ON THE NUMBER OF k-INDEPENDENT SETS
IN SOME PRODUCTS OF GRAPHS

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Abstract: A subset $S \subseteq V(G)$ is $k$-independent if for each two distinct vertices from $S$ the distance between them is at least $k$. In this paper we determine the number of all $k$-independent sets in some product of graphs.

AMS Subject Classification: 05C20
Key Words: $k$-independent set, counting, graph products

1. Introduction

In general we use the standard terminology and notation of graph theory, see Berge [1] and Diestel [3]. Only simple, undirected, connected graphs are considered. $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. The length of the shortest path joining vertices $x$ and $y$ in $G$ will be denoted by $d_G(x,y)$. By $P_n$, $n \geq 2$ we mean the graph with the vertex set $V(P_n) = \{t_1, \ldots, t_n\}$ and the edge set $E(P_n) = \{\{t_i, t_{i+1}\}; \; i = 1, \ldots, n-1\}$, $n \geq 1$. Moreover $P_1$ is a graph with $V(P_1) = \{t_1\}$ and $P_0$ is a graph with $V(P_0) = \emptyset$. By $K_n$ we will denote the complete graph on $n$ vertices, $n \geq 1$. Let $G$ be a graph on $V(G) = \{t_1, \ldots, t_n\}$, $n \geq 1$ and $H$ be a graph on $V(H) = \{y_1, \ldots, y_m\}$, $m \geq 1$. By Cartesian product of two graphs $G$ and $H$ we mean a graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(t_i, y_p), (t_j, y_q)\}; \; t_i = t_j$ and $\{y_p, y_q\} \in E(H)$ or $\{t_i, t_j\} \in E(G)$ and $y_p = y_q$. Let $G$ be a graph on $V(G) = \{t_1, \ldots, t_n\}$, $n \geq 2$ and $h_n = (H_i)_{i \in \{1, \ldots, n\}}$ be a sequence of vertex

Received: April 17, 2007 © 2007, Academic Publications Ltd.

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disjoint graphs on \( V(H_i) = V = \{y_1, \ldots, y_x\}, \ x \geq 1 \). By generalized lexicographic product of \( G \) and \( h_n = (H_i)_{i \in \{1, \ldots, n\}} \) we mean a graph \( G[h_n] \) such that \( V(G[h_n]) = V(G) \times V \) and \( E(G[h_n]) = \{(t_i, y_p), (t_j, y_q)\}; \ t_i = t_j \) and \( (y_p, y_q) \in E(H_i) \) or \( \{t_i, t_j\} \in E(G) \). If \( H_i = H, \ i = 1, \ldots, n \), then \( G[h_n] = G[H] \), where \( G[H] \) is a lexicographic product of two graphs.

A subset \( S \subseteq V(G) \) is said to be \( k \)-independent of \( G \) if for each two distinct vertices \( x, y \in S, d_G(x, y) \geq k \). In addition the empty set and a subset containing only one vertex also are meant as a \( k \)-independent sets of \( G \). Note that for \( k = 2 \) the definition reduces to the definition of an independent set of the graph \( G \). The \( k \)-independent sets and the total number of \( k \)-independent sets of graph were studied in [4]-[10]. Prodinger et al [7] initiated the study of the number of independent sets in a graph. The problem of counting the number of independent sets in graph is NP-complete (see for instance Roth [8]). However for certain types of graphs the problem of determining their numbers of independent sets is polynomial. The literature includes many papers dealing with the theory of counting of \( k \)-independent sets in graphs, see [2, 5, 6, 9].

The total number of independent sets of graph \( G \) was named by Prodinger et al [7] as Fibonacci number of a graph \( G \). They denote it by \( F(G) \). Let \( |V(G)| = n \). If \( f_G(n, p) \) denotes the number of all \( p \)-elements independent sets of \( G \), then \( F(G) = \sum f_G(n, p) \). It is interesting to know that \( F(P_n) = F_n = \sum_{p \geq 0} \binom{n - p + 1}{p} \), so it is equal to the Fibonacci numbers, see Berge [1]. The Fibonacci numbers has also the recurrence form \( F_n = F_{n-1} + F_{n-2} \) with the initial conditions \( F_0 = 1, \ F_1 = 2 \). Kwaśnik et al [5] defined more general concept, namely generalized Fibonacci number of a graph \( G \). This number was defined as the total number of \( k \)-independent sets of a graph \( G \) and it was denoted by \( F_k(G) \). If \( f_G(k, n, p) \) denotes the number of all \( p \)-elements \( k \)-independent sets of \( G \), then \( F_k(G) = \sum_{p \geq 0} f_G(k, n, p) \). It was proved:

**Theorem 1.** (see [5]) Let \( k \geq 2, \ n \geq 0, \ 0 \leq p \leq n \) be integers. Then

\[
F_k(P_n) = \sum_{p \geq 0} f_{P_n}(k, n, p) = \sum_{p \geq 0} \binom{n - p - (p - 1)(k - 2) + 1}{p}.
\]

The \( F_k(P_n) \) generalize the Fibonacci numbers \( F_n \) and we put notation \( F_k(P_n) = F(k, n) \). Evidently \( F(2, n) = F_n \). The numbers \( F(k, n) \) has also the recurrence form:

**Theorem 2.** (see [5]) Let \( k \geq 2, \ n \geq 0 \) be integers. Then numbers \( F(k, n) \)
satisfy the following recurrence: $F(k, n) = F(k, n-1) + F(k, n-k)$ for $n \geq k$ with the initial conditions $F(k, n) = n+1$ for $n = 0, 1, \ldots, k-1$.

For others classes of graphs the total number of $k$-independent sets were determined, see [4]-[10].

**Theorem 3.** (see [4]) Let $n \geq 0$, $p \geq 0$, $x \geq 1$ be integers. Then for an arbitrary graph $G$ on $n$ vertices $F(G[K_x]) = \sum_{p \geq 0} f_G(n, p)x^p$.

**Theorem 4.** (see [10]) Let $k \geq 3$, $x \geq 1$, $p \geq 0$ be integers. Then for an arbitrary graph $G$ on $n, n \geq 2$, vertices and for an arbitrary sequence of vertex disjoint graphs $h_n = (H_i)_{i \in \{1, \ldots, n\}}$ such that $|V(H_i)| = x$ for $i = 1, \ldots, n$, $F_k(G[h_n]) = \sum_{p \geq 0} f_G(k, n, p)x^p$.

2. Main Results

Now we consider the graph $P_n \times K_m, n \geq 0$, $m \geq 1$ and we present numbers $F((P_n \times K_m)[K_x])$ and $F_k((P_n \times K_m)[h_n])$, where $h_n = (H_i)_{i \in \{1, \ldots, n\}}$ is an arbitrary sequence of graphs. Firstly we calculate the numbers $f_{P_n \times K_m}(k, mn, p)$. In this paper for convenience we denote these numbers by $f(k, n, p)$.

**Theorem 5.** Let $k \geq 2, p \geq 2$ be integers. If $n < (p-1)\tau + 1$, then $f(k, n, p) = 0$, where $\tau = \begin{cases} k-1 & \text{if } m > 1, \\ k & \text{if } m = 1. \end{cases}$

**Proof.** Let $n < (p-1)\tau + 1$. We shall prove that $f(k, n, p) = 0$. Assume on the contrary that $f(k, n, p) > 0$. This means that there exists a $p$-element $k$-independent set of $P_n \times K_m$. If $m = 1$, then it is clear that graph $P_n \times K_1$ is isomorphic to $P_n$ and to construct a $k$-independent set $S$ of $P_n$ having $p$-elements, $p \geq 2$, we need at least $(p-1)k+1$ vertices, hence $S = \{t_1, t_{k+1}, \ldots, t_{(p-1)k+1}\}$, contradiction the assumption. If $m > 1$, then from the definition of graph $P_n \times K_m$ and by fact that $K_m$ is a complete graph on $m$ vertices we deduce that to construct a $k$-independent set $S'$ of $P_n \times K_m$ we need at least $(p-1)(k-1) + 1$ vertices. Then $S'$ has the following form $S' = \{(t_1, y_i), (t_k, y_j), \ldots, (t_{(p-1)(k-1)+1}, y_q)\}$, where if $(t_r, y_i), (t_s, y_j) \in S'$ and $s = r + k - 1$, then $i \neq j$. Hence $n \geq (p-1)(k-1) + 1$, contradiction. Consequently from the above cases if $n < (p-1)\tau + 1$, then $f(k, n, p) = 0$, where $\tau = \begin{cases} k-1 & \text{if } m > 1, \\ k & \text{if } m = 1. \end{cases}$ Thus the theorem is proved.
Theorem 6. Let $k \geq 2, n \geq 0, m \geq 1, p \geq 0$ be integers. Then the numbers $f(k, n, p)$ satisfy the following recurrence relations: $f(k, n, 0) = 1$, $f(k, n, 1) = mn$, for $p \geq 2$ $f(k, n, p) = 0$ if $n < (p - 1)\tau + 1$ and for $n \geq (p - 1)\tau + 1$ we have $f(k, n, p) = f(k, n - 1, p) + mB^n_1$ and $B^n_1 = f(k, n - k, p - 1) + (m - 1)B^{p-1}_{n-(k-1)}$, where $B^n_1 = 1$ and $\tau = \begin{cases} k - 1 & \text{if } m > 1, \\ k & \text{if } m = 1. \end{cases}

Proof. Let $k, n, p, m$ be as in the statement of the theorem. If $p = 0$, then the empty set is the unique $k$-independent set of the graph $P_n \times K_m$. So $f(k, n, 0) = 1$. If $p = 1$, then every vertex of the graph $P_n \times K_m$ is a $k$-independent set of the graph $P_n \times K_m$. Consequently $f(k, n, 1) = m \cdot n$. Let now $p \geq 2$. If $n < (p - 1)\tau + 1$, where $\tau = \begin{cases} k - 1 & \text{if } m > 1, \\ k & \text{if } m = 1, \end{cases}$ then by Theorem 5 we have that $f(k, n, p) = 0$. Assume that $n \geq (p - 1)\tau + 1$. Let $S_1$ be a family of all $p$-element $k$-independent sets $S \subseteq V(P_n \times K_m)$ such that $(t_n, y_m) \notin S$ and let $S_2$ be a family of all $p$-elements $k$-independent sets $S \subseteq V(P_n \times K_m)$ such that $(t_n, y_m) \in S$. By the general rule for counting $k$-independent sets $f(k, n, p) = |S_1| + |S_2|$. Assume that the number of all $p$-element $k$-independent sets of $P_n \times K_m$ containing a vertex $(t_n, y_i)$, where $i$ is one from $1, \ldots, m$ is equal to $B^n_{p,y_i}$. Moreover by the definition of the Cartesian product and by fact that $K_m$ is a complete graph we deduce that for every $1 \leq i, j \leq m, B^n_{p,y_i} = B^n_{p,y_j}$. Consequently we put notation $B^n_{p,y_i} = B^n_{p,y_j}$. Of course $B^n_1 = 1$. Let $S \in S_1$. If $(t_n, y_i) \notin S, i = 1, \ldots, m - 1$, then $S = S^*$, where $S^*$ is an arbitrary $p$-element $k$-independent set of the graph $P_{n-1} \times K_m$. Hence we have $f(k, n - 1, p)$ $k$-independent sets $S$ having $p$-elements. If there exists $1 \leq i \leq m - 1$ such that $(t_n, y_i) \in S$ by our assumption we have $B^n_i$ such subsets. By the fact that we can choose the vertex $(t_n, y_i)$ belonging to $S$ on $(m - 1)$ ways we obtain that there are $(m - 1)B^n_i$ $k$-independent sets $S$ with $p$-elements. So $|S_1| = f(k, n - 1, p) + (m - 1)B^n_i$ in this case. Now we calculate the number $|S_2|$. By previous considerations $|S_2| = B^n_i$, so we have to determine the number $B^n_i$. Since $(t_n, y_m) \in S$, then $(t_n, y_i) \notin S, i = 1, \ldots, m - 1, (t_n, y_{j}) \notin S$, where $r = n - 1, \ldots, n - (k - 2), j = 1, \ldots, m$ and $(t_{n-(k-1)}, y_m) \notin S$. If $(t_{n-(k-1)}, y_i) \notin S, i = 1, \ldots, m - 1$, then $S = S' \cup \{(t_n, y_m)\}$, where $S'$ is an arbitrary $(p - 1)$-element $k$-independent set of the graph $P_{n-k} \times K_m$, so we have $f(k, n - k, p - 1)$ such sets $S$. If there exists $1 \leq i \leq m - 1$ such that $(t_{n-(k-1)}, y_i) \in S$, then by our assumption we have $B^{p-1}_{n-(k-1)}$ such subsets. By fact that we can choose the vertex $(t_{n-(k-1)}, y_i)$ on $(m - 1)$ ways we obtain $(m - 1)B^{p-1}_{n-(k-1)}$ $p$-element $k$-independent sets $S$. Consequently $B^n_i = f(k, n - k, p - 1) + (m - 1)B^{p-1}_{n-(k-1)}$. Finally from the above cases for $p \geq 2$ we have that
Theorem 7. Let \( k \geq 2, n \geq 0, m \geq 1 \). Then for \( n \geq k \) numbers \( F_k(P_n \times K_m) \) satisfy the following recurrence relations: \( F_k(P_n \times K_m) = F_k(P_{n-1} \times K_m) + mB_{n-1} + B_m \) and \( B_n = F_k(P_{n-k} \times K_m) + (m-1)B_{n-(k-1)} \), with the initial conditions:

\[
F_k(P_n \times K_m) = mn + 1, \text{ for } n = 0, 1, \ldots, k-1; \quad B_n = 1, \quad n = 1, \ldots, k-1.
\]

Thus the theorem is proved. □

Corollary 1. Let \( n \geq 0, p \geq 0, k \geq 2 \) be integers. If \( m = 1 \), then \( \sum_{p \geq 0} f(k, n, p) = F(k, n) \).

Proof. If \( 0 \leq n \leq k \), then \( \sum_{p \geq 0} f(k, n, p) = f(k, n, 0) + f(k, n, 1) = n + 1 = F(k, n) \) in this case. If \( n \geq k + 1 \), then

\[
\sum_{p \geq 0} f(k, n, p) = f(k, n, 0) + f(k, n, 1) + \sum_{p \geq 2} f(k, n, p)
= 1 + n + \sum_{p \geq 2} (f(k, n-1, p) + f(k, n-2, p-1))
= 1 + n + \sum_{p \geq 2} f(k, n-1, p) + \sum_{p \geq 2} f(k, n-2, p-1)
= (n-1) + 1 + \sum_{p \geq 2} f(k, n-1, p) + 1 + \sum_{r=p-1 \geq 1} f(k, n-k, r)
= f(k, n-1, 0) + f(k, n-1, 1) + \sum_{p \geq 2} f(k, n-1, p) + f(k, n-k, 0)
+ \sum_{r \geq 1} f(k, n-k, r) = \sum_{p \geq 0} f(k, n-1, p) + \sum_{r \geq 0} f(k, n-k, r)
= F(k, n-1) + F(k, n-k) = F(k, n),
\]

which ends the proof. □

Corollary 2. Let \( n \geq 1, m \geq 1, x \geq 1 \) be integers. Then \( F((P_n \times K_m)[K_x]) = \sum_{p \geq 0} f(2, n, p)x^p \).

Corollary 3. Let \( k \geq 3, n \geq 2, m \geq 1, x \geq 1 \) be integers. Then for an arbitrary sequence of vertex disjoint graphs \( h_n = (H_i)_{i \in \{1, \ldots, n\}} \) such that \( |V(H_i)| = x \), for \( i = 1, \ldots, n \) we have \( F_k((P_n \times K_m)[h_n]) = \sum_{p \geq 0} f(k, n, p)x^p \).
Proof. If $n = 0$, then also $p = 0$ and this implies $F_k(P_n \times K_m) = f(k, 0, 0) = 1$, by the definition of $F_k(P_n \times K_m)$.

If $n = 1, \ldots, k - 1$, then $p = 0$ or $p = 1$ so $F_k(P_n \times K_m) = f(k, n, 0) + f(k, n, 1)$ and using Theorem 6 we have $F_k(P_n \times K_m) = 1 + mn$.

Let $n \geq k$. Then using Theorem 6, $F_k(P_n \times K_m) = \sum_{p \geq 0} f(k, n, p) = \sum_{p \geq 0} (f(k, n - 1, p) + mB_n^p) = \sum_{p \geq 0} f(k, n - 1, p) + m \sum_{p \geq 0} B_n^p = F_k(P_{n-1} \times K_m) + m \sum_{p \geq 0} B_n^p$.

Let $\sum_{p \geq 0} B_n^p = B_n$. Evidently if $n = 1, \ldots, k-1$, then $p = 1$ and $B_n = B_n^1 = 1$.

In our case $n \geq k$ we obtain $F_k(P_n \times K_m) = F_k(P_{n-1} \times K_m) + mB_n$. Moreover by Theorem 6, $B_n = \sum_{p \geq 0} B_n^p = \sum_{p \geq 0} (f(k, n - k, p - 1) + (m - 1)B_n^{p-1}_{n-(k-1)}) = \sum_{p \geq 0} f(k, n - k, p - 1) + (m - 1) \sum_{p \geq 0} B_n^{p-1}_{n-(k-1)}$. Because for $p = 0$ the numbers $f(k, n - k, p - 1)$ and $B_n^{p-1}_{n-(k-1)}$ there no exist we can put $B_n = \sum_{r = p-1 \geq 0} f(k, n - k, r) + (m - 1) \sum_{r = p-1 \geq 0} B_n^{r}_{n-(k-1)} = F_k(P_{n-k} \times K_m) + (m - 1)B_n^{0}_{n-(k-1)}$, which ends the proof.

References


