping and β is open, then f is (γ, β)-continuous iff for each β-open set V in Y, $f^{-1}(V)$ is γ-open in X. We use this characterization and prove:

Theorem 2. *The (γ, β)-continuous image of γ-connected space is γ-connected, where β is open.*

Proof. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be (γ, β)-continuous from a γ-connected space from (X, τ_1) onto a space (Y,τ_2). Suppose that Y is γ-disconnected and (A, B) is a γ-disconnection of Y. Since f is (γ, β)-continuous, therefore $f^{-1}(A), f^{-1}(B)$ are both γ-open in X. Clearly $f^{-1}(A), f^{-1}(B))$ is a pair of γ-disconnection of X, a contradiction. Hence Y is γ-connected. This completes the proof. □

Next, we characterize γ-connectedness in terms of γ-boundary as.

Theorem 3. *A space X is γ-connected iff every nonempty proper subspace has a nonempty γ-boundary.*

Proof. Suppose that a nonempty proper subspace A of a γ-connected space X has empty γ-boundary. Then A is γ-open and $\mathrm{cl}_\gamma(A) \cap \mathrm{cl}_\gamma(X - A) = \phi$. Let p be a γ-limit point of A. Then $p \in \mathrm{cl}_\gamma(A)$ but $p \notin \mathrm{cl}_\gamma(X - A)$. In particular $p \notin X - A$ and so $p \in A$. Thus A is γ-closed and γ-open. By Theorem 1, X is γ-disconnected. This contradiction proves that A has a nonempty γ-boundary.

Conversely, suppose X is γ-disconnected. Then by Theorem 1, X has a proper subspace A which is both γ-closed and γ-open. Then $\mathrm{cl}_\gamma(A) = A, \mathrm{cl}_\gamma(X - A) = (XA)$ and $\mathrm{cl}_\gamma(A) \cap \mathrm{cl}_\gamma(X - A) = \phi$. So A has empty γ-boundary, a contradiction. Hence X is γ-connected. This completes the proof. □

Definition 2. A two point discrete space D = $\{a, b\}$ is called γ-discrete if $\tau_\gamma = \tau$.

Theorem 4. *If a space X is γ-connected, then there does not exist a surjective (γ, β)-continuous function f from X onto two point γ-discrete space, where β is open.*

Proof. Suppose there exists a (γ, β)-continuous from a γ-connected space X onto a two point γ-discrete space D $= \{a, b\}$. Then (γ, β)-continuity of f implies A$= f^{-1}\{a\}$ and $B = f^{-1}\{b\}$ are γ-open in X. Clearly (A, B) is a γ-disconnection of X. This contradiction proves the theorem. □

Definition. (see [4]) Let X be a space and $A \subseteq X$. Then the class of γ-open sets in A is defined in a natural way as:

$$\tau_{\gamma_A} = \{A \cap O : O \in \tau_\gamma\},$$

where τ_γ is the class of γ-open sets of X. That is, G is γ-open in A iff G $= A \cap O$, where O is a γ-open set in X.

Theorem 5. *Let (A, B) be a γ-disconnection of a space X and C be a γ-connected subspace of X. Then C is contained in A or B.*

Proof. Suppose that C is neither contained in A nor in B. Then $C \cap A, C \cap B$ are both nonempty γ-open subsets of C such that

$$(C \cap A) \cap (C \cap B) = \phi, \text{ and } (C \cap A) \cup (C \cap B) = C.$$

This gives that $(C \cap A, C \cap B)$ is a γ-disconnection of C. This contradiction proves the theorem. □

Theorem 6. *Let X $=\cup_{\alpha \in I}\{X_\alpha\}$, where each X_α is γ-connected and $\cap_{\alpha \in I}\{X_\alpha\} \neq \phi$. Then X is γ-connected.*

Proof. Suppose on the contrary that (A, B) is a γ-disconnection of X. Since each X_α is γ-connected, therefore by Theorem 5, $X_\alpha \subseteq A$ or $X_\alpha \subseteq B$. Since $\cap X_\alpha \neq \phi$, therefore all X_α are contained in A or in B. This gives that, if $X \subseteq A$, then B$=\phi$ or if $X \subseteq B$, then $A = \phi$.

This contradictions proves that X is γ-connected. Hence the proof. □

Using Theorem 6, we characterize γ-connectedness as:

Theorem 7. *A space X is γ-connected iff for every pair of points x, y in X, there is a γ-connected subset of X which contains both x and y.*

Proof. The necessity is immediate since the γ-connected space itself contains these two points.

For the sufficiency, suppose that for any two points x, y; there is a γ-connected subspace $C_{x,y}$ of X such that $x, y \in C_{x,y}$. Let $a \in X$ be a fixed point and $\{C_{a,x}, x \in X\}$ be a class of all γ-connected subsets of X which contain a and $x \in X$. Then $X = \cup_{x \in X}\{C_{a,x}\}$ and $\cap_{x \in X}\{C_{a,x}\} = \phi$. Therefore by Theorem 6, X is γ-connected. This completes the proof. □

Theorem 8. *Let C be a γ-connected subset of a space X and $A \subseteq X$ such that $C \subseteq A \subseteq \mathrm{Cl}_\gamma(C)$. Then A is γ-connected.*

Proof. It is sufficient to show that $\mathrm{Cl}_\gamma(C)$ is γ-connected. On the contrary, suppose that $\mathrm{Cl}_\gamma(C)$ is γ-disconnected. Then there exists a γ- disconnection (H, K) of $\mathrm{Cl}_\gamma(C)$. That is, there are $H \cap C, K \cap C$ γ-open sets in C such that

$$(H \cap C) \cap (K \cap C) = (H \cap K) \cap C = \phi,$$

and

$$(H \cap C) \cup (K \cap C) = (H \cup K) \cap C = C.$$

This gives that $(H \cap C, K \cap C)$ is a γ-disconnection of C, a contradiction. This proves that $\mathrm{Cl}_\gamma(C)$ is γ-connected. □

3. γ-Components in Space X

Definition 3. A maximal γ-connected subset of a space X is called a γ-component of X. If X is itself γ-connected, then X is the only γ-component of X.

Next we study the properties of γ-components of a space X:

Theorem 9. *Let X be a space. Then:*

(1) For each $x \in X$, there is exactly one γ-component of X containing x.

(2) Each γ-connected subset of X is contained in exactly one γ-component of X.

(3) A γ-connected subset of X which is both γ-open and γ-closed is a γ-component, if γ is regular.

(4) Every γ-component of X is γ-closed in X.

Proof. (1) Let $x \in X$ and $\{C_\alpha : \alpha \in I\}$ a class of all γ-connected subsets of X containing x. Put $C = \cup_{\alpha \in I} C$, then by Theorem 6, C is γ-connected and $x \in X$. Suppose $C \subseteq C^*$ for some γ-connected subset C^* of X. Then $x \in C^*$ and hence C^* is one of the C_α's and hence $C^* \subseteq C$. Consequently C $= C^*$. This proves that C is a γ-component of X which contains x.

(2) Let A be a γ-connected subset of X which is not a γ-component of X. Suppose that C_1, C_2 are γ-components of X such that $A \subseteq C_1, A \subseteq C_2$. Since $C_1 \cap C_2 = \phi$, $C_1 \cup C_2$ is another γ-connected set which contains C_1 as well as C_2, a contradiction to the fact that C_1 and C_2 are γ-components. This proves that A is contained in exactly one γ-component of X.

(3) Suppose that A is γ-connected subset of X which is both γ-open and γ-closed. By (2), A is contained is exactly one γ-component C of X. If A is a

proper subset of C, and since γ is regular, therefore C $=(C\cap A)\cup(C\cap(X-A))$ is a γ-disconnection of C, a contradiction. Thus A = C.

(4) Suppose a γ-component C of X is not γ-closed. Then by Theorem 8, $\mathrm{cl}_\gamma(C)$ is γ-connected containing γ-component C of X. This implies C $=\mathrm{cl}_\gamma(C)$ and hence C is γ-closed. This completes the proof. □

4. γ-Locally Connected Spaces

Definition 4. A space X is said to be γ-locally connected if for any point $x\in X$ and any γ-open set U containing x, there is a γ-connected γ-open set V such that $x\in V\subseteq U$.

Example. Let X = a, b, c, $\tau=\{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. For a $\in$ X, define an operation $\gamma:\tau\to P(X)$ such that

$$\gamma(A)=\begin{cases} A, & \text{if } a\in A\,,\\ cl(A, & \text{if } a\notin A\,. \end{cases}$$

Calculation gives that ϕ, X, $\{a\}, \{b\}, \{a,b\}$ are the only γ-open sets. Clearly X is γ-locally connected but X in not locally connected.

Theorem 10. *If X is a γ-locally connected space, then X has a γ-nbd base comprising γ-connected γ-open sets.*

Proof. Let β be the class of all γ-connected γ-open subsets of a γ-locally connected space X. We show that β is a γ-nbd base for a topology τ on X. Let U be γ-open subset in X and $x\in U$. Since X is γ-locally connected space, therefore there exists a γ-connected γ-open set $B\in\beta$ such that $x\in B\subseteq U$. This implies that each γ-open set in X is the union of members of β. Consequently β is a γ-nbd base for τ. Hence the proof. □

The following theorem shows that γ-locally connectedness is a γ-open hereditary property.

Theorem 11. *A γ-open subset of γ-locally connected space is γ-locally connected.*

Proof. Let U be a γ-open subset of a γ-locally connected space X. Let $x\in U$ and V a γ-open nbd of x in U. Then V is a γ-open nbd of x in X. Since X is γ-locally connected, therefore there exists a γ-connected, γ-open nbd W of x such that $x\in W\subseteq V$. In this way W is also a γ-connected γ-open nbd x in U such that $x\in W\subseteq U\subseteq V$ or $x\in W\subseteq V$. This proves that U is γ-locally connected. Hence the proof. □

Definition. (see [1]) A mapping $f:(X,\tau_1)\to(Y,\tau_2)$ is said to be (γ,β)-closed (resp. (γ,β)-open), if for any γ-closed (γ-open) set A of X, f(A) is β-closed (respt. β-open) in Y.

Theorem 12. *A (γ, β)-continuous (γ, β)-open surjective image of γ-locally connected space is γ-locally connected space, where β is open.*

Proof. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be (γ, β)-continuous,(γ, β)-open from a γ-locally connected space X to a space Y. We show that Y = f (X) is γ-locally connected space. Let $y \in Y$ and choose $x \in X$ such that f (x) = y. Let U be β-open set containing x. Since X is γ-locally connected, therefore there exists a γ-connected, γ-open set V containing x such that

$$x \in V \subseteq f^{-1}(U).$$

This gives that $f(x) \in f(V) \subseteq ff^{-1}(U) = U$ or $y \in f(V) \subseteq U$. Since f is (γ, β)-continuous, therefore f(V) is γ-open. Moreover f(V) is γ-connected. This proves that Y is γ-locally connected. Hence the proof. □

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