International Journal of Pure and Applied Mathematics
Volume 39 No. 2 2007, 193-201

ON DETERMINATION OF DEVELOPABLE AND MINIMAL SURFACES

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Abstract: The present manuscript presents a determination of developable and minimal surfaces. Given a curve c in $\Re^3$, $T(c)$ is the surface of the tangent fibre bundle of c. The aim of this paper is to obtain some solutions in order to determine these surfaces, which are developable and also minimal.

AMS Subject Classification: 53A04, 53A05, 53A99
Key Words: developable surfaces, minimal surfaces, fundamental magnitudes, tangent fibre bundle of curve

1. Introduction

A minimal surface is one with zero mean curvature. This includes, but is not limited to, surfaces of minimum area subject to constraints on the location of their boundary [1], [8].

Minimal surfaces are defined by the property that they locally minimize area. Physical models of area-minimizing minimal surfaces can be made by dipping a wire frame into a soap solution, forming a soap film. Mathematically, this condition is expressed as a differential equation which has locally as many solutions as you want. However, it is not clear how a locally defined solution extends to infinity.

A plane is a trivial minimal surface, and the first nontrivial examples, the catenoid and helicoid, (Figure 1) were found by Meusnier in 1776 (Meusnier 1785), see [10]. Note that while a sphere is a "minimal surface" in the sense that

Received: May 10, 2007

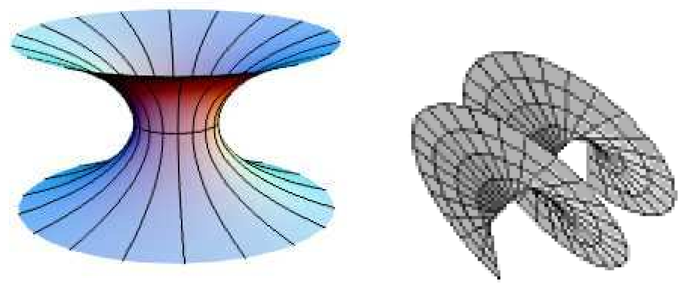

Figure 1: The catenoid and the helicoid

it minimizes the surface area-to-volume ratio, it does not qualify as a minimal surface in the sense used by mathematicians. Catenoid is a minimal surface made by rotating a catenary once around the axis, see [3]. Helicoid is a surface swept out by a line rotating with uniform velocity around an axis perpendicular to the line and simultaneously moving along the axis with uniform velocity, see [11]. The (circular) helicoid is the minimal surface having a (circular) helix as its boundary. It is the only ruled minimal surface other than the plane (Catalan 1842, do Carmo 1986). For many years, the helicoid remained the only known example of a complete embedded minimal surface of finite topology with infinite curvature. However, in 1992 a second example, known as Hoffman's minimal surface and consisting of a helicoid with a hole, was discovered (Sci. News 1992). The helicoid is the only non-rotary surface which can glide along itself (Steinhaus 1999, p. 231).

Recent work in minimal surfaces has identified new completely embedded minimal surfaces, that is minimal surfaces which do not intersect. In particular Costa's minimal surface (Figure 2) was first described mathematically in 1982 by Celso Costa and later visualized by Jim Hoffman. This was the first such surface to be discovered in over a hundred years. Jim Hoffman, David Hoffman and William Meeks III, then extended the definition to produce a family of surfaces with different rotational symmetries [10].

Besides the surfaces of finite type there are many periodic minimal surfaces. In Figure 3 we can see a part of the triply periodic Schwarz P-surface. It devides space into two congruent regions!

There are also singly and doubly periodic surfaces. Below are two such surfaces found by Scherk. They look quite different but are locally isomet-

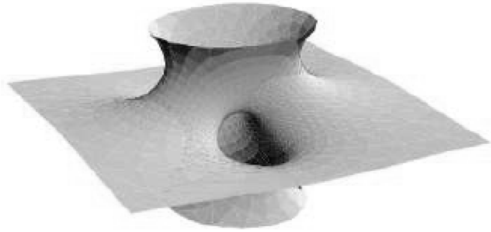

Figure 2: The Costa minimal surface

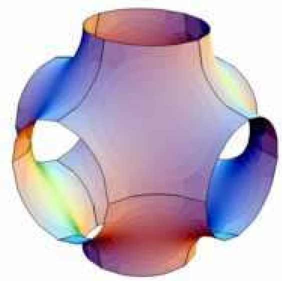

Figure 3: The Schwarz P-surface

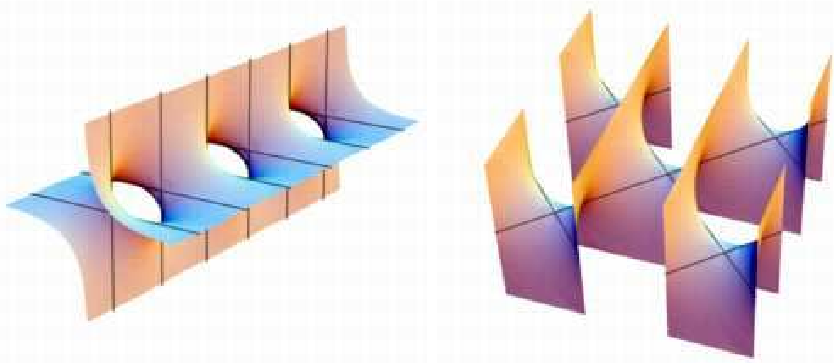

Figure 4: Scherk's minimal surfaces

ric (Figure 4). Surfaces are called isometric when there exists a continuous deformation relating them. The helicoid and the catenoid are isometric [9].

Finally, other examples of minimal surfaces include Bour's surface, Catalan's minimal surface, Henneberg's surface, Richmond's surface, and the Enneper's surface. Minimal surfaces have become an area of intense mathematical and scientific study over the past 15 years, specifically in the areas of molecular engineering and materials science, due to their anticipated nanotechnology applications.

The definition of minimal surfaces can be extended to cover constant mean curvature surfaces.

2. Basic Elements of the Theory of Surfaces

Let c be a curve in the three dimensional space $\Re^3$. The tangents of c form a developable surface, see [4], [5], [6]. The developable surfaces are ruled surfaces such that all points on a ruling have the same tangent plane. The developable surfaces can be unrolled on to a plane without distortion, see [4]. Hence they are surfaces which can be obtained by bending plane regions. Because of this property developable surfaces are often used in industry. The purpose of this paper is to determine the curves such that the tangents to each of these curves form a developable and minimal surface (see [1], [2]) and to obtain some solutions to the problem.

Recall that any c be a surface in $\Re^3$ which is referred to the orthogonal coordinate system Oxyz [7]. The surface S can be represented with respect to Oxyz as follows:

$$\overrightarrow{r} = \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}.$$

Using the fundamental magnitude of the first order [4] the unit vector field $\overrightarrow{\ell_o}$ to the surface S has the form

$$\overrightarrow{\ell_o} = \frac{\overrightarrow{\ell}}{\left|\overrightarrow{\ell}\right|} = \frac{\frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v}}{\left|\frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v}\right|} = \frac{1}{\sqrt{EG-F^2}} \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

The fundamental magnitudes of the second order of the surface S [4] are given by

$$L = \frac{d^2\overrightarrow{r}}{du^2} \cdot \overrightarrow{\ell_o} = \frac{1}{\sqrt{EG-F^2}} \begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix},$$

$$M = \frac{d^2\overrightarrow{r}}{dudv} \cdot \overrightarrow{\ell_o} = \frac{1}{\sqrt{EG-F^2}} \begin{vmatrix} \frac{\partial^2 x}{\partial u\partial v} & \frac{\partial^2 y}{\partial u\partial v} & \frac{\partial^2 z}{\partial u\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix},$$

$$N = \frac{d^2\overrightarrow{r}}{dv^2} \cdot \overrightarrow{\ell_o} = \frac{1}{\sqrt{EG-F^2}} \begin{vmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

The surface S is developable (see [4], [5]), which means is isometric onto a plane, iff the Gaussian curvature is zero, that means

$$LN - M^2 = 0\,. \tag{1}$$

The surface is minimal, that means it has the minimal area between the others, which pass through a curve, iff the mean curvature is zero [7], that is

$$EN - 2FM + LG = 0\,. \tag{2}$$

3. Determination of Developable and Minimal Surfaces

Let c a given curve in $\Re^3$ defined by the vector exactions.

$$\overrightarrow{r} = \overrightarrow{r}\,(s) = x\,(s)\,\overrightarrow{i} + y\,(s)\,\overrightarrow{j} + z\,(s)\,\overrightarrow{k}\,, \tag{3}$$

where s is the arch of the curve

The tangent bundle $T\,(c)$ of this curve forms a surface, which is a tangent developable surface, see [5], [6].

The vector equation of the developable surface $T\,(c)$ has the form

$$\overrightarrow{R} = \overrightarrow{R}\,(s,\lambda) = \overrightarrow{r}\,(s) + \lambda\,\dot{\overrightarrow{r}}\,(s) = \Big(x\,(s) + \lambda\dot{x}\,(s)\Big)\,\overrightarrow{i}$$
$$+ \Big(y\,(s) + \lambda\dot{y}\,(s)\Big)\,\overrightarrow{j} + \Big(z\,(s) + \lambda\dot{z}\,(s)\Big)\,\overrightarrow{k}\,. \tag{4}$$

The fundamental magnitudes of the first order of $T\,(c)$ are (see [6])

$$E = \left(\frac{\partial\,\overrightarrow{R}}{\partial s}\right)^2 = \Big(\dot{x} + \lambda\ddot{x}\Big)^2 + \Big(\dot{y} + \lambda\ddot{y}\Big)^2 + \Big(\dot{z} + \lambda\ddot{z}\Big)^2\,, \tag{5}$$

$$F = \frac{\partial\,\overrightarrow{R}}{\partial s}\cdot\frac{\partial\,\overrightarrow{R}}{\partial \lambda} = \dot{x}\,\Big(\dot{x} + \lambda\ddot{x}\Big) + \dot{y}\,\Big(\dot{y} + \lambda\ddot{y}\Big) + \dot{z}\,\Big(\dot{z} + \lambda\ddot{z}\Big)\,, \tag{6}$$

$$G = \left(\frac{\partial\,\overrightarrow{R}}{\partial \lambda}\right)^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2\,. \tag{7}$$

Choosing in relation (7) the functions $x\,(s)\,,\ y\,(s)$ and $z\,(s)$ to satisfy the condition:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 1\,, \tag{8}$$

we get $F = 1$ because $\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{1}{2}\frac{d}{ds}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}\frac{d(1)}{ds} = 0$ and the fundamental magnitudes of the first order of $T(c)$ become:

$$E = 1 + \lambda^2\left(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2\right), \; F = 1, \; G = 1\,. \tag{9}$$

Therefore the expression $\sqrt{EG - F^2}$ becomes:

$$\sqrt{EG - F^2} = \lambda\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}\,.$$

Furthermore, the fundamental magnitudes of second order of the surface $T(c)$ have the form

$$L = \frac{1}{\lambda\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}\begin{vmatrix} \ddot{x} + \lambda\dddot{x} & \ddot{y} + \lambda\dddot{y} & \ddot{z} + \lambda\dddot{z} \\ \dot{x} + \lambda\ddot{x} & \dot{y} + \lambda\ddot{y} & \dot{z} + \lambda\ddot{z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}, \tag{10}$$

$$M = \frac{1}{\lambda\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}\begin{vmatrix} \ddot{x} & \ddot{y} & \ddot{z} \\ \dot{x} + \lambda\ddot{x} & \dot{y} + \lambda\ddot{y} & \dot{z} + \lambda\ddot{z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}, \tag{11}$$

$$N = \frac{1}{\lambda\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}\begin{vmatrix} 0 & 0 & 0 \\ \dot{x} + \lambda\ddot{x} & \dot{y} + \lambda\ddot{y} & \dot{z} + \lambda\ddot{z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = 0\,. \tag{12}$$

From the relations (10), (11) and (12) taking under consideration the properties of determinants imply

$$L = \frac{1}{\lambda\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}\begin{vmatrix} \dddot{x} & \dddot{y} & \dddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}, \quad M = 0 \text{ and } N = 0\,. \tag{13}$$

The above relation implies $LN - M^2 = 0$ that means the surface $T(c)$ is developable.

In order this surface to be also minimal, relation (2) must also valid. Substituting fundamental magnitudes of the first order (8) and second order (13) in relation (2) we obtain:

$$L = 0 \text{ or } \begin{vmatrix} \dddot{x} & \dddot{y} & \dddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = 0\,. \tag{14}$$

From the above, we have

Proposition 1. *Let* $c:\overrightarrow{r}(s)=x(s)\overrightarrow{i}+y(s)\overrightarrow{j}+z(s)\overrightarrow{k}$ *be a curve in* $\Re^3$. *The surface T(c) of the tangent fibre bundle of c, is minimal, iff the functions* $x(s)$, $y(s)$ *and* $z(s)$, *satisfy the system of differential equations (8) and the second part of (14).*

We try to find some solutions of the system (8) and (14).

The equation (14) can be written

$$\begin{vmatrix} \dddot{x}+a\ddot{x}+b\dot{x} & \dddot{y}+a\ddot{x}+b\dot{y} & \dddot{z}+a\ddot{z}+b\dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}=0\,, \tag{15}$$

where $a,\ b\ \in\Re$

If we put

$$\dddot{x}+a\ddot{x}+b\dot{x}=0,\ \ \dddot{y}+a\ddot{x}+b\dot{y}=0,\ \ \dddot{z}+a\ddot{z}+b\dot{z}=0\,, \tag{16}$$

then (14) is satisfied.

The three differential equations (16) have the same form, which is

$$\dddot{\phi}+a\ddot{\phi}+b\dot{\phi}=0\,, \tag{17}$$

whose characteristic equation is

$$r^3+ar^2+br=0\,, \tag{18}$$

whose roots

$$r_1=0,\ \ r_2=\frac{-a+\sqrt{a^2-4b}}{2},\ \ r_3=\frac{-a-\sqrt{a^2-4b}}{2}\,. \tag{19}$$

The general solution of (17) has the form

$$\phi=c_1+c_2e^{r_1s}+c_3e^{r_2s}\,, \tag{20}$$

where $c_1,\ c_2,\ c_3$ are arbitrary constants. For $x,\ y,\ z$ we give $c_1,\ c_2,\ c_3$ different values.

Therefore the parametric equations of the curve has the form

$$\begin{aligned} &x=\alpha_1+\alpha_2e^{r_1s}+\alpha_3e^{r_2s},\ \ y=\beta_1+\beta_2e^{r_1s}+\beta_3e^{r_2s},\\ &z=\gamma_1+\gamma_2e^{r_1s}+\gamma_3e^{r_2s}\,, \end{aligned} \tag{21}$$

with the condition (8).

The developable surface $T(c)$ of the curve c defined by relations (21) is minimal [7].

From the condition (8) we have

$$\dot{x} = \sqrt{1 - \dot{y}^2 - \dot{z}^2} \tag{22}$$

which imply

$$\ddot{x} = \frac{-\left(\dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}\right)}{\sqrt{1 - \dot{x}^2 - \dot{y}^2}}\,, \tag{23}$$

$$\dddot{x} = \frac{-\left(\dot{y}\,\dddot{y} + \dot{z}\,\dddot{z} + \ddot{y}^2 + \ddot{z}^2\right)\left(1 - \dot{y}^2 - \dot{z}^2\right) - \left(\dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}\right)^2}{\sqrt{\left(1 - \dot{x}^2 - \dot{y}^2\right)^3}}\,. \tag{24}$$

Therefore the relation (14) by means of the (22), (23) and (24) takes the form

$$\left[\left(\dot{y}\,\dddot{y} + \dot{z}\,\dddot{z} + \ddot{y}^2 + \ddot{z}^2\right)\left(1 - \dot{y}^2 - \dot{z}^2\right) + \left(\dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}\right)^2\right]\left(\ddot{z}\,\dot{y} - \ddot{y}\,\dot{z}\right) +$$

$$\left(\dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}\right)\left(1 - \dot{y}^2 - \dot{z}^2\right)\left(\dddot{z}\,\dot{y} - \dot{z}\,\dddot{y}\right) + \left(1 - \dot{y}^2 - \dot{z}^2\right)^2\left(\dddot{y}\,\ddot{z} - \ddot{y}\,\dddot{z}\right) = 0. \tag{25}$$

From the above we have the theorem

Theorem 2. *To every pair of functions $y = y(s)$ and $z = z(s)$ which satisfy the equation (25) corresponds a curve c whose parametric equation is $(x(s), y(s), z(s))$, where $x(s)$ is given by (22). The tangents to this curve form minimal surface which is also developable.*

4. Conclusion

To study the surfaces S in $\Re^3$ which are developable and minimal, plays an important role in many applications in technology (see [5], [3]). We refer the car industry, aeronautical industry and shoes industry. It is known that in many branches of constructions we need pieces of surfaces which pass through a closed curve having minimal area and the same time these surfaces are coming from a plane form. From these requirements it is obvious that we need surfaces which are minimal and the same time developable, which are isometric onto a plane. Therefore we have to determine such surfaces. The previous theory solves this problem.

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