LINEAR SPACES OF MATRICES, SYMMETRIC MATRICES OR HERMITIAN MATRICES WITH A FIXED RANK OVER A FINITE FIELD

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Abstract: Here we raise several questions concerning linear spaces of matrices with fixed rank over $\mathbb{F}_q$.

AMS Subject Classification: 15A30, 12E20, 14N05
Key Words: linear spaces of matrices, rank, Hermitian matrices over a finite field

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Here we raise the following questions.

**Question 1.** Fix integers $m \geq n \geq k > 0$, $e \geq 1$, and a prime power $q$. What is the maximal integer $a(q, m, n, k)$ (resp. $b(q, m, n, k)$) such that there is an $a(q, m, n, k)$-dimensional (resp. $d(q, m, n, k)$-dimensional) $\mathbb{F}_q$-vector space of $n \times m$ matrices over $\mathbb{F}_q$ such that each non-zero element of it has rank $k$ (resp. $\leq k$)? What is the maximal integer $a(q, m, n, k, e)$ (resp. $b(q, m, n, k)$) such that there is an $a(q, m, n, k, e)$-dimensional (resp. $b(q, m, n, k, e)$-dimensional) $\mathbb{F}_{q^e}$-vector space of $n \times m$ matrices with a basis defined over $\mathbb{F}_q$ such that each non-zero element of it has rank $k$ (resp. $\leq k$)? Say something about the linear subspaces with maximal dimension.

**Question 2.** Fix integers $n \geq k > 0$, $e \geq 1$, and a prime power $q$. What is the maximal integer $c(q, n, k)$ (resp. $d(q, n, k)$) such that there is a $c(q, n, k)$-dimensional (resp. $d(c, q, n, k)$-dimensional) $\mathbb{F}_q$-vector space of symmetric $n \times n$
matrices over \( \mathbb{F}_q \) such that each non-zero element of it has rank \( k \) (resp. \( \leq k \))? What is the maximal integer \( e(q, n, k) \) (resp. \( f(q, n, k) \)) such that there is an \( e(q, m, m, k, e) \)-dimensional \( \mathbb{F}_{q^e} \)-vector space of \( n \times m \) matrices with a basis defined over \( \mathbb{F}_q \) such that each non-zero element of it has rank \( k \) (resp. \( \leq k \))?

Say something about the linear subspaces with maximal dimension.

**Question 3.** Fix integers \( n \geq k > 0 \) and a prime power \( q \). What is the maximal integer \( c(q, n, k, e) \) (resp. \( d(q, m, n, k) \)) such that there is an \( a(q, m, m, k, e) \)-dimensional \( \mathbb{F}_{q^e} \)-vector space of \( n \times m \) matrices with a basis defined over \( \mathbb{F}_q \) such that each non-zero element of it has rank \( k \) (resp. \( \leq k \))?

**Question 4.** Fix integers \( m \geq n \geq k > 0 \) and a prime power \( q \) such that \( 0 < t < a(q, m, n, k) \). Show the existence of a \( t \)-dimensional \( \mathbb{F}_{q^t} \)-vector space of \( n \times m \) matrices over \( \mathbb{F}_q \) such that each non-zero element of it has rank \( k \), but that it is not contained in a bigger linear space with the same property. Hopefully, prove the same guess for the other integers introduced in Questions 1, 2 and 3.

For background on Hermitian matrices over a finite field, see [6], Chapter 23; however, here we use the notation \( \mathbb{F}_q \) (resp. \( \mathbb{F}_{q^2} \)) instead of \( \mathbb{F}_{q^m} \) resp. \( \mathbb{F}_q \). For similar problems for the real, complex and quaternionic division rings, see [1], [2], [3], [4], [5] and [7].

**Question 5.** Fix as the base field \( F \) either the real or the complex or the quaternionic division ring. Take the set-up of Questions 1 or 2 for \( e = 1 \). Is there a maximal dimension linear subspace defined over the rational field?

**Remark 1.** Fix as the base field \( F \) either the real or the complex or the quaternionic division ring. Take the case \( m = n \) and \( k = n - 1 \). The existence parts of the proofs in [1], [2] and [3] use an induction on \( n \) in which all the 3 division rings play simultaneously. In this way we get at least an explicit upper bound for the degree of the extension \( L \) of \( \mathbb{Q} \) on which a maximal dimension linear subspace may be defined over \( L \).

**Remark 2.** Take the case \( k = n \). Let \( A, A_1, \ldots, A_t \) be \( n \times m \) matrices over \( \mathbb{F}_q \). \( \text{rank}(A) < n \) if and only if \( m - n + 1 \) \( n \times n \) submatrices of \( A \) have zero-determinant. Hence the set \( B(A_1, \ldots, A_t) = \{ (x_1, \ldots, x_t) \in \mathbb{F}_{q^t}^{\oplus t} : \text{rank}(x_1 A_1 + \cdots + x_t A_t) < n \} \) is defined by \( m - n + 1 \) homogeneous degree \( n \) equations with coefficients in \( \mathbb{F}_q \). By Chevalley-Warning Theorem for systems of homogeneous equations we get \( a(q, m, n, n) \leq n(m - n + 1) \). The same theorem gives an upper bound for the integer \( a(q, m, n, k) \) for any \( k < n \). However, these upper bounds do not use the structure of the determinantal equations and hence they probably are very bad.
Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References


