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AN IMPROVEMENT OF MORI'S CONSTANT IN THE THEORY OF SPACE QUASICONFORMAL MAPPINGS

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Abstract: Let $A(n,K)=\{f \mid f$ is a K-quasiconformal mapping which maps unit ball B^n onto B^n with $f(0)=0\}$, and

$$M(n,K)=\sup_{\substack{f\in A(n,K)\\ x,y\in \overline{B}^n, x\neq y}} \frac{|f(y)-f(x)|}{|y-x|^{\alpha}}, \qquad \alpha=K^{\frac{1}{1-n}}.$$

In this paper, the authors prove that $M(n,K)\leq 2\lambda_n^{1-\alpha}m^{\alpha}<3\lambda_n^2$, where $m=\frac{1+\{1+[1+(\frac{3}{2})^{\beta}\lambda_n^{1+\beta}]^2\}^{\frac{1}{2}}}{2}$, $\beta=\frac{1}{\alpha}$, and $\lambda_n\in[4,2e^{n-1}]$ is the Grötzsch ring constant of R^n.

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1. Introduction

For any integer $n\geq 2$ and for $1\leq K<\infty$, let $A(n,K)$ denote the class of $K-$quasiconformal mappings f which

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map unit ball B^n onto itself with $f(0)=0$. Define

$$M(n,K)=\sup_{\substack{f\in A(n,K)\\ x,y\in\overline{B}^n, x\neq y}}\frac{|f(y)-f(x)|}{|y-x|^{\alpha}}, \qquad \alpha=K^{\frac{1}{1-n}}. \tag{1.1}$$

The constant $M(n,K)$ in (1.1) is called Mori's constant in the theory of quasiconformal mappings. For $n=2$, A. Mori [9] first obtained that $M(2,K)\leq 16$ in 1956. Later, V. Gavrilov [4] conjectured that $M(2,K)=16^{1-\frac{1}{K}}$ in 1961, although this problem is still open, but a lot of improved values for $M(2,K)$ are obtained. For example, H.L. Qu [12] got the following result in 1985: If $1\leq K<2$, then $M(2,K)<256^{1-\frac{1}{K}}$ and $\lim\limits_{K\to 1}M(2,K)=1$; if $K\geq 2$, then $M(2,K)<2\cdot 8^{1-\frac{1}{K}}$ and $\lim\limits_{K\to\infty}M(2,K)=16$. V.I. Semënov proved that $M(2,K)\leq 124^{1-\frac{1}{K}}$ in 1987, S.L. Qiu [11] got that $16^{1-\frac{1}{K}}\leq M(2,K)\leq 64^{1-\frac{1}{K}}$ in 1992, and Z. Li and G.Z. Cui [7] also obtained an improved result for $M(2,K)$ in 1993.

For $n\geq 3$, Mori's constant $M(n,K)$ of the $K-$ quasiconformal mappings were estimated by E.D.Callender [3], Yu.G. Reshetnyak [13], [14], B.V. Shabat [17], and O. Martio, S. Rickman and J. Väisälä [8]. In each of these estimates Mori's constant $M(n,K)$ depends both upon the dimension n and the maximal dilatation K. In fact, we can get $M(n,K)\leq 4\lambda_n^2$, where $\lambda_n\in[4,2e^{n-1}]$ is the Grötzsch ring constant which depends only on n, this fact was proved by F.W. Gehring in his *special distortion theorem* [12, Theorem 14, p. 387], and the same method was extended to give the results for all $n\geq 3$. A local variant of Gehring's result was obtained by G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [1], [2], with a constant depending only upon K.

The main aim of this paper is to improve Gehring's result, and prove the following theorem.

Theorem.

$$M(n,K)\leq 2\lambda_n^{1-\alpha}m^{\alpha}<3\lambda_n^2,$$

where $m=\frac{1+\{1+[1+(\frac{3}{2})^{\beta}\lambda_n^{1+\beta}]^2\}^{\frac{1}{2}}}{2}$, $\beta=\frac{1}{\alpha}$, *and* $\lambda_n\in[4,2e^{n-1}]$ *is the Grötzsch ring constant of* R^n.

2. Preliminary Knowledge and Terminology

We shall use the relatively standard notation and terminology from [18]. A ring $R=R(C_0,C_1)$ is a domain (open connected set) in R^n whose complement in

$\overline{R}^n$ consists of two components C_0 and C_1, where C_0 is bounded. The conformal capacity cap$R$ of a ring R is defined by

$$\text{cap}R = \inf_u \int_R |\nabla u|^n dm, \tag{2.1}$$

where the infimum is taken over all continuous functions $u : \overline{R}^n \to R$ such that u is ACT (absolutely continuous in the sense of Tonelli [15]) in R^n, $u = 0$ on C_0 and $u = 1$ on C_1. The modulus mod$R$ is then

$$\text{mod}R = \left(\frac{\omega_{n-1}}{\text{cap}R}\right)^{\frac{1}{n-1}}, \tag{2.2}$$

where ω_{n-1} is the $n-1$ dimension Lebesgue measure of unit sphere ∂B^n. A ring is said to be extremal if it has the maximum modulus among all rings with a certain geometric property.

Two extremal rings are particularly important in our study. The first is the Grötzsch ring $R_{G,n}(t)$, $t > 1$, whose complementary components are the closed unit ball $\overline{B}^n$ and the ray $[t, \infty)$ along the x_1- axis. The other is the Teichmüller ring $R_{T,n}(t)$, $t > 0$, whose complementary components are the segment $[-1, 0]$ and the ray $[t, \infty]$ along the x_1- axis. Their conformal capacities are denoted as in [1] by

$$\gamma_n(t) = \text{cap}R_{G,n}(t), \qquad t > 1 \tag{2.3}$$

and

$$\tau_n(t) = \text{cap}R_{T,n}(t), \qquad t > 0. \tag{2.4}$$

These functions satisfy the following functional identity and inequality [19]

$$\gamma_n(t) = 2^{n-1}\tau_n(t^2 - 1), \quad t > 1 \tag{2.5}$$

and

$$\gamma_n(t) \geq \omega_{n-1}(\log \lambda_n t)^{1-n}, \quad t > 1. \tag{2.6}$$

Next, for $K > 0$ and $n \geq 2$, we define a homeomorphism $\varphi_{K,n} : [0,1] \to [0,1]$ by $\varphi_{K,n}(0,) = 0$, $\varphi_{K,n}(1) = 1$ and

$$\varphi_{K,n} = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))} \tag{2.7}$$

when $0 < t < 1$. If let $\alpha = K^{\frac{1}{1-n}}$, $\beta = \frac{1}{\alpha}$, by [1], [2], [19], for all $n \geq 2$ and $K \geq 1$,

$$\varphi_{K,n}(t) \leq \lambda_n^{1-\alpha} t^{\alpha} \tag{2.8}$$

and

$$\varphi_{1/K,n}(t) \geq \lambda_n^{1-\beta} t^{\beta}. \tag{2.9}$$

We shall require the Poincaré metric $\rho(x,y)$ on B^n defined by

$$\tanh^2 \frac{\rho(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + (1-|x|^2)(1-|y|^2)} \tag{2.10}$$

for all $x, y \in B^n$ or, equivalently, by

$$\sinh^2 \frac{\rho(x,y)}{2} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} \tag{2.11}$$

for $x, y \in B^n$.

The following two inequalities about the Poincaré metric $\rho(x,y)$ were proved in [2], [19]:

$$|x-y| \leq 2\tanh(\frac{1}{4}\rho(x,y)), \qquad x, y \in B^n \tag{2.12}$$

and

$$\tanh(\frac{1}{2}\rho(f(x), f(y))) \leq \varphi_{K,n}(\tanh(\frac{1}{2}\rho(x,y))), \tag{2.13}$$

where f is a $K-$ quasiconformal mapping of B^n into B^n in (2.13).

3. The Proof of Theorem

Proof of Theorem. Let f be a $K-$ quasiconformal mapping which maps B^n onto B^n with $f(0) = 0$, for any $x, y \in B^n$, $x \neq y$. The following two cases will complete the proof of theorem:

Case 1. $|x-y|^2 + (1-|x|^2)(1-|y|^2) \geq \frac{1}{m^2}$. Then we can get

$$\tanh(\frac{1}{2}\rho(x,y)) \leq m|x-y| \tag{3.1}$$

by (2.10).

Combing (2.12) and (2.13) we have

$$\begin{aligned} |f(x)-f(y)| &\leq 2\tanh(\frac{1}{4}\rho(f(x), f(y)) \leq 2\tanh(\frac{1}{2}\rho(f(x), f(y)) \\ &\leq 2\varphi_{K,n}(\tanh(\frac{1}{2}\rho(x,y))). \end{aligned} \tag{3.2}$$

The above inequalities (2.8), (3.1) and (3.2) imply

$$|f(x)-f(y)| \leq 2\lambda_n^{1-\alpha}(\tanh(\frac{1}{2}\rho(x,y))^{\alpha} \leq 2\lambda_n^{1-\alpha}m^{\alpha}|x-y|^{\alpha}. \tag{3.3}$$

Case 2. $|x-y|^2+(1-|x|^2)(1-|y|^2) \leq \frac{1}{m^2}$. From this we obtain that $|x-y| \leq \frac{1}{m}$ and $(1-|x|^2)(1-|y|^2) \leq \frac{1}{m^2}$. Without loss of generality, we may assume $1-|x|^2 \leq \frac{1}{m}$, $|x| \geq \frac{\sqrt{m(m-1)}}{m}$. It is easy to verify that $m > \frac{1+\sqrt{5}}{2}$ by the expression of m in the theorem, this ensure

$$\frac{2\sqrt{m(m-1)}-1}{2m} > \frac{1}{2m} \geq \frac{1}{2}|x-y|. \tag{3.4}$$

According to (3.4) we consider the following circular ring R:

$$R = R(C_0, C_1) = \{z \in R^n \mid \frac{1}{2}|x-y| < |z-\frac{1}{2}(x+y)| < \frac{2\sqrt{m(m-1)}-1}{2m}\}, \tag{3.5}$$

where C_0 and C_1 denote the bounded and unbounded complement components of R respectively. It is easy to see that $x, y \in C_0$ and $\infty \in C_1$, we can conclude that $0 \in C_1$ by

$$\frac{1}{2}|x+y| = |x-\frac{1}{2}(x-y)| \geq |x| - \frac{1}{2}|x-y| \geq \frac{2\sqrt{m(m-1)}-1}{2m}. \tag{3.6}$$

Since f is a $K-$ quasiconformal mapping which maps B^n onto B^n, it is well-known that f has an extension to a homeomorphism of $\overline{B}^n$ onto $\overline{B}^n$, see [18], [10]. We retain the notation f for the extended homeomorphism, constructing homeomorphism F of $\overline{R}^n$ onto $\overline{R}^n$ as following:

$$F(x) = \begin{cases} f(x), & x \in \overline{B}^n, \\ \infty, & x = \infty, \\ \frac{f(\frac{x}{|x|^2})}{|f(\frac{x}{|x|^2})|^2}, & x \in R^n \setminus \overline{B}^n. \end{cases} \tag{3.7}$$

The above homeomorphism F is a $K-$ quasiconformal mapping also [6], [20], $F(x), F(y) \in F(C_0)$ and $0, \infty \in F(C_1)$. By making use of [17, Lemma 7.35] we get

$$\mathrm{cap} F(R) \geq \tau_n \left(\frac{|F(y)|}{|F(x)-F(y)|} \right). \tag{3.8}$$

On the other hand, the basic properties of quasiconformal mapping [18], [19] yield

$$\mathrm{cap} F(R) \leq K \mathrm{cap} R = K\omega_{n-1} \left(\log \frac{2\sqrt{m(m-1)}-1}{m|x-y|} \right)^{1-n}. \tag{3.9}$$

From (3.8), (2.5) and (2.6) we have

$$\begin{aligned}\mathrm{cap}F(R) &\geq 2^{1-n}\gamma_n\left(\sqrt{\frac{|F(y)-F(x)|+|F(y)|}{|F(y)-F(x)|}}\right)\\ &\geq 2^{1-n}\omega_{n-1}\left(\log\lambda_n\sqrt{\frac{|F(y)-F(x)|+|F(y)|}{|F(y)-F(x)|}}\right)^{1-n}. \end{aligned} \tag{3.10}$$

Combing (3.9) and (3.10) we obtain

$$\alpha\log\frac{2\sqrt{m(m-1)}-1}{m|x-y|}\leq 2\log\left(\lambda_n\sqrt{\frac{|F(y)-F(x)|+|F(y)|}{|F(y)-F(x)|}}\right), \tag{3.11}$$

since $F(x), F(y)\in B^n$, $|F(x)-F(y)|+|F(y)|\leq 3$, this and (3.11) implies

$$|F(y)-F(x)|\leq 3\lambda_n^2\left(\frac{m}{2\sqrt{m(m-1)}-1}\right)^{\alpha}|y-x|^{\alpha}, \tag{3.12}$$

by the construction of F in (3.7) we know that

$$|f(y)-f(x)|\leq 3\lambda_n^2\left(\frac{m}{2\sqrt{m(m-1)}-1}\right)^{\alpha}|y-x|^{\alpha}, \tag{3.13}$$

It is not difficult to verify that $2\lambda_n^{1-\alpha}m^{\alpha}=3\lambda_n^2\left(\frac{m}{2\sqrt{m(m-1)}-1}\right)^{\alpha}<3\lambda_n^2$ by the expression of m in the theorem.

At last, we conclude that $M(n,K)\leq 2\lambda_n^{1-\alpha}m^{\alpha}$ by Case 1, Case 2 and the randomicity of f. □

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