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A NOTE ON MINIMAL HYPERSURFACES OF THE UNIT SPHERE

Sharief Deshmukh[1], Falleh R. Al-Solamy[2 §]

[1]Department of Mathematics
College of Science
King Saud University
P.O. Box 2455, Riyadh, 11451, KINGDOM OF SAUDI ARABIA
e-mail: shariefd@ksu.edu.sa

[2]Department of Mathematics
King Abdul Aziz University
P.O. Box 80015, Jeddah, 21589, KINGDOM OF SAUDI ARABIA
e-mail: falleh@hotmail.com

Abstract: In this note, we show that for compact and connected minimal hypersurfaces of constant scalar curvature in the unit sphere $S^{n+1}(1)$ the set of values that the square of the length of the shape operator can assume is discrete.

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1. Introduction

One of the interesting questions in the geometry of minimal hypersurfaces of constant scalar curvature in the unit sphere is to show that the set of values of the constant scalar curvature for these type of hypersurfaces in the unit sphere is discrete. Let M be an orientable compact minimal hypersurface of constant scalar curvature in the unit sphere $S^{n+1}(1)$ and B be its Weingarten map (shape operator). Lawson [4] has proved that the first and second values of $\|B\|^2$ are 0 and n (that is, it is proved that if $\|B\|^2 < n$, then $\|B\|^2 = 0$). On the third value of $\|B\|^2$ Peng and Terng [7] have proved that if $\|B\|^2 > n$,

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§Correspondence author

then $\|B\|^2 > n + c(n)$, where $c(n) > \frac{1}{12n}$ is a positive constant. Also for $n = 3$ these authors proved that $\|B\|^2 \geq 6$ and consequently they conjectured that the third value of $\|B\|^2$ should be $2n$, as there exist Cartan's isoparametric minimal hypersurfaces in the unit sphere $S^{n+1}(1)$ satisfying $\|B\|^2 = 2n$. Then Yang and Cheng (cf. [9], [10]) improved the result of Peng and Terng by proving $c(n) > \frac{2}{7}n - \frac{9}{14}$. These authors in [11] further improved this result by proving if $\|B\|^2 > n$, then $\|B\|^2 \geq \frac{1}{3}(4n + 1)$. Also recently in [5], the authors have proved that for Willmore hypersurfaces in dimension four the set of values that $\|B\|^2$ takes is discrete. In this short note, we prove the following:

Theorem. *Let M be an orientable compact and connected minimal hypersurface of constant scalar curvature in the unit sphere $S^{n+1}(1)$. Then the square of the length of the second fundamental form $\|B\|^2 \in Spec(M)$, where $Spec(M)$ is the spectrum of the of the Laplacian operator Δ acting on smooth functions on M.*

Since the $Spec(M)$ is the discrete set the above theorem implies that the set of values that $\|B\|^2$ can assume it is discrete.

2. Preliminaries

Let M be an n-dimensional immersed submanifold in the Euclidean space R^{n+p}. We denote by g and $\overline{\nabla}$ the Euclidean metric and the Euclidean connection on R^{n+p}. We denote by the same letter g and by ∇ the induced metric and the Riemannian connection on the submanifold M. Then we have the following fundamental equations for the submanifold

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X N = -A_N(X) + \nabla^{\perp}_X N\,, \tag{2.1}$$

$X, Y \in \mathfrak{X}(M)$, $N \in \Gamma(\upsilon)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M, $\Gamma(\upsilon)$ is the space of smooth sections of the normal bundle υ of M, h is the second fundamental form, A_N is the Weingarten map with respect to the normal vector field N and $\nabla^{\perp}$ is the connection in the normal bundle υ. For a local orthonormal frame $\{e_1, \ldots, e_n\}$ the mean curvature H of the submanifold is given by $H = \frac{1}{n}\sum h(e_i, e_i)$.

For a constant vector field $\xi \in \mathfrak{X}(R^{n+p})$ we write $\xi = u + \varsigma$, where $u \in \mathfrak{X}(M)$ is the tangential component and $\varsigma \in \Gamma(\upsilon)$ is the normal component of ξ. If we denote by $A = A_\varsigma$ the Weingarten map with respect to the normal vector field ς, then using equations (2.1) we immediately have

$$\nabla_X u = A(X), \quad \nabla^{\perp}_X \varsigma = -h(X,u), \quad X \in \mathfrak{X}(M)\,. \tag{2.2}$$

Let M be an orientable compact minimal hypersurface of the unit sphere $S^{n+1}(1)$ and $F : M \to S^{n+1}(1)$ be its immersion. If $i : S^{n+1}(1) \to R^{n+2}$ is the standard imbedding, then $\psi = i \circ F : M \to R^{n+2}$ makes M submanifold of R^{n+2} with parallel mean curvature vector field H. Let N be the unit normal vector field of M in $S^{n+1}(1)$ and $\overline{N}$ be that of $S^{n+1}(1)$ in R^{n+2}. Then the second fundamental form of M in R^{n+2} is expressed as

$$h(X,Y) = g(BX,Y)N - g(X,Y)\overline{N}, \quad X,Y \in \mathfrak{X}(M)\,, \tag{2.3}$$

where B is the Weingarten map (shape operator) of M in $S^{n+1}(1)$. Since M is the minimal hypersurface of $S^{n+1}(1)$, from equation (2.3) it follows that the mean curvature vector field H of M as submanifold of R^{n+2} is given by

$$H = -\overline{N}\,. \tag{2.4}$$

3. Proof of the Theorem

Let M be an n-dimensional compact and connected minimal hypersurface of constant scalar curvature in $S^{n+1}(1)$ and B be its shape operator. Throughout this section we assume that M is non totally geodesic, consequently that $\|B\|^2$ is a positive constant. For a smooth function $\phi : M \to R$, let $\Delta\phi = \operatorname{div}(\nabla\phi)$ be the Laplacian of ϕ, where $\nabla\phi$ is the gradient of the function ϕ. Let $\xi \in \mathfrak{X}(R^{n+2})$ be a constant unit vector field. Then by equation (2.4) the gradient of the smooth function $\varphi = g(H,\xi)$ is given by $\nabla\varphi = -u$, where $\xi = u + \varsigma$, $u \in \mathfrak{X}(M)$ and $\varsigma \in \Gamma(\upsilon)$, υ being normal bundle of M in R^{n+2}. Note that for the Weingarten map $A = A_\varsigma$ we have

$$\operatorname{tr}A = \sum_i g(Ae_i, e_i) = \sum_i g(h(e_i,e_i),\varsigma) = n\varphi,$$

and consequently the equation $\nabla\varphi = -u$ together with equation (2.2) gives

$$\Delta\varphi = -n\varphi. \tag{3.1}$$

Let $f : M \to R$ be the smooth function defined by $f = g(\varsigma, N)$, N being the unit normal vector field to M in $S^{n+1}(1)$. Then we have

$$\varsigma = fN - \varphi\overline{N}, \tag{3.2}$$

and

$$A = fB + \varphi I. \tag{3.3}$$

Note that $\overline{\nabla}_X \overline{N} = X$ and consequently $\nabla^{\perp}_X \overline{N} = 0$, $X \in \mathfrak{X}(M)$. Thus taking covariant derivative with normal connection in the equation (3.2) and using equation (2.2) we arrive at

$$-h(X,u) = X(f)N + f\nabla^{\perp}_X N - X(\varphi)\overline{N}.$$

Taking inner product with N in above equation and using equation (2.3) we get the following expression for the gradient of f

$$\nabla f = -Bu. \tag{3.4}$$

Note that as M is a minimal hypersurface of $S^{n+1}(1)$, we have tr$B = 0$ and consequently $\sum_i (\nabla B)(e_i, e_i) = 0$, where $(\nabla B)(X,Y) = \nabla_X BY - B(\nabla_X Y)$. Thus using equations (2.2) and (3.4) and a pointwise constant local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, we compute

$$\begin{aligned} \Delta f = -\sum_i e_i g(u, Be_i) = -\sum_i g(Ae_i, Be_i) \\ = -\sum_i g(fBe_i + \varphi e_i, Be_i) = -f\,\|B\|^2. \end{aligned} \tag{3.5}$$

Since M is not totally geodesic and the scalar curvature $\tau = n(n-1) - \|B\|^2$ is a constant, $\|B\|^2$ is a nonzero constant. We claim that the functions φ and f are not both simultaneously constant. If both are constants, by equations (3.1) and (3.5) we will have $\varphi = 0$ and $f = 0$ and consequently the equations (3.2) and $\nabla\varphi = -u$ will give $\varsigma = 0$, $u = 0$. This would mean $\xi = 0$ a contradiction to the fact that ξ is a constant unit vector field.

Next suppose that f is a constant function, then by (3.5), $f = 0$ and as seen above φ is not a constant function. Then by equation (3.3) we get $A = \varphi I$, and consequently the equation $\nabla\varphi = -u$ together with equation (2.2) would imply that $H_\varphi(X,Y) = -\varphi g(X,Y)$, $X, Y \in \mathfrak{X}(M)$, where H_φ is the Hessian of the function φ. This is Obata's differential equation for non constant function φ (cf. [6]), which confirms that $M = S^n(1)$, that is, M is totally geodesic which is contrary to our assumption that M is not totally geodesic. Hence the function f is not a constant and by equation (3.5), it is an eigenfunction of the Laplacian operator Δ. Thus for the minimal hypersurface M which is not totally geodesic we get $\|B\|^2 \in$Spec(M). Since $0 \in$Spec(M), the totally geodesic case is automatically covered. This complete the proof of our theorem.

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