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THE SPHERICAL COORDINATES IN TAXICAB SPACE

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Abstract: In this paper the inner-product and norm has been defined in taxicab space and we give the representation of every point with respect to spherical coordinates using taxicab trigonometric functions.

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1. Introduction

In [2] Krause has defined a new metric instead of well known Euclidean metric which called taxicab metric as follows: Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are different points in Euclidean analytical plane R^2. The distance between point A to point B is defined by

$$d_T(A, B) = |a_1 - b_1| + |a_2 - b_2|$$

and so, this plane is called metric by taxicab plane or two dimensional taxicab space. This taxicab plane is denoted by R_T^2. Thus the difference between R_T^2 and real Euclidean plane is only distance function, so that the points and lines are same and the angles are measured in the same way.

If $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ are different points of n-

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dimensional real space, the taxicab distance between A and B is defined as follows:

$$d_T(A,B) = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| \ .$$

This paper appeared by a inspirational study, see [1] Ekici et al, have defined the inner-product and norm in taxicab plane and they give geometrical approach. In this paper, firstly we will present inner-product and norm in three dimensional taxicab plane R_T^3 and then give the spherical coordinates of a point using taxicab trigonometric functions.

2. The Inner-Product

All of the vectors which we are dealing with passing through the origin. Let $u = (u_1, u_2, u_3)$ a vector in R_T^3 and we call u is belong to first, second, third and fourth octant if all components are positive, second and third components are positive, only third component is positive, first and third components are positive, respectively. Therefore, the symmetries of these octants with respect to $u_3 = 0$ plane, we call these octants fifth, sixth, seventh and eighth octants respectively. Notice that the intersections of any two octants may be coplanar, collinear or consist of a point, (i.e. origin). In this study these octants called by neighbor octants, diagonal neighbor octants and opposite octants respectively.

Definition. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be two vectors in R_T^3. The function $\langle u, v\rangle_T$ defines the inner-product in the R_T^3.

$$\langle u,v\rangle_T = \begin{cases} |u_1v_1| + |u_2v_2| + |u_3v_3| \quad ; & u \text{ and } v \text{ are in the same octant}, \\ -|u_1v_1| + |u_2v_2| + |u_3v_3| \quad ; & u \text{ and } v \text{ are in neighbor octants}, \\ & u_1v_1 < 0,\ u_2v_2 > 0,\ u_3v_3 > 0, \\ |u_1v_1| - |u_2v_2| + |u_3v_3| \quad ; & u \text{ and } v \text{ are in neighbor octants}, \\ & u_1v_1 > 0,\ u_2v_2 < 0,\ u_3v_3 > 0, \\ |u_1v_1| + |u_2v_2| - |u_3v_3| \quad ; & u \text{ and } v \text{ are in neighbor octants}, \\ & u_1v_1 > 0,\ u_2v_2 > 0,\ u_3v_3 < 0, \\ -|u_1v_1| - |u_2v_2| + |u_3v_3| \quad ; & u \text{ and } v \text{ are in diagonal neighbor octants}, \\ & u_1v_1 < 0,\ u_2v_2 < 0,\ u_3v_3 > 0, \\ -|u_1v_1| + |u_2v_2| - |u_3v_3| \quad ; & u \text{ and } v \text{ are in diagonal neighbor octants}, \\ & u_1v_1 < 0,\ u_2v_2 > 0,\ u_3v_3 < 0, \\ |u_1v_1| - |u_2v_2| - |u_3v_3| \quad ; & u \text{ and } v \text{ are in diagonal neighbor octants}, \\ & u_1v_1 > 0,\ u_2v_2 < 0,\ u_3v_3 < 0, \\ -|u_1v_1| - |u_2v_2| - |u_3v_3| \quad ; & u \text{ and } v \text{ are in opposite octants}. \end{cases}$$

	First Positions of u, v	Cases	New Positions
$r > 0$	Same Octant	8	Same Octant
$r > 0$	Neighbor Octant	24	Neighbor Octant
$r > 0$	Diag. Neighbor Octant	24	Diag. Neighbor Octant
$r > 0$	Opposite Octant	8	Opposite Octant
$r < 0$	Same Octant	8	Opposite Octant
$r < 0$	Neighbor Octant	24	Diag. Neighbor Octant
$r < 0$	Diag. Neighbor Octant	24	Neighbor Octant
$r < 0$	Neighbor Octant	8	Same Octant

Table 2.1:

The inner-product of two vectors in taxicab geometry is positive definite, symmetric and two-linear. It is easy to see that positive definite and symmetry properties holds, but we want to deliberate on the two-linear and homogenity properties.

a) *Homogenity Property:* In order to show that the homogenity property of $\langle \quad \rangle_T$ with respect to scalar r (i.e. show the equality $\langle ru, v\rangle_T = r\,\langle u, v\rangle_T$), we have to consider 64 cases for each positive $r \in R^+$ and 64 cases for each negative $r \in R^-$. Notice that for each u, v which are chosen from arbitrarily octants, ru and v will be same, neighbor, diagonal neighbor or opposite octants for each scalar $r \in R$. All the cases presented by Table 2.1.

b) *Additivity Property:* In order to show the linearity property (i.e. show the equality $\langle u + v, w\rangle_T = \langle u, w\rangle_T + \langle v, w\rangle_T$), we have to consider many cases, too. Let we take any three vectors u, v and w from R^3_T. These vectors can be placed 512 types in the octants. So these all cases must be considered one by one. These cases will not examined in this paper, but for each cases, it is easy to see the property holds. As we mentioned before, like the octant of a vector u may be change for ru ($r \in R$), the summation of two vectors may belong to any octant. The magnitudes of vectors designate the octant of summation vector. All the cases presented by Table 2.2.

Remark. If we take u, v and w from the first, second and third octant respectively, and then $u + v$ belong the either first octant or second octant. If we take u and v from the first and seventh octant, $u + v$ maybe belong to anywhere. Consequently, some cases have subcases, i.e., for instance, in 336 cases, 144 cases have 2 subcases, 144 cases have 4 subcases and 48 cases have 8 subcases. The authors believe that the following problem will be very enjoyable for readers.

Problem. If we consider n-dimensional real vector space instead of 3-

First position of u, v and w	Cases	Position of summation vector with respect to other one
All the vectors from same octants	8	Same octants
Two of them from the same octant and the other one from neighbor octant	72	Same octants Neighbor octants
Two of them from the same octant and the other one from diag. neighbor octant	72	Same octants Neighbor octants Diag. neighbor octants
Two of them from the same octant and the other one from opposite octant	24	Same octants Neighbor octants Diag. neighbor octants Opposite octants
All the vectors from opposite octants	336	Same octants Neighbor octants Diag. neighbor octants Opposite octants

Table 2.2:

dimensional real vector space how many cases and subcases we have to consider totaly to show the homogenity and additivity properties of inner-product?

3. Norm

Let u be a vector as form $u = (u_1, u_2, u_3)$ in 3-dimensional taxicab space and then the norm defined as follows:

$$\begin{aligned} \|u\|_T &= \sqrt{\langle u, u\rangle_T + 2|u_1u_2| + 2|u_2u_3| + 2|u_1u_3|} \\ &= \sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2 + 2|u_1u_2| + 2|u_2u_3| + 2|u_1u_3|} \\ &= |u_1| + |u_2| + |u_3| = d_T(u, 0) \ . \end{aligned}$$

Geometrically the norm of an vector means the summations of distances of components to coordinate axes. In higher dimensional taxicab spaces, the taxicab norm and its properties considered by Akça et al, see [4].

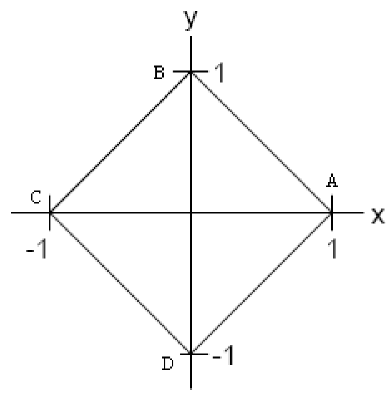

Figure 1: The taxicab unit circle

4. The Spherical Coordinates in 3-Dimensional Taxicab Space

Let $P = (x, y)$ be a point in R_T^2 taxicab plane then the r shows the radial distance from the origin i.e. $\left\|\overrightarrow{OP}\right\|_T = r$ and θ is the angle made by $\overrightarrow{OP}$ and the positive x-axis in the counter-clockwise direction. The ratio of the circumference to radius length of the taxicab circle is 4, i.e. $\pi_T = 4$, see [5].

$$\begin{array}{llll} \theta = \dfrac{d_T(P,A)}{r} & , \quad \theta \in [0,2) \quad ; & \theta = \dfrac{d_T(P,B)+2r}{r} & , \quad \theta \in [2,4) \\ r = x + y & & r = -x + y & \end{array}$$

$$\begin{array}{llll} \theta = \dfrac{d_T(P,C)+4r}{r} & , \quad \theta \in [4,6) \quad ; & \theta = \dfrac{d_T(P,D)+6r}{r} & \quad \theta \in [6,8) \\ r = -x - y & & r = x - y & \end{array}$$

The solutions of these four system of the equation as follows:

For $\theta \in [0,4)$, then $x = r(1 - \frac{\theta}{2})$; and for $\theta \in [4,8)$ and then $x = r(-3 + \frac{\theta}{2})$. Thus

$$\frac{x}{r} = \cos_T(\theta) = \begin{cases} 1 - \dfrac{\theta}{2}; & \theta \in [0,4) \\ -3 + \dfrac{\theta}{2}; & \theta \in [4,8) \end{cases}$$

As the same way:

For $\theta \in [0,2)$ then $y = r\frac{\theta}{2}$; for $\theta \in [2,6)$ implies $y = r(2 - \frac{\theta}{2})$; and then for

Position of $P=(x,y,z)$	Spherical Coordinates of $P=(x,y,z)$
First Octant	$\sqrt{\langle\overrightarrow{OP},\overrightarrow{OP}\rangle_T+2(\lvert xy\rvert+\lvert xz\rvert+\lvert yz\rvert)}(\frac{\varphi}{2}(1-\frac{\theta}{2}),\ \frac{\varphi}{2}\frac{\theta}{2},\ 1-\frac{\varphi}{2})$
Second Octant	$\sqrt{\langle\overrightarrow{OP},\overrightarrow{OP}\rangle_T+2(\lvert xy\rvert+\lvert xz\rvert+\lvert yz\rvert)}(\frac{\varphi}{2}(1-\frac{\theta}{2}),\ \frac{\varphi}{2}(2-\frac{\theta}{2}),\ 1-\frac{\varphi}{2})$
Third Octant	$\sqrt{\langle\overrightarrow{OP},\overrightarrow{OP}\rangle_T+2(\lvert xy\rvert+\lvert xz\rvert+\lvert yz\rvert)}(\frac{\varphi}{2}(-3+\frac{\theta}{2}),\ \frac{\varphi}{2}(2-\frac{\theta}{2}),\ 1-\frac{\varphi}{2})$
Fourth Octant	$\sqrt{\langle\overrightarrow{OP},\overrightarrow{OP}\rangle_T+2(\lvert xy\rvert+\lvert xz\rvert+\lvert yz\rvert)}(\frac{\varphi}{2}(-3+\frac{\theta}{2}),\ \frac{\varphi}{2}(-4+\frac{\theta}{2}),\ 1-\frac{\varphi}{2})$
Fifth Octant	$\sqrt{\langle\overrightarrow{OP},\overrightarrow{OP}\rangle_T+2(\lvert xy\rvert+\lvert xz\rvert+\lvert yz\rvert)}((2-\frac{\varphi}{2})(1-\frac{\theta}{2}),\ (2-\frac{\varphi}{2})\frac{\theta}{2},\ 1-\frac{\varphi}{2})$
Sixth Octant	$\sqrt{\langle\overrightarrow{OP},\overrightarrow{OP}\rangle_T+2(\lvert xy\rvert+\lvert xz\rvert+\lvert yz\rvert)}((2-\frac{\varphi}{2})(1-\frac{\theta}{2}),\ (2-\frac{\varphi}{2})(2-\frac{\theta}{2}),\ 1-\frac{\varphi}{2})$
Seventh Octant	$\sqrt{\langle\overrightarrow{OP},\overrightarrow{OP}\rangle_T+2(\lvert xy\rvert+\lvert xz\rvert+\lvert yz\rvert)}((2-\frac{\varphi}{2})(-3+\frac{\theta}{2}),\ (2-\frac{\varphi}{2})(2-\frac{\theta}{2}),\ 1-\frac{\varphi}{2})$
Eighth Octant	$\sqrt{\langle\overrightarrow{OP},\overrightarrow{OP}\rangle_T+2(\lvert xy\rvert+\lvert xz\rvert+\lvert yz\rvert)}((2-\frac{\varphi}{2})(-3+\frac{\theta}{2}),\ (2-\frac{\varphi}{2})(-4+\frac{\theta}{2}),\ 1-\frac{\varphi}{2})$

Table 4.1:

$\theta\in[6,8)$, $y=r(-4+\frac{\theta}{2})$ obtained. Therefore,

$$\frac{y}{r}=\sin_T(\theta)=\begin{cases}\dfrac{\theta}{2}; & \theta\in[0,2)\ ,\\ 2-\dfrac{\theta}{2}; & \theta\in[2,6)\ ,\\ -4+\dfrac{\theta}{2}; & \theta\in[6,8)\ ,\end{cases}$$

for the more information about taxicab trigonometry, see [5].

To define spherical coordinates in 3-dimensional real vector space we take an axis usually called polar axis, and a perpendicular plane known as the equatorial plane, on which we choose a ray (the initial ray) originating at the intersection of the plane and the axis (the origin O). To transform from rectangular Cartesian coordinates (x,y,z) to spherical ones and back, use the following formulas:

$$x=r\sin\varphi\cos\theta\,,\quad y=r\sin\varphi\sin\theta\,,\quad z=r\cos\varphi\,.$$

As seen above this give the three coordinates (r,θ,φ). This coordinates defines as the first coordinate of any point P is the distance r of P from the pole O, the second coordinate of P is the angle theta from the polar axis to the projection of $\overrightarrow{OP}$ into the plane and the third coordinate of P is the angle phi between that positive part of the vertical axis and the line segment $\overrightarrow{OP}$. Thus these spherical coordinates take values like as $r \in [0, \infty)$, $\theta \in [0, 2\pi)$ and $\varphi \in [0, \pi)$.

In 3-dimensional taxicab space we know that the first spherical coordinate is that $\left\|\overrightarrow{OP}\right\|_T = |x| + |y| + |z|$, but notice that in the taxicab plane the functions $\cos_T$ and $\sin_T$ cannot be written as a unique form. So that if we associate the separate of R_T^3 (separating to octants) with partial taxicab trigonometric functions we cannot state spherical coordinates of a point as a unique form. So that we determine 8 distinct types and give in Table 4.1.

Thus we give the spherical coordinates of an point in R_T^3.

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