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TOWARDS A PARSIMONIOUS CALCULATION OF JACK POLYNOMIALS

Leigh Roberts
School of Economics and Finance
Victoria University
P.O. Box 600, Wellington, NEW ZEALAND
e-mail: leigh.roberts@vuw.ac.nz

Abstract: We prove the following new result concerning coefficients of expansions of Jack polynomials in monomial and elementary symmetric functions. Up to normalisation, we show that those coefficients are unchanged upon removing the first row or column of the Ferrers diagrams of the partitions used for indexing, or upon reflecting the Ferrers diagrams in a sufficiently large grid. Our results exhibit a strong duality between expansion of Jack polynomials in monomial and elementary symmetric functions; and provide an important step towards the efficient calculation of Jack polynomials through their parsimonious expansion in those symmetric functions.

AMS Subject Classification: 05E05
Key Words: Jack polynomials, monomial symmetric functions, elementary symmetric functions, Ferrers diagram

1. Introduction

Jack polynomials $J_\rho^{(\alpha)}$, indexed by partitions ρ and with $\alpha > 0$, are of interest to physicists and engineers, among many others. To statisticians Zonal polynomials $Z_\rho = J_\rho^{(2)}$ (see [3, p. 408]) are of particular relevance, since they arise in transformations of covariance matrices in statistical inference (e.g. [5, Section 2.5.1]). It was in this context that Takemura [8, Chapter 4] investigated the expansion of Zonal polynomials in terms of monomial and elementary symmetric polynomials, inter alia. Takemura found that up to normalisation the coefficients in these expansions were unaltered if one removed the first row or column of the Ferrers diagram of the indexing partitions; or if one reflected

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those Ferrers diagrams in a sufficiently large rectangular grid. We refer to these operations as slicing and reflection operations respectively.

In principle Takemura's results simplified the calculation of Zonal polynomials. Expanding these polynomials in monomial and elementary symmetric functions is appealing because, of the commonly used bases for the symmetric functions, only these functions result in triangular coefficient matrices. Takemura's results applied to both of these types of symmetric function, and meant that some of the coefficients of Z_ρ for partitions ρ of higher weight were very simply related to coefficients for partitions ρ of lower weight, as indicated in the example provided in Section 5. Nevertheless there have been few attempts to calculate Zonal polynomials, and Takemura's results seem not to have been applied: see [5, Section 1.3] for a discussion.

The triangularity of the coefficient matrices for expansions in terms of the monomial and elementary symmetric functions persists for general $\alpha > 0$, and the principal result of this paper extends Takemura's results for $\alpha = 2$ to Jack polynomials, viz. $\alpha > 0$. Our proofs are simpler than those in [8], in addition to which our methods highlight the complementarity of these results, which is less clear from Takemura's work. Our results are an important step towards utilising monomial and elementary symmetric functions to calculate Jack polynomials, which functions provide far more parsimonious representations of these polynomials than the power sum functions conventionally used for this purpose (see [3, p. 23], i.e., for the definition of the power sums, and [2] for low order expansions of this nature).

We utilise the expansion of Jack polynomials as determinants of matrices which assume the form of a single column superimposed on an otherwise triangular matrix. The determinants of such matrices involve summing over characteristic zigzag terms, illustrated in Figures 1 and 2. For the applications in this paper the outstanding column is firstly the vector of monomial symmetric functions m_λ, and then the vector of elementary symmetric functions e_λ, as illustrated schematically in Figures 2(a) and (b) respectively.

The twin expansions of Jack polynomials are closely related, as indicated in Section 3.1. Simultaneous development of both expansions involves little more effort than calculating either individually.

The calculations are complementary in the sense that expansion in monomial symmetric functions results in terms of the one sign, and tends to involve partitions low in the reverse lexicographic ordering (RLO); while expansion in elementary symmetric functions results in terms of alternating signs, and tends to involve partitions high in the RLO. See [5, Section 8.2] for a discussion concerning possible advantages and disadvantages of the two expansions.

When Jack polynomials are expanded in terms of monomial and elementary symmetric functions, the normalisation effectively imposed in this paper is that the leading coefficient is unity. The constant needed to convert these functions to the conventional normalisation is given in [3], and reproduced in Section 3.

Notation and definitions are given in Section 2. In Theorem 8 of Section 3 we give the desired forms of the coefficients of the expansions of the Jack polynomials, which are illustrated in Figure 2. Lemmas supporting the main result are provided in Section 4, and the paper concludes with the principal theorem and examples.

2. Notation and Definitions

2.1. Partitions and Matrices Indexed by Partitions

Unless stated otherwise, partitions are assumed to be in standard form, i.e. listed as positive non-increasing integers with no trailing zeroes. Then $\lambda = (l_1, l_2, \ldots, l_r)$ has weight $w(\lambda) = \sum_j l_j$, and we write $\lambda \vdash w(\lambda)$. The length of λ is given by $\ell(\lambda) = r$, and (following [8]) the height of λ is $h(\lambda) = l_1$. The conjugate partition to λ is denoted by $\lambda' = (l_1', l_2', \ldots)$, and for partitions $\lambda, \kappa, \rho, \sigma, \tau, \ldots$, it is understood that $w = w(\lambda) = w(\kappa) = \ldots$ unless otherwise specified. The multiplicity of i in λ is $m_i^{(\lambda)}$, so that $l_j' - l_{j+1}' = m_j^{(\lambda)}$.

The dominance or majorisation partial ordering is denoted by $\leq$: thus $\kappa = (k_1, k_2, \ldots) \geq \lambda = (l_1, l_2, \ldots) \Leftrightarrow k_1 + k_2 + \ldots + k_i \geq l_1 + l_2 + \ldots + l_i$ for all i, provided that $w(\kappa) = w(\lambda)$. For partitions κ and λ not necessarily of the same weight, we define $\kappa + \lambda = (k_1, k_2, \ldots) + (l_1, l_2, \ldots) = (k_1 + l_1, k_2 + l_2, \ldots)$, with trailing zeroes for the shorter partition; and the parts of $\kappa \cup \lambda$ to contain all the parts of κ together with those of λ. According to [3, p. 5], these operations are dual in the sense that $(\kappa \cup \lambda)' = \kappa' + \lambda'$.

The conventional total ordering of partitions, viz. the reverse lexicographic ordering (RLO), is denoted by $\overset{R}{\leq}$: hence $(4) \overset{R}{\geq} (3, 1)$. The majorisation partial ordering is consistent with the RLO: $\kappa \geq \lambda \Rightarrow \kappa \overset{R}{\geq} \lambda$.

Matrices may have rows and columns indexed either conventionally, or by partitions in RLO. In the latter case the top left element of a matrix $A = (a_{\kappa,\lambda})$ is $a_{(w),(w)}$ while the element to its right is $a_{(w),(w-1,1)}$, etc. Let $[\kappa, \lambda]$ denote the interval $\{\tau : \kappa \geq \tau \geq \lambda\}$. For a matrix A with rows and columns indexed by partitions in RLO, $A_{[\kappa,\lambda]}$ represents the submatrix indexed by rows and columns in the interval shown. The identity matrix over the same indexing set is denoted by $I_{[\kappa,\lambda]}$, and the transpose of a matrix A is denoted by A^T.

Following [3] and [7], monomial and elementary symmetric functions are denoted by m_λ and e_λ respectively, and are defined on variates $x_1, x_2, \ldots, x_n$. These functions are stacked into column vectors M and E respectively in RLO, so that the listing of the indices of the vector elements from the top is $(w), (w-1,1), (w-2,2), \ldots$. Thus

$$M^T = \left(m_{(w)}, m_{(w-1,1)}, \ldots, m_{(1^w)}\right) , \quad E^T = \left(e_{(w)}, e_{(w-1,1)}, \ldots, e_{(1^w)}\right) ,$$

while for $\kappa \geq \lambda$,

$$M^T_{[\kappa,\lambda]} = (m_\kappa, \ldots, m_\lambda) , \quad E^T_{[\kappa,\lambda]} = (e_\kappa, \ldots, e_\lambda) ,$$

where these vectors reduce to a single element in the case for which $\kappa = \lambda$. $y_{[\kappa,\lambda]} = y$ say is a column vector with elements indexed by $[\kappa, \lambda]$, such that the first element $y_\kappa = 1$ and the remaining elements zero. Analogously, $z_{[\kappa,\lambda]} = z$ is a column vector indexed by $[\kappa, \lambda]$, with final element $z_\lambda = 1$ and all previous elements zero. For a column vector v, let $(v; 0)$ denote a square matrix with first column v and remaining elements zero, and $(0; v)$ a square matrix with final column v, again with other elements zero.

2.2. Definition of a d2 Series of Partitions

Definition 1. When $w(\kappa) = w(\lambda)$ we define $\kappa \overset{d2}{>} \lambda$ to mean that:

1. $\kappa > \lambda$; and

2. In some possibly non-standard ordering of the partitions, and perhaps with an additional zero appended to κ, κ and λ differ in exactly two elements.

Definition 2. A d2 series of partitions from κ to λ of length t is a sequence

$$\kappa \overset{d2}{>} \sigma_1 \overset{d2}{>} \ldots \overset{d2}{>} \sigma_{t-1} \overset{d2}{>} \lambda \,. \tag{1}$$

We write $\{\sigma_j\} \in d2 : [\kappa, \lambda]$ as shorthand for (1); we further write $|\{\sigma_j\}| = t-1$ to indicate a d2 series of length t.

2.3. Reflection of Partitions

Definition 3. If $r \geq h(\lambda)$ and $s \geq \ell(\lambda)$, then λ is said to be (r^s)-reflectible, since its reflection in the $s \times r$ grid (r^s) is then well defined.

For (r^s)-reflectible $\lambda \vdash w$, define the (r^s)-reflection $\lambda^* \vdash (rs - w)$ by

$$\lambda + \lambda^* = (r^s) \,.$$

That is, if $\lambda = (l_1, l_2, \ldots)$ and $\lambda^* = (l_1^*, l_2^*, \ldots)$, then $l_i^* = r - l_{s+1-i}$ for $1 \le i \le s$, with possible trailing zeroes in the listings of the partitions.

3. The Basic Form of the j^m and j^e Coefficients

The main result of this section is Theorem 8, which provides the coefficients $j^m_{\rho,\lambda}$ and $j^e_{\rho,\lambda}$ as determinantal expansions, illustrated in Figure 2. The principal tools needed for this theorem are Lemmas 4 and 5, adapted from a result in Lapointe et al [1, Property 4], and which are illustrated in Figure 1.

Lemma 4. *Let $B = (b_{ij})$ be a $u \times u$ matrix with conventional ordering of rows and columns, such that $b_{ij} = 0$ whenever $i > j$ and $j \ne r$, where $1 \le r < u$. Then*

$$\det B = \sum_{s \ge r} \sum_{t \ge 0} (-1)^t \sum_{\substack{r<i_j<s \\ |\{i_j\}|=t-1}} b_{s,r}\, b_{r,i_1}\, b_{i_1,i_2} \ldots b_{i_{t-2},i_{t-1}}\, b_{i_{t-1},s} \prod_{\substack{k \ne r,s, \\ k \notin \{i_j\}}} b_{k,k}$$

in which the third summation is taken over all $(t-1)-$ subsets $\{i_j\}$ of $\{i : r < i < s\}$, and $i_j < i_{j+1}$ for each j. The case $s = r$ corresponds to the product of the diagonal elements of the matrix, while the elements $b_{s,r}$ for $s < r$ do not appear.

Proof. Choosing the (s,r)-th term from the single column below the diagonal forces the characteristic zigzag pattern of elements chosen from above the diagonal, as indicated in Figure 1(a). All terms in the expansion of $\det B$ are obtained by varying the number $t-1$ of such elements chosen above the diagonal between the r-th and s-th columns, and then by choosing all possible combinations of the given number of elements between those limits. □

The analogous result when an aberrant column is superimposed on a lower triangular matrix is given in the next lemma, and is illustrated in Figure 1(b).

Lemma 5. *Let $B = (b_{ij})$ be a $u \times u$ matrix such that $b_{ij} = 0$ whenever $i < j$ and $j \ne r$, where $1 < r \le u$. Then*

$$\det B = \sum_{s \le r} \sum_{t \ge 0} (-1)^t \sum_{\substack{s<i_j<r \\ |\{i_j\}|=t-1}} b_{i_1,s}\, b_{i_2,i_1}\, b_{i_3,i_2} \ldots b_{r,i_{t-1}}\, b_{s,r} \prod_{\substack{k \ne r,s, \\ k \notin \{i_j\}}} b_{k,k}\,.$$

Proof. The proof is similar to that of the preceding lemma. □

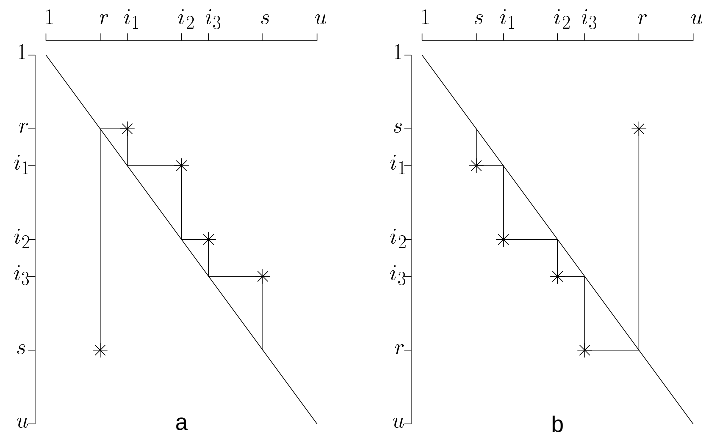

Figure 1: Illustrative terms in $\det B$ for $t = 4$ from Lemmas 4 and 5

The Jack polynomial $J_\rho^{(\alpha)}$ is an eigenfunction of the quasi Laplace-Beltrami operator (see [6]):

$$\mathcal{L}^{(\alpha)}\ J_\rho^{(\alpha)} = c_\rho^{(\alpha)}\ J_\rho^{(\alpha)}\,. \tag{2}$$

For $\rho = (r_1, r_2, \ldots)$, the eigenvalue is given by

$$c_\rho^{(\alpha)} = \left(n - \frac{\alpha}{2}\right) w - \sum\ j\ r_j + \frac{\alpha}{2} \sum\ r_j^2\,, \tag{3}$$

while the operator assumes the form

$$\mathcal{L}^{(\alpha)} = \frac{\alpha}{2} \sum_{i=1}^{n}\ x_i^2 D_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{n}\ \frac{x_i^2}{x_i - x_j}\ D_i \tag{4}$$

in which $D_i = \partial/\partial x_i$. We write c_ρ for $c_\rho^{(\alpha)}$, J_ρ for $J_\rho^{(\alpha)}$ and $\mathcal{L}$ for $\mathcal{L}^{(\alpha)}$.

Theorem 6. *Define operator matrices* $\Omega^m = \Omega^m_{[(w),(1^w)]} = \left(\omega^m_{\kappa,\lambda}\right)$ *and* $\Omega^e = \Omega^e_{[(w),(1^w)]} = \left(\omega^e_{\kappa,\lambda}\right)$ *by the relations:*

$$\mathcal{L}\, M = \Omega^m \qquad \text{and} \qquad \mathcal{L}\, E = \Omega^e E\,. \tag{5}$$

The elements $\omega^m_{\kappa,\lambda}$ *and* $\omega^e_{\kappa,\lambda}$ *are then as follows. When* $\kappa \overset{d\,2}{>} \lambda$*, let* $\lambda = (l_i, l_j, \ldots)$ *and* $\kappa = (l_i + s, l_j - s, \ldots)$*, in which* $s > 0$*;* $l_i \geq l_j \geq s$*; the listing of parts* *in κ and λ is not necessarily in the standard non-increasing order; and the remaining elements in the listings of κ and λ are identical. Then*

$$\omega^m_{\kappa,\lambda} = \begin{cases} (l_i - l_j + 2s)\, m^{(\lambda)}_{l_i}\, m^{(\lambda)}_{l_j}\,, & \text{when} \quad l_i > l_j\,, \\ (l_i - l_j + 2s)\, m^{(\lambda)}_{l_i} \left(m^{(\lambda)}_{l_i} - 1 \right)/2\,, & \text{when} \quad l_i = l_j\,, \end{cases} \tag{6}$$

so that $\omega^m_{\kappa,\lambda} > 0$ if $\kappa \overset{d2}{>} \lambda$. Should $\kappa \overset{d2}{\not\geq} \lambda$, then $\omega^m_{\kappa,\lambda} = 0$; and the diagonal elements of Ω^m are given by

$$\omega^m_{\lambda,\lambda} = c_\lambda\,. \tag{7}$$

The diagonal elements of Ω^e are given by

$$\omega^e_{\lambda,\lambda} = c_{\lambda'} \tag{8}$$

and whenever $\lambda \neq \kappa$,

$$\omega^e_{\lambda,\kappa} = -\alpha\, \omega^m_{\kappa,\lambda}\,. \tag{9}$$

Proof. The form of Ω^m is given in [1], [4, Theorem 2.4] or [5, Section 3.5], and as a closely related exercise, in [3, p. 327, Example 3(c)]. Equation (9) is derived in [6]. □

The Ω^m matrix is sparse: according to Roberts [5, Section 4.2], the proportion of non-zero elements above the diagonal is $O\left(w^3\, e^{-K\sqrt{w}}\right)$ as $w \to \infty$, with $K = \pi\sqrt{2/3}$.

Theorem 7. *With c_ρ defined in (3) and Ω^m, Ω^e given in (5),*

$$J_\rho = j^m_{\rho,\rho}\, \frac{\det\left(\Omega^m_{[\rho,(1^w)]} - c_\rho\; I_{[\rho,(1^w)]} + \left(M_{[\rho,(1^w)]};0\right)\right)}{\det\left(\Omega^m_{[\rho,(1^w)]} - c_\rho\; I_{[\rho,(1^w)]} + \left(y_{[\rho,(1^w)]};0\right)\right)} \tag{10}$$

$$= j^e_{\rho,\rho'}\, \frac{\det\left(\Omega^e_{[(w),\rho']} - c_\rho\; I_{[(w),\rho']} + \left(0;E_{[(w),\rho']}\right)\right)}{\det\left(\Omega^e_{[(w),\rho']} - c_\rho\; I_{[(w),\rho']} + \left(0;z_{[(w),\rho']}\right)\right)} \tag{11}$$

in which $j^m_{\rho,\rho}$ and $j^e_{\rho,\rho'}$ are the non-zero leading coefficients in (12). The determinants in the denominators of (10) and (11) reduce to the products of negative terms, save for one element of unity in each case.

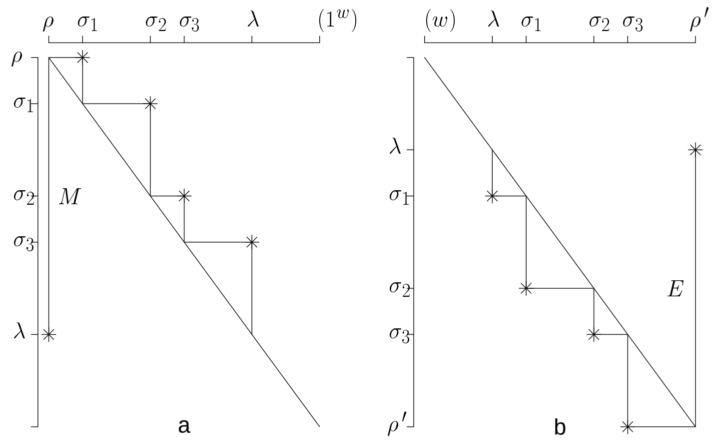

Figure 2: Illustrative terms in $j^m_{\rho,\lambda}$ and $j^e_{\rho,\lambda}$ for $t = 4$ in (13) and (14).

Proof. See [6, Theorem 4] for the proofs of (10) and (11). The denominator in each of (10) and (11) is the determinant of a triangular matrix, and reduces to the product of diagonal elements.

In the case of (10), each of these diagonal terms is negative save the first, which is unity, by (7) and Theorem 14. As for (11), each of the diagonal terms in the denominator matrix is negative apart from the last, which is unity, by virtue of (8) and Theorem 14 on the one hand, and the property that $\kappa > \lambda$ iff $\lambda' > \kappa'$ on the other. ☐

Takemura [8, Section 4.2] derives an explicit expression for $j^e_{\rho,\rho'}$ when $\alpha = 2$.

Theorem 8. *Let*

$$J_\rho = \sum_{\lambda \vdash w} j^m_{\rho,\lambda}\, m_\lambda = \sum_{\lambda \vdash w} j^e_{\rho,\lambda}\, e_\lambda\,. \tag{12}$$

Then $j^m_{\rho,\lambda} = 0$ *unless* $\rho \geq \lambda$; *and when* $\rho \geq \lambda$,

$$j^m_{\rho,\lambda} = j^m_{\rho,\rho} \sum_{t \geq 0} \sum_{\substack{\{\sigma_j\} \in d2:[\rho,\lambda] \\ |\{\sigma_j\}| = t-1}} \frac{\prod_{j=1}^t \omega^m_{\sigma_{j-1},\sigma_j}}{\prod_{j=1}^t \left(c_\rho - c_{\sigma_j}\right)}\,. \tag{13}$$

The second summation in (13) is taken over all distinct d2 series of length t between ρ and λ; and $\sigma_{j-1} > \sigma_j$ for all j.

Analogously, $j^{e}_{\rho,\lambda} = 0$ *unless* $\lambda \geq \rho'$*; and when* $\lambda \geq \rho'$

$$j^{e}_{\rho,\lambda} = j^{e}_{\rho,\rho'} \sum_{\substack{\{\sigma_j\} \in d\,2:[\lambda,\rho'] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^{t} \ \omega^{e}_{\sigma_j,\sigma_{j-1}}}{\prod_{j=1}^{t} \left(c_\rho - c_{\sigma_j'}\right)} . \tag{14}$$

Each term in each product in the denominators of (13) and (14) is positive.

Proof. We apply Theorem 7, and first show that in the expansion of the determinant in the numerator of (10), for the term containing m_λ one may restrict the range of the matrices involved in both the numerator and denominator of (10) to $[\rho, \lambda]$. Recall from Theorem 6 that for $\omega^{m}_{\kappa,\lambda} \neq 0$, it is necessary that $\kappa \overset{d\,2}{\geq} \lambda$. Then the zigzag patterns in Figure 2 contributing non-zero terms to $j^{m}_{\rho,\lambda}$ can involve only d2 series between ρ and λ. The transitivity of the $>$ relation implies that all intermediate partitions like σ_j in Figure 2 belong to $[\rho, \lambda]$. All partitions σ such that $\sigma \in [\rho, (1^w)]$ but $\sigma \notin [\rho, \lambda]$ give rise to terms which cancel in the expansion of the determinants in the numerator and denominator of (10).

A similar argument for (11) allows one to restrict the range of partitions to $[\lambda, \rho']$, and one may write

$$j^{m}_{\rho,\lambda} = j^{m}_{\rho,\rho} \ \frac{\det\left(\Omega^{m}_{[\rho,\lambda]} - c_\rho \ I_{[\rho,\lambda]} + \left(z_{[\rho,\lambda]};0\right)\right)}{\det\left(\Omega^{m}_{[\rho,\lambda]} - c_\rho \ I_{[\rho,\lambda]} + \left(y_{[\rho,\lambda]};0\right)\right)} , \tag{15}$$

$$j^{e}_{\rho,\lambda} = j^{e}_{\rho,\rho'} \ \frac{\det\left(\Omega^{e}_{[\lambda,\rho']} - c_\rho \ I_{[\lambda,\rho']} + \left(0;y_{[\lambda,\rho']}\right)\right)}{\det\left(\Omega^{e}_{[\lambda,\rho']} - c_\rho \ I_{[\lambda,\rho']} + \left(0;z_{[\lambda,\rho']}\right)\right)} . \tag{16}$$

Applying Lemmas 4 and 5, and noting that the factors $(-1)^t$ cancel with the negative terms in the denominators of (10) and (11), completes the proof of (13) and (14). The final remark on the positivity of the terms in the denominators follows from Theorem 14. ☐

In this paper we are essentially normalising the Jack polynomials to have leading coefficient unity, labelled as $P^{(\alpha)}_\lambda$ in [3, p. 378]. The constant multiple needed to convert the $P^{(\alpha)}_\lambda$ back to the $J^{(\alpha)}_\lambda$ is given in equation (10.21) in [3, p. 381]:

$$j^{m}_{\rho,\rho} = j^{e}_{\rho,\rho'} = \prod_{s\in\lambda} \left[\alpha\, a(s) + l(s) + 1\right] ,$$

where $a(s)$ is the arm-length and $l(s)$ the leg-length, and the product is taken over the squares in the Ferrers diagram of λ: see [3, p. 337].

3.1. Comparing j^e and j^m Coefficients

From Theorem 8,

$$j^e_{\rho,\lambda} = j^e_{\rho,\rho'} \sum_{\substack{\{\sigma_j\}\in d\,2:[\lambda,\rho'] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^{t} \ \omega^e_{\sigma_j,\sigma_{j-1}}}{\prod_{j=1}^{t}\left(c_\rho - c_{\sigma'_j}\right)} \tag{17}$$

$$= j^e_{\rho,\rho'} \sum_{\substack{\{\sigma_j\}\in d\,2:[\lambda,\rho'] \\ |\{\sigma_j\}|=t-1}} (-\alpha)^t \ \frac{\prod_{j=1}^{t} \ \omega^m_{\sigma_{j-1},\sigma_j}}{\prod_{j=1}^{t}\left(c_\rho - c_{\sigma'_j}\right)}\,, \tag{18}$$

where we have applied (9). In contrast with (18):

$$j^m_{\lambda,\rho'} = j^m_{\lambda,\lambda} \sum_{\substack{\{\sigma_j\}\in d\,2:[\lambda,\rho'] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^{t} \ \omega^m_{\sigma_{j-1},\sigma_j}}{\prod_{j=1}^{t}\left(c_\lambda - c_{\sigma_j}\right)}\,. \tag{19}$$

The numerators of (18) and (19) are identical, and algorithms for calculating j^m and j^e coefficients are closely related.

4. Supporting Lemmas: The Impact of Slicing and Reflection Operators

For our final theorem we need to consider the impact on the numerators and denominators of (13) and (14) of slicing and reflection operators on partitions. In Section 4.2 we look at the effect which these operators have on the ω^m coefficients, and apply these results to d2 series in Section 4.3. In Section 4.4 we consider the impact of these operators on the eigenvalues of the quasi Laplace-Beltrami operator.

4.1. Preliminary Results

Lemma 9. *For any positive integer r,*

$$\kappa > \lambda \text{ iff } \kappa + (1^r) > \lambda + (1^r) \text{ iff } \kappa \cup (r) > \lambda \cup (r)$$

Proof. The first proposition is immediate from the definition of majorisation, given in Section 2.1. Applying this result to the second proposition, one has

$$\kappa > \lambda \text{ iff } \lambda' > \kappa' \text{ iff } \lambda' + (1^r) > \kappa' + (1^r) \text{ iff } \kappa \cup (r) > \lambda \cup (r)\,. \qquad \square$$

Lemma 10. *If κ and λ are both (r^s)-reflectible, then $\kappa > \lambda$ iff $\kappa^* > \lambda^*$.*

Proof. The proof is immediate. $\square$

Lemma 11. *(a) If $q \geq h(\kappa)$, then $\kappa \overset{d2}{>} \lambda$ iff $\kappa \cup (q) \overset{d2}{>} \lambda \cup (q)$.*
(b) If $p \geq \ell(\lambda)$, then $\kappa \overset{d2}{>} \lambda$ iff $\kappa + (1^p) \overset{d2}{>} \lambda + (1^p)$.
(c) If κ and λ are each (r^s)-reflectible, then $\kappa \overset{d2}{>} \lambda$ iff $\kappa^* \overset{d2}{>} \lambda^*$.

Proof. These results are simple consequences of Lemmas 9 and 10. $\square$

4.2. The Impact on ω^m Coefficients

Theorem 12. *Suppose that $\kappa \neq \lambda$.*
(a) If $q \geq h(\kappa)$, then

$$\omega^m_{\kappa\cup(q),\lambda\cup(q)} = \omega^m_{\kappa,\lambda}\,. \tag{20}$$

(b) If $p \geq \ell(\lambda)$, then

$$\omega^m_{\kappa+(1^p),\lambda+(1^p)} = \omega^m_{\kappa,\lambda}\,. \tag{21}$$

(c) If κ and λ are each (r^s)-reflectible, then

$$\omega^m_{\kappa^*,\lambda^*} = \omega^m_{\kappa,\lambda}\,. \tag{22}$$

Proof. For each of (20), (21) and (22), if one side vanishes so does the other, according to Theorem 6 and Lemma 11.

(a) For the listing of κ and λ in the enunciation of Theorem 6, $q > l_i \geq l_j$, so that the multiplicities in (6) are unchanged upon the insertion or deletion of the first row of the Ferrers diagram.

(b) To move from κ and λ to $\kappa + (1^p)$ and $\lambda + (1^p)$ respectively, the multiplicities transform as $m_i^{(\lambda)} = m_{i+1}^{(\lambda+(1^p))}$, and the expressions (6) are unchanged, whence (21).

(c) Finally, $m_i^{(\lambda)} = m_{q-i}^{(\lambda^*)}$. The expressions (6) are again unchanged, whence (22). $\square$

4.3. The Impact on d2 Series

Lemma 13. *Suppose $\kappa > \lambda$, and recall the definition in (1).*
(a) If $q \geq h(\kappa)$, then

$$\kappa \overset{d2}{>} \ldots \overset{d2}{>} \sigma_j \overset{d2}{>} \ldots \overset{d2}{>} \lambda \text{ iff } \kappa \cup (q) \overset{d2}{>} \ldots \overset{d2}{>} \sigma_j \cup (q) \overset{d2}{>} \ldots \overset{d2}{>} \lambda \cup (q)\,.$$

(b) If $p \geq \ell(\lambda)$, then

$$\kappa \overset{d2}{>} \ldots \overset{d2}{>} \sigma_j \overset{d2}{>} \ldots \overset{d2}{>} \lambda \text{ iff } \kappa + (1^p) \overset{d2}{>} \ldots \overset{d2}{>} \sigma_j + (1^p) \overset{d2}{>} \ldots \overset{d2}{>} \lambda + (1^p)\,.$$

(c) If κ and λ are each (r^s)-reflectible, then

$$\kappa \overset{d2}{>} \ldots \overset{d2}{>} \sigma_j \overset{d2}{>} \ldots \overset{d2}{>} \lambda \text{ iff } \kappa^* \overset{d2}{>} \ldots \overset{d2}{>} \sigma_j^* \overset{d2}{>} \ldots \overset{d2}{>} \lambda^*\,.$$

Proof. These results are simple consequences of Lemmas 9 and 10. □

4.4. The Impact on the Eigenvalues c_ρ

Theorem 14. *When $\kappa > \lambda$, then $c_\kappa > c_\lambda$.*

Proof. See [6, Theorem 3] or [5, Section 3.2.1]. □

Lemma 15. *Recall the definition of c_ρ in (3) on p. 176. Let $q \geq h(\rho)$ and $p \geq \ell(\rho)$, where $\rho \vdash w$. Then*

$$c_{\rho \cup (q)} - c_\rho = nq + \alpha\, \frac{q(q-1)}{2} - (q+w)$$

and

$$c_{\rho + (1^p)} - c_\rho = np - \frac{p(p+1)}{2} + \alpha w\,.$$

Proof. The proof is immediate. □

Lemma 16. *For $q \geq h(\kappa)$ and $q \geq h(\lambda)$,*

$$c_{\kappa \cup (q)} - c_{\lambda \cup (q)} = c_\kappa - c_\lambda \tag{23}$$

while for $p \geq \ell(\kappa)$ and $p \geq \ell(\lambda)$

$$c_{\kappa + (1^p)} - c_{\lambda + (1^p)} = c_\kappa - c_\lambda\,. \tag{24}$$

Proof. The proof of (23) follows from the first result in Lemma 15, which shows that $c_{\rho\cup(q)} - c_\rho$ is independent of the partition ρ. The proof of (24) follows similarly from the second result. □

Lemma 17. *If κ and λ are both (r^s)-reflectible, then $c_\kappa - c_\lambda = c_{\kappa^*} - c_{\lambda^*}$*

Proof. Again from (3), and for $\lambda = (l_1, l_2, \ldots)$,

$$c_{\lambda^*} = \left(n - \frac{\alpha}{2}\right)(sr - w) - \sum_{j=1}^{s}(s+1-j)(r-l_j) + \frac{\alpha}{2}\sum_{j=1}^{s}(r-l_j)^2 \ .$$

Simplification yields that the difference $c_\lambda - c_{\lambda^*}$ does not depend on the parts l_j, i.e. it is independent of the partition λ. □

5. The Principal Theorem and Examples

Theorem 18. *If κ is (q^p)-reflectible, then the (q^p)-reflection of κ is denoted by $\overline{\kappa}$. Should κ be (p^q)-reflectible, then the (p^q)-reflection is denoted by $\widetilde{\kappa}$.*

(a) Let $q \geq h(\rho)$. Then

$$j^m_{\rho\cup(q),\lambda\cup(q)} / j^m_{\rho\cup(q),\rho\cup(q)} = j^m_{\rho,\lambda} / j^m_{\rho,\rho} \,. \tag{25}$$

(b) Let $p \geq \ell(\lambda)$. Then

$$j^m_{\rho+(1^p),\lambda+(1^p)} / j^m_{\rho+(1^p),\rho+(1^p)} = j^m_{\rho,\lambda} / j^m_{\rho,\rho} \,. \tag{26}$$

(c) Let $q \geq h(\rho)$ and $p \geq \ell(\lambda)$. Then

$$j^m_{\overline{\rho},\overline{\lambda}} / j^m_{\overline{\rho},\overline{\rho}} = j^m_{\rho,\lambda} / j^m_{\rho,\rho} \,. \tag{27}$$

(d) Let $q \geq h(\rho)$. Then

$$j^e_{\rho\cup(q),\lambda+(1^q)} / j^e_{\rho\cup(q),\rho'+(1^q)} = j^e_{\rho,\lambda} / j^e_{\rho,\rho'} \,. \tag{28}$$

(e) Let $p \geq h(\lambda)$. Then

$$j^e_{\rho+(1^p),\lambda\cup(p)} / j^e_{\rho+(1^p),\rho'\cup(p)} = j^e_{\rho,\lambda} / j^e_{\rho,\rho'} \,. \tag{29}$$

(f) Let $q \geq h(\lambda)$ and $p \geq h(\rho)$. Then

$$j^e_{\widetilde{\rho},\overline{\lambda}} / j^e_{\widetilde{\rho},\overline{\rho'}} = j^e_{\rho,\lambda} / j^e_{\rho,\rho'} \,. \tag{30}$$

Proof. We prove (25), (28) and (30), the other proofs being analogous. These three proofs all utilise Theorem 8 and Theorem 12; the first two additionally apply Lemma 16, while the final proof applies Lemma 17.

If $\rho \not\geq \lambda$, then both sides of (25) vanish, from Theorem 8 and Lemma 9. If $\rho > \lambda$, then $q \geq h(\rho) \geq h(\lambda)$, and we can write

$$\begin{aligned}\frac{j^m_{\rho,\lambda}}{j^m_{\rho,\rho}} &= \sum_{t\geq 0} \sum_{\substack{\{\sigma_j\}\in d\,2:[\rho,\lambda] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^t \ \omega^m_{\sigma_{j-1},\sigma_j}}{\prod_{j=1}^t \left(c_\rho - c_{\sigma_j}\right)} \\ &= \sum_{t\geq 0} \sum_{\substack{\{\sigma_j\}\in d\,2:[\rho,\lambda] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^t \ \omega^m_{\sigma_{j-1}\cup(q),\sigma_j\cup(q)}}{\prod_{j=1}^t \left(c_{\rho\cup(q)} - c_{\sigma_j\cup(q)}\right)} \\ &= \sum_{t\geq 0} \sum_{\substack{\{\sigma_j\}\in d\,2:[\rho\cup(q),\lambda\cup(q)] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^t \ \omega^m_{\sigma_{j-1},\sigma_j}}{\prod_{j=1}^t \left(c_{\rho\cup(q)} - c_{\sigma_j}\right)} = \frac{j^m_{\rho\cup(q),\lambda\cup(q)}}{j^m_{\rho\cup(q),\rho\cup(q)}}.\end{aligned}$$

Similarly, if $\lambda \not\geq \rho'$, then both sides of (28) vanish, again from Theorem 8 and Lemma 9. If $\lambda \geq \rho'$, then $q \geq h(\rho) \geq \ell(\lambda)$, and we can write

$$\begin{aligned}\frac{j^e_{\rho,\lambda}}{j^e_{\rho,\rho'}} &= \sum_{t\geq 0} \sum_{\{\sigma_j\}\in d\,2:[\lambda,\rho']\,|\,|\{\sigma_j\}|=t-1} \frac{\prod_{j=1}^t \ \omega^e_{\sigma_j,\sigma_{j-1}}}{\prod_{j=1}^t \left(c_\rho - c_{\sigma'_j}\right)} \\ &= \sum_{t\geq 0} \sum_{\{\sigma_j\}\in d\,2:[\lambda,\rho']\,|\,|\{\sigma_j\}|=t-1} \frac{\prod_{j=1}^t \ \omega^e_{\sigma_j+(1^q),\sigma_{j-1}+(1^q)}}{\prod_{j=1}^t \left(c_{\rho\cup(q)} - c_{\sigma'_j\cup(q)}\right)} \\ &= \sum_{t\geq 0} \sum_{\{\sigma_j\}\in d\,2:[\lambda+(1^q),\rho'+(1^q)]\,|\,|\{\sigma_j\}|=t-1} \frac{\prod_{j=1}^t \ \omega^e_{\sigma_j,\sigma_{j-1}}}{\prod_{j=1}^t \left(c_{\rho\cup(q)} - c_{\sigma'_j}\right)} = \frac{j^e_{\rho\cup(q),\lambda+(1^q)}}{j^e_{\rho\cup(q),\rho'+(1^q)}}.\end{aligned}$$

As for (30), note that $\widetilde{\lambda'} = \overline{\lambda}'$. When $\lambda \not\geq \rho'$, then both sides of (30) vanish, from Theorem 8 and Lemma 10. If $\lambda \geq \rho'$, then $q \geq h(\lambda) \geq \ell(\rho)$ and $p \geq h(\rho) \geq \ell(\lambda)$, so that ρ is (p^q)-reflectible and λ is (q^p)-reflectible. For (q^p)-reflectible κ one can write

$$\frac{j^e_{\kappa',\lambda}}{j^e_{\kappa',\kappa}} = \sum_{t\geq 0} \sum_{\substack{\{\sigma_j\}\in d\,2:[\lambda,\kappa] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^t \ \omega^e_{\sigma_j,\sigma_{j-1}}}{\prod_{j=1}^t \left(c_{\kappa'} - c_{\sigma'_j}\right)}$$

	ρ	λ	common value	ρ	λ	from
$j^m_{\rho,\lambda}/j^m_{\rho,\rho}$	$(2,2,2)$	$(2,2,1,1)$	$2/3$	$(2,2)$	$(2,1,1)$	(25)
	$(2,2,2)$	$(2,1,1,1,1)$	1	$(2,2)$	$(1,1,1,1)$	
	$(2,2,1,1)$	$(2,1,1,1,1)$	$12/5$	$(2,1,1)$	$(1,1,1,1)$	
$''$	$(5,1)$	$(4,2)$	$4/7$	(4)	$(3,1)$	(26)
	$(5,1)$	$(3,3)$	$18/35$	(4)	$(2,2)$	
	$(4,2)$	$(3,3)$	$2/3$	$(3,1)$	$(2,2)$	
$''$	$(7,4)$	$(7,3,1)$	$4/7$	$(7,3)$	$(6,4)$	(30)
	$''$	$(7,2,2)$	$18/35$	$''$	$(5,5)$	
	$''$	$(6,5)$	$3/5$	$''$	$(7,2,1)$	
	$''$	$(5,4,2)$	$45/49$	$''$	$(5,3,2)$	
$j^e_{\rho,\lambda}/j^e_{\rho,\rho'}$	$(2,2,2)$	$(5,1)$	$1/3$	$(2,2)$	(4)	(28)
	$(2,2,2)$	$(4,2)$	$-4/3$	$(2,2)$	$(3,1)$	
	$(2,2,1,1)$	$(5,1)$	$-8/5$	$(2,1,1)$	(4)	
$''$	$(5,1)$	$(2,2,2)$	$48/35$	(4)	$(2,2)$	(29)
	$(5,1)$	$(2,2,1,1)$	$-24/7$	(4)	$(2,1,1)$	
	$(4,2)$	$(2,2,2)$	$-4/3$	$(3,1)$	$(2,2)$	

Table 1:

$$= \sum_{t\geq 0} \sum_{\substack{\{\sigma_j\}\in d\,2:[\lambda,\kappa] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^{t} \ \omega^e_{\overline{\sigma}_j,\overline{\sigma}_{j-1}}}{\prod_{j=1}^{t}\left(c_{\widetilde{\kappa'}} - c_{\widetilde{\sigma'_j}}\right)} = \sum_{t\geq 0} \sum_{\substack{\{\sigma_j\}\in d\,2:[\lambda,\kappa] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^{t} \ \omega^e_{\overline{\sigma}_j,\overline{\sigma}_{j-1}}}{\prod_{j=1}^{t}\left(c_{\overline{\kappa}'} - c_{\overline{\sigma}'_j}\right)}$$

$$= \sum_{t\geq 0} \sum_{\substack{\{\sigma_j\}\in d\,2:[\overline{\lambda},\overline{\kappa}] \\ |\{\sigma_j\}|=t-1}} \frac{\prod_{j=1}^{t} \ \omega^e_{\sigma_j,\sigma_{j-1}}}{\prod_{j=1}^{t}\left(c_{\overline{\kappa}'} - c_{\sigma'_j}\right)} = \frac{j^e_{\overline{\kappa}',\overline{\lambda}}}{j^e_{\overline{\kappa}',\overline{\kappa}}}.$$

Now set $\rho = \kappa'$. □

Example 19. We set $\alpha = 2$ in the following examples (Table 1).

Corresponding to (25), for instance, one has the following expansions of $J^{(\alpha)}_\rho$:

$$J^{(\alpha)}_{(2,2,2)} = (1+\alpha)(4+2\alpha)(9+3\alpha)m_{(2,2,2)} + 2(4+2\alpha)(9+3\alpha)m_{(2,2,1,1)}$$
$$+ 4!(9+3\alpha)m_{(2,1^4)} + 6!m_{(1^6)}\,,$$

$$J^{(2)}_{(2,2,2)} = \frac{6!}{2}\left[m_{(2,2,2)} + \frac{2}{3}\, m_{(2,2,1,1)} + m_{(2,1^4)} + 2\, m_{(1^6)}\right],$$

$$J^{(\alpha)}_{(2,2)} = (1+\alpha)(4+2\alpha)m_{(2,2)} + 2(4+2\alpha)m_{(2,1,1)} + 4! m_{(1^4)},$$

$$J^{(2)}_{(2,2)} = 4!\left[m_{(2,2)} + \frac{2}{3}\, m_{(2,1,1)} + m_{(1^4)}\right],$$

$$J^{(\alpha)}_{(2,2,1,1)} = (3+\alpha)(8+2\alpha)m_{(2,2,1,1)} + 12(8+2\alpha)m_{(2,1^4)} + 360\, m_{(1^6)},$$

$$J^{(2)}_{(2,2,1,1)} = 60\left[m_{(2,2,1,1)} + \frac{12}{5}\, m_{(2,1^4)} + 6\, m_{(1^6)}\right],$$

$$J^{(\alpha)}_{(2,1,1)} = (3+\alpha)m_{(2,1,1)} + 12\, m_{(1^4)},$$

$$J^{(2)}_{(2,1,1)} = 5\left[m_{(2,1,1)} + \frac{12}{5}\, m_{(1^4)}\right].$$

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