

A REPRESENTATION OF REAL AND COMPLEX
NUMBERS IN QUANTUM THEORY

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Abstract: A quantum theoretic representation of real and complex numbers is described as equivalence classes of Cauchy sequences of quantum states of finite strings of qubits. There are 4 qubit types, each with associated single qubit annihilation creation operators that give the state and location of each qubit type on a 2 dimensional integer lattice. The string states, defined as finite products of creation operators acting on the vacuum state, correspond to complex rational numbers with real and imaginary components. These states span a Fock space \mathcal{F} . Arithmetic relations and operations are defined for the string states. Cauchy sequences of these states are defined, and the arithmetic relations and operations lifted to apply to these sequences. Based on these, equivalence classes of these sequences are seen to have the requisite properties of real and complex numbers. The representations have some interesting aspects. Quantum equivalence classes are larger than their corresponding classical classes, but no new classes are created. There exist Cauchy sequences where each state in the sequence is an entangled superposition of the real and imaginary components, yet the sequence is a real number. Except for superposition state coefficients, the construction is done with no reference to the real and complex number base of \mathcal{F} .

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1. Introduction

Real and complex numbers are very important to physics in several different ways. They form the basis of all physical theories in that the theories are mathematical structures based on the real and complex numbers. Real numbers are also used to represent the space time manifold as R^4 . All theoretical predictions to be tested by experiment are, or can be cast, as real number solutions to equations.

On the other hand outputs of experiments are rational numbers. This is based on the observation that they are or can be represented as states of finite strings of kits or qukits in some base $k \geq 2$. Also all computers, both classical and quantum, work with states of finite strings of kits or qukits. Usually they are base 2 rational numbers as states of finite bit or qubit strings.

Comparison between experiment outputs and computer outputs as a comparison between theory and experiment depends on the fact that rational numbers are dense in the set of real numbers. Also those rational numbers expressible as states of finite base k kit or qukit strings, are dense in the set of all rational numbers. Because of this computer outputs, as states of finite kit or qukit strings, represent, to arbitrary accuracy, theoretical predictions. Also they can be directly compared to experimental predictions.

However, as noted, physical theories are based on real and complex numbers and not on rational numbers. The completeness properties of real and complex numbers play an essential role in theoretical predictions. This follows from the observation that all theoretical predictions are theorems, i.e. theoretical statements in the theory language that are provable in the physical theory. The proofs of these statements, which are based on the mathematical properties of the theory as an axiomatizable mathematical theory, depend essentially on the properties of the real and complex number base of the theory.

These considerations show the basic importance of the different types of numbers to physics and mathematics. Yet they leave open the deeper question of the relationship between the foundations of mathematics and physics and why mathematics is relevant to physics. This question, which was first described by [30] and commented on by others [12], [9], is especially acute if one accepts the Platonic view of mathematical existence. In this view, which seems to be accepted at least implicitly by many, mathematical objects have an ideal abstract existence. This seems completely unrelated to the physical existence of objects that both exist in and determine the properties of space time.

There are several different approaches to understanding this relationship [25], [26], [21], [29]. The approach underlying this paper is to work towards

a coherent theory of physics and mathematics together [1, 2]. Such a theory, by treating both physics and mathematics together in one theory, may help to understand how mathematics and physics are related. It may also help to answer some of the basic outstanding questions in physics.

In this paper a step in this direction is taken by describing quantum representations of real and complex numbers. The use of quantum rather than classical representations is done because this brings both the treatment of physical systems and numbers into the same general theory. Quantum theory is the basic theory underlying the description of physical systems. Using the same basic theory to describe both physical systems and numbers as mathematical systems should help in bringing together descriptions of physical and mathematical systems. An alternative approach to a coherent theory, that questions the relevance of real and complex numbers in physics [10], [14], will not be treated here.

The other main point is that all physical representations of numbers are as states of finite strings of physical systems. This will be taken over here in that the only systems available for representations of numbers are states of finite strings of kits or qukits. Since quantum theory is taken to be the basic underlying theory for both physical and mathematical systems, quantum representations of real and complex numbers will be based on states of finite strings of qukits.

The importance of qukits lies in the fact that they are the basic units of quantum information just as kits are the basic units of classical information. The importance is based on the observation that qukits, as units with k orthogonal states for any $k > 2$, can be used for either quantum representations of numbers, or for representations of quantum mechanical systems in physics.

States of finite strings of qukits are quite useful to represent the natural numbers N , the integers, I , and the rational numbers Ra . However they do not represent either real or complex numbers¹. Thus some way to connect these representations of rational numbers to real and complex numbers must be found.

The method used here follows the one in some mathematical analysis textbooks [6] that describe real numbers as equivalence classes of Cauchy sequences of rational numbers. The other equivalent description as equivalence classes of Dedekind cuts is not used here. Also complex numbers are usually described as

¹They can be represented by states of infinite strings of qukits, but these are not describable as states in a separable Hilbert space. Even a field theoretic description seems problematic even though systems with an infinite number of degrees of freedom are described in quantum field theory.

ordered pairs (x, y) (or (x, iy)) of real numbers. However here, the procedure used in computers that work with complex rational numbers as ordered pairs of rational numbers, (ra_1, ira_2) will be followed.

The goal of this paper is to give quantum theory representations of real and complex numbers as equivalence classes of Cauchy sequences of quantum representations of real and complex rational numbers as states of finite strings of qukits. To avoid inessential complications, the description will be limited $k = 2$ or states of strings of qubits.

The use of quantum theory to study representations of numbers and other mathematical systems is not new [16], [7], [28], [24], [8], [27], [11], [20], [15]. Of particular note is work on quantum set theory represented as an orthomodular lattice valued set theory [24], [8], [27], [11], [20], [15]. In this work natural numbers, integers, and rational numbers have representations that are either similar to the usual ones in mathematical analysis [24], [8], [27] or are based on a categorical approach [20], [15]. However the work in these references differs from the approach taken here in that, here, states of finite qubit strings are used as the rational number base of real and complex numbers.

Steps taken in this paper in this include, descriptions of rational numbers as states of finite strings of qubits by use of annihilation creation operator strings acting on the qubit vacuum state, and the description of the basic arithmetic relations and operations on these rational string states. This is done in the next section.

Cauchy sequences of these states are defined in Section 3. The definitions are based on the description of the arithmetic relations and operations. Some examples are given including some that have no classical counterpart. The definition of Cauchy convergence is also extended to sequences of states that are linear superpositions of rational string states.

Section 4 describes the basic properties and operations on Cauchy sequences. This includes lifting of the basic arithmetic relations and operations to apply to the sequences. Section 5 describes the real and complex number representations as equivalence classes of the Cauchy sequences. The proof that they, or equivalently, representatives of each class satisfy the real number axioms is outlined. It is seen that the quantum equivalence classes are larger than the classical ones but that no new classes are created.

It is useful to note terminology used in the paper. Following standard usage real, imaginary, and complex real numbers will be referred to as real, imaginary, and complex numbers. Real, imaginary and complex rational numbers will be referred to as noted. Rational string states are states of finite qukit strings that represent rational numbers. Complex rational string states are states of pairs

of finite qubit strings that represent complex rational numbers.

The last section contains a discussion of these results. It is emphasized again that, except for the description of linear superpositions of string states and Cauchy sequences of these states, the quantum theoretic description of real and complex numbers is independent of R and C . It is also noted that, since the equivalence classes of Cauchy sequences of quantum states are complex numbers, they can be used as the complex number field for any physical theory that is a mathematical structure based on the complex number field. This encompasses most of physics, since theories such as quantum mechanics, QED, QCD, string theory, and general relativity are structures of this type. The section also has a brief discussion of the relation between this complex number field and C .

2. Complex Rational String States

It is useful to define states of pairs of qubit strings on a two dimensional integer lattice $I \times I$. One dimension indicates the string direction and the other allows for an arbitrary number of strings as these are needed for the discussion of n -ary operations on the strings.

For this paper it should be noted that it is not necessary to consider $I \times I$ as a lattice of points in 2 or more dimensional physical space. This would be suitable for physical representations of the mathematical model being considered here. Here the lattice $I \times I$ is a convenient method to represent the minimal conditions needed here. These are:

— I is an infinite set of points with the order type of the integers:

1. No largest or smallest point,
2. Ordering is discrete,
3. Each point has just one nearest neighbor above and below.

— A denumerable set of pairs of discrete points whose purpose is to distinguish the qubits in different strings. This is especially important for fermionic qubits.

— No metric distance between pairs of lattice points is assumed or needed here.

Note that ordering of the labels of the different strings is convenient, and is represented by the second I component of the lattice. But it is not necessary.

For the purposes of this paper it is immaterial whether the string pairs needed to represent a complex rational number consist of one string of two different types of qubits or two adjacent strings of the same type of qubit that

are distinguished by their different positions on the lattice. Here the model consisting of one string of two different qubit types will be used.

A compact representation of numbers is used here that combines the location of the sign and the “binal” point. The representation is suitable for real and complex natural numbers and integers as well as rational numbers. As examples the usual decimal numbers, -63.71 , 459 , -0.0753 would be expressed here as $63 - 71$, $459+$, $0 - 0753$.

Since strings with different numbers of qubits are being considered, it is useful to describe them using annihilation creation (AC) operators. Here four types are needed: $a_{\alpha,j,h}^\dagger, a_{\alpha,j,h}; b_{\beta,j,h}^\dagger, b_{\beta,j,h}; c_{\gamma,j,h}^\dagger, c_{\gamma,j,h}; d_{\gamma,j,h}^\dagger, d_{\gamma,j,h}$. The c and d qubits denote the signs of the real and imaginary strings, where $\gamma = +, -$, and the a and b qubits are the 0, 1 components of the real and imaginary qubit strings. Here α and β take values in $\{0, 1\}$ and (j, h) is a point in the lattice.

Complex rational string states are represented by

$$|\gamma, (a^\dagger)^s; \gamma', (b^\dagger)^t, (m, h)\rangle = c_{\gamma,m,h}^\dagger (a^\dagger)_{[l,u]}^s; d_{\gamma',m,h}^\dagger (b^\dagger)_{[l',u']}^t |0\rangle, \quad (1)$$

where the state $|0\rangle$ denotes the qubit vacuum. Here m, h denotes the location of the sign qubits, $[l, u]$ and $[l', u']$ denote lattice intervals $(l, h), (l+1, h), \dots, (u, h)$ and $(l', h), (l'+1, h), \dots, (u', h)$, where $l \leq m \leq u$ and $l' \leq m \leq u'$. Finally

$$\begin{aligned} (a^\dagger)_{[l,u]}^s &= a_{s(u),u,h}^\dagger a_{s(u-1),u-1,h}^\dagger \cdots a_{s(l),l,h}^\dagger, \\ (b^\dagger)_{[l',u']}^t &= b_{t(u'),u',h}^\dagger b_{t(u'-1),u'-1,h}^\dagger \cdots b_{t(l'),(l',h)}^\dagger, \end{aligned} \quad (2)$$

where s and t are 0, 1 valued functions with integer interval domains $[l, u]$ and $[l', u']$ respectively. It is easy to generalize to let s and t be functions that depend also on h , but this will not be done here.

Note that the states of qubit strings described here represent colocated strings of two types of qubits with sign qubits at the location (m, h) . Each string location (j, h) , other than (m, h) , contains up to two qubits, none, or one a and/or one b type. The (m, h) site contains the same a and b type qubits and two sign c and d type qubits.

For the purposes of this work it is immaterial whether the qubits are bosons or fermions as the representation is sufficiently inclusive to incorporate both. Fermion AC operators satisfy the anticommutation relations,

$$\begin{aligned} \{a_{\alpha,j,h}, a_{\alpha',k,h'}\} &= \{b_{\beta,j,h}, b_{\beta',k,h'}\} \\ &= \{a_{\alpha,j,h}^\dagger, a_{\alpha',k,h'}^\dagger\} = \{b_{\beta,j,h}^\dagger, b_{\beta',k,h'}^\dagger\} = 0; \\ \{a_{\alpha,j,h}, a_{\alpha',k,h'}^\dagger\} &= \delta_{\alpha,\alpha'} \delta_{j,k} \delta_{h,h'}; \\ \{b_{\beta,j,h}, b_{\beta',k,h'}^\dagger\} &= \delta_{\beta,\beta'} \delta_{j,k} \delta_{h,h'}, \end{aligned} \quad (3)$$

where $\{x, y\} = xy + yx$. As a consequence a specific ordering of the AC operators of each type must be used. This will be implicitly assumed here to be that shown in equations (1) and (2). For bosons the same relations hold if $\{, \}$ is replaced by commutation relations $[,]$, where $[x, y] = xy - yx$. Since the a, b, d, c systems are all different all six pairs of these AC operators commute. For these systems the ordering of the AC operators does not matter.

The Fock space spanned by states of the form $|\gamma, (a^\dagger)^s; \gamma', (b^\dagger)^t, (m, h)\rangle$ is denoted by $\mathcal{F}_{m,h}$. This is the space of all complex rational string states with the "binal" point at (m, h) . The space

$$\mathcal{F} = \bigoplus_{(m,h) \in I \times I} \mathcal{F}_{m,h} \tag{4}$$

is spanned by all complex² rational string states located anywhere on $I \times I$.

The arithmetic and numerical properties of the states $|\gamma, (m, h), (a^\dagger)^s; \gamma', (b^\dagger)^t, \rangle$ will be described here independent of the corresponding numerical value in the complex number base C of the space \mathcal{F} . Nevertheless it is useful to define an operator \tilde{N} whose eigenvalues correspond to the complex numbers associated with the states $|\gamma, (m, h), (a^\dagger)^s; \gamma', (b^\dagger)^t, \rangle$. In particular \tilde{N} serves as a check on the consistency of the definitions of the basic arithmetic relations and operations.

\tilde{N} is the sum of two operators \tilde{N}^R, \tilde{N}^I for the real and imaginary component states. Each of the two operators is in turn a product of two commuting operators, a sign scale operator \tilde{N}_{ss}^X , and a value operator \tilde{N}_v^X for $X = R, I$. One has

$$\tilde{N} = \tilde{N}^R + \tilde{N}^I, \tag{5}$$

where

$$\tilde{N}^X = \tilde{N}_{ss}^X \tilde{N}_v^X \tag{6}$$

and

$$\tilde{N}_{ss}^X = \begin{cases} \sum_{\gamma,m} \gamma 2^{-m} c_{\gamma,m}^\dagger c_{\gamma,m} & \text{if } X = R, \\ \sum_{\gamma',m} \gamma' 2^{-m} d_{\gamma',m}^\dagger d_{\gamma',m} & \text{if } X = I, \end{cases} \tag{7}$$

and

$$\tilde{N}_v^X = \begin{cases} \sum_{\alpha,j,h} \alpha 2^j a_{\alpha,(j,h)}^\dagger a_{\alpha,(j,h)} & \text{if } X = R, \\ \sum_{\beta,j,h} i\beta 2^j b_{\beta,(j,h)}^\dagger b_{\beta,(j,h)} & \text{if } X = I. \end{cases} \tag{8}$$

Note that, because of the presence of strings of leading or trailing 0s, the eigenspaces of \tilde{N} are infinite dimensional. Also \tilde{N} is unbounded and has complex eigenvalues. The eigenspace for the number 0 includes all states of the form

²The term "complex" includes both real and imaginary components.

$|\gamma, (a^\dagger)^s; \gamma', (b^\dagger)^t, (m, h)\rangle$ for all (m, h) where s and t are constant 0 functions on integer intervals that include m . The signs can be either $+$ or $-$. It is useful to designate these states by the simple form $|+, 0\rangle$.

2.1. Basic Arithmetic Relations

There are two basic arithmetic relations, equality $=_A$ and ordering \leq_A . Two states $|\gamma, (m, h), (a^\dagger)^s; \gamma', (b^\dagger)^t\rangle$ and $|\gamma_1, (m, h), (a^\dagger)^{s'}; \gamma'_1, (b^\dagger)^{t'}\rangle$ are arithmetically equal if the real and imaginary parts are the same up to leading and trailing 0's. That is

$$|\gamma, (m, h), (a^\dagger)^s; \gamma', (b^\dagger)^t\rangle =_A |\gamma_1, (m, h), (a^\dagger)^{s'}; \gamma'_1, (b^\dagger)^{t'}\rangle \quad (9)$$

if for all j in $D(s) \cap D(s')$ $s(j) = s'(j)$ and for all j in $D(s) - D(s')$, $s(j) = 0$ and for all j in $D(s') - D(s)$, $s'(j) = 0$ with a similar condition holding for t and t' . Here $D(s)$ is the integer domain of s . Also $\gamma = \gamma_1$ and $\gamma' = \gamma'_1$.

Because one is working with quantum states with boson or fermion properties, the definition of such an obvious relation as arithmetic equality has some unexpected features. These are related to the fact that A equality depends only on the properties of s, s', t, t' and not on the other variables such as the positions of the qubit sequences in the lattice.

To see these features, consider a restricted definition of A equality $=_{A, (m, h), (m', h')}$ that applies only to those sequence pairs with the sign qubits at (m, h) and (m', h') and is undefined elsewhere. One aspect is that the diagonal definition $=_{A, (m, h), (m, h)}$ is meaningless and is not defined anywhere. This is a result of the fact that for either boson or fermion qubit sequences which overlap at some lattice sites, there is no way to determine which of the two qubits in the lattice overlap sites belongs to which sequence. Also $=_{A, (m, h), (m', h')}$ and $=_{A, (m', h'), (m, h)}$ are identical as they have the same domains of definition. Note too that A equality of two states does not implies quantum theory equality. Two states can be quite different quantum theoretically yet be the same arithmetically.

These properties of A equality are mirrored in the properties of the associated quantum projection operators for these definitions. For the restricted $=_{A, (m, h), (m', h')}$ the associated projection operator is a product of projection operators for the real and imaginary components,

$$\tilde{P}_{=_{A, (m, h), (m', h')}} = \tilde{P}_{=_{A, (m, h), (m', h')}}^R \tilde{P}_{=_{A, (m, h), (m', h')}}^I \cdot \quad (10)$$

Here

$$\tilde{P}_{=_{A, (m, h), (m', h')}}^R = \sum_{\gamma, s \neq 0} \tilde{P}_{\gamma, (m, h), [s]} \tilde{P}_{\gamma, (m', h'), [s_{\Delta m}]} \quad (11)$$

and a similar expression for $\tilde{P}^I_{=_{A,(m,h),(m',h')}}.$ Here $s^{\neq 0}$ means that the sum is restricted to those s with no leading or trailing 0s and $[s]$ denotes the set of all s' that are equal to s up to leading or trailing 0s. $[s_{\Delta m}]$ is the set of all s' equal to $[s_{\Delta m}]$ up to leading or trailing 0s and $s_{\Delta m}(j') = s(j)$, where $j' - m' = j - m$ or $j' = j + \Delta m$.

$\tilde{P}_{\gamma,(m,h),[s]}$ is given by

$$\tilde{P}_{\gamma,(m,h),[s]} = \sum_{s' \sim_0 s} \tilde{P}_{|\gamma,(m,h),s',l,u}. \tag{12}$$

In terms of A-C operators $\tilde{P}_{|\gamma,(m,h),s',l,u}$ can be expressed as

$$\tilde{P}_{|\gamma,(m,h),s',l,u} = c_{\gamma,(m,h)}^\dagger (a^\dagger)_{[(l,h),(u,h)]}^{s'} c_{\gamma,(m,h)} a_{[(l,h),(u,h)]}^{s'}. \tag{13}$$

Here $(a^\dagger)_{[(l,h),(u,h)]}^{s'} = a_{s'(u,h),(u,h)}^\dagger \cdots a_{s'(l,h),(l,h)}^\dagger$ with a similar expression for $a_{[(l,h),(u,h)]}^{s'}$. In equation (12) the sum over $s' \sim_0 s$ is over all s' that differ only by leading or trailing 0's. The dependence of l and u on s' is implied.

A global definition of A equality, $=_A$, is defined by

$$=_A \leftrightarrow \exists(m, h), (m', h') [(m, h) \neq (m', h') \wedge =_{A,(m,h),(m',h')}]. \tag{14}$$

That is two states are A equal if and only if they are $=_{A,(m,h),(m',h')}$ for some pairs $(m, h), (m', h')$. The corresponding projection operator $\tilde{P}_{=,A}$ is defined by

$$\tilde{P}_{=,A} = \sum_{(m,h) \neq (m',h')} \tilde{P}_{=_{A,(m,h),(m',h')}}. \tag{15}$$

As seen in equation (10) $\tilde{P}_{=,A}$ can be expressed as a product of projection operators for the real and imaginary components.

To save on notation the pair (m, h) will often be deleted in the following. Thus $|\gamma, (a^\dagger)^s; \gamma', (b^\dagger)^t, (m, h)\rangle$ will be represented as $|\gamma, (a^\dagger)^s; \gamma', (b^\dagger)^t\rangle$ or even in the shorter form $|\gamma, s; \gamma', t\rangle$.

The ordering relation \leq_A is defined separately for the real and imaginary components of the rational state pairs. For positive real components one has

$$|+, (m, h), s, t\rangle \leq_{A,R} |+, (m', h'), s', t'\rangle \leftrightarrow s \sim_{0_{\Delta m}} s' \text{ or } s <_{\Delta m} s'. \tag{16}$$

The relations $\sim_{0_{\Delta m}}$ and $<_{\Delta m}$ are defined by

$$s \sim_{0_{\Delta m}} s' \text{ if } \begin{cases} \forall j \in [l, u] (s(j) = 1 \leftrightarrow s'(j + \Delta m) = 1, \\ \text{and } \forall j' \in [l', u'] (s'(j') = 1 \leftrightarrow s(j - \Delta m) = 1)), \end{cases} \tag{17}$$

$$s <_{\Delta m} s' \text{ if } \begin{cases} \exists j \in [l, u] \cap [l', u'] (s(j) = 0, s'(j + \Delta m) = 1, \\ \text{and } s_{[j+1, u]} \sim_{0_{\Delta m}} s'_{[j+1+\Delta m, u']}). \end{cases}$$

The definitions of $s \sim_{0_{\Delta m}} s'$ and $s <_{\Delta m} s'$ state that s is equal to or less than s' when differences in m, m' are taken into account. These locations matter because they are used to determine the powers of 2 associated with the values of s and s' .

The extension to zero and negative states is given by

$$\begin{aligned} &|+, \bar{0}, t\rangle \leq_{A,R} |+, s', t'\rangle \text{ for all } s', \\ &|+, s, t\rangle \leq_{A,R} |+, s', t'\rangle \rightarrow |-, s', t\rangle \leq_{A,R} |-, s, t'\rangle. \end{aligned} \quad (18)$$

Equation (18) holds for any pair $(m, h), (m', h')$ where $(m, h) \neq (m', h')$. As was the case for A equality, one can define a projection operator $\tilde{P}_{\leq_{A,R}}$. Here the definition is slightly more complex as the signs of the two components to be compared must be taken into account. Similar relations hold for the imaginary components of the rational string states.

The definitions of $=_A$ and \leq_A can also be applied to states ψ, ψ' that are linear superpositions of rational string states. The probability, $P_{\psi=_A \psi'}$, that $\psi =_A \psi'$ is given by

$$\begin{aligned} P_{\psi=_A \psi'} &= \sum_{\gamma, s; \gamma', t} \sum_{\gamma_1 s'; \gamma'_1, t'} \{ |\langle \gamma, s; \gamma', t | \psi \rangle|^2 |\langle \gamma_1 s'; \gamma'_1, t' | \psi' \rangle|^2 \\ &\quad \times \langle \gamma, s; \gamma', t | \tilde{P}_{=A} | \gamma_1 s'; \gamma'_1, t' \rangle \}, \end{aligned} \quad (19)$$

where

$$\tilde{P}_{=A} = \tilde{P}_{=A,R} \tilde{P}_{=A,I} \quad (20)$$

is a product of projection operators for the real and imaginary components. The validity of this expression also depends on the fact that $\tilde{P}_{=A,R}$ and $\tilde{P}_{=A,I}$ commute. Expressions for the probability that $\psi \leq_A \psi'$ will not be given here as they are similar.

2.2. Arithmetic Operations

The basic arithmetic operations to be described are addition, subtraction, multiplication, and division to arbitrary accuracy. As is well known states of finite qubit strings are not closed under division. However, one can implement division to any finite accuracy on these states. Let \tilde{O}_A be a unitary operator for describing these arithmetic operations. One has

$$\tilde{O}_A |\gamma, s; \gamma', t\rangle |\gamma_1 s'; \gamma'_1, t'\rangle = |\gamma, s; \gamma', t\rangle |\gamma_1 s'; \gamma'_1, t'\rangle |\gamma'_1, s''; \gamma''_1, t''\rangle, \quad (21)$$

where

$$|\gamma'_1, s''; \gamma''_1, t''\rangle =_A |(\gamma, s; \gamma', t) O_A (\gamma_1 s'; \gamma'_1, t')\rangle. \quad (22)$$

The expression $|(\gamma, s; \gamma', t)O_A(\gamma_1 s'; \gamma'_1, t')\rangle$ is a useful notation that uses O inside $|_, _ \rangle$ to represent the rational string state resulting from carrying out the operation O_A . To ensure unitarity for \tilde{O}_A the first two states are repeated with the resultant state created. Also the lattice location (m'', h'') of this state differs from those of the first two states.

Application of the arithmetic operations to states that are linear superpositions of the string states creates entangled states. One has

$$\tilde{O}_A \psi \psi' = \sum_{\alpha \gamma, s, \gamma', t} \sum_{\gamma_1, s', \gamma'_1, t'} \langle \gamma, s; \gamma', t | \psi \rangle \langle \gamma_1 s'; \gamma'_1, t' | \psi' \rangle |(\gamma, s; \gamma', t) | \gamma_1 s'; \gamma'_1, t')\rangle |(\gamma, s; \gamma', t) O_A(\gamma_1 s'; \gamma'_1, t')\rangle. \tag{23}$$

Taking the trace over the ψ and ψ' components gives a mixed state

$$\rho_{\psi O_A \psi'} = \sum_{\gamma, s, \gamma', t} \sum_{\gamma_1, s', \gamma'_1, t'} |\langle \gamma, s; \gamma', t | \psi \rangle|^2 \times |\langle \gamma_1 s'; \gamma'_1, t' | \psi' \rangle|^2 \rho_{(\gamma, s; \gamma', t) O_A(\gamma_1 s'; \gamma'_1, t')} \tag{24}$$

to represent the result of the operation.

The definitions of $\tilde{O}_A = \tilde{+}_A, \tilde{-}_A, \tilde{\times}_A, \tilde{\div}_{A,\ell}$ are all based on the use of successor operators, one for each j . To save on notation, $x_{i,j,h}^\dagger, x_{i,j,h}$ will denote $a_{i,j,h}^\dagger, a_{i,j,h}$ or $b_{i,j,h}^\dagger, b_{i,j,h}$ for the real $X = R$ or imaginary $X = I$ components of the state. The successor operator for each X, j, h is defined by an iterative expression

$$\begin{aligned} \tilde{N}_{X,j,h} &= x_{1,j,h}^\dagger x_{0,j,h} + x_{1,j+1,h}^\dagger x_{0,j,h}^\dagger x_{1,j,h} P_{X,unocc,j+1,h} \\ &+ P_{X,occ,j+2,h} P_{X,occ,j+1,j} \tilde{N}_{X,j+1,h} x_{0,j,h}^\dagger x_{1,j,h}. \end{aligned} \tag{25}$$

Here $P_{X,occ,j+1,h}, P_{X,unocc,j+1,h}$ are projection operators for finding site $j + 1, h$ occupied or unoccupied by a type x qubit. The adjoint $\tilde{N}_{X,j,h}^\dagger$ is defined by

$$\begin{aligned} \tilde{N}_{X,j,h}^\dagger &= x_{0,j,h}^\dagger x_{1,j,h} + P_{X,unocc,j+1,h} x_{1,j,h}^\dagger x_{0,j,h} x_{1,j+1,h} \\ &+ x_{1,j,h}^\dagger x_{0,j,h} \tilde{N}_{X,j+1,h}^\dagger P_{X,occ,j+1,h} P_{X,occ,j+2,j}. \end{aligned} \tag{26}$$

The action of $\tilde{N}_{X,j,h}$ on a state $|\gamma, (m, h), s; \gamma', t\rangle$ in which site j, h is occupied, creates a new state that corresponds to arithmetic addition of 2^{j-m} for $X = R$ and $i2^{j-m}$ for $X = I$. The action of the adjoint $\tilde{N}_{X,j,h}^\dagger$ on the same state corresponds to subtraction of 2^{j-m} for $X = R$ and $i2^{j-m}$ for $X = I$. It is useful to note, too, that

$$(\tilde{N}_{X,j,h})^2 |\gamma, s; \gamma', t\rangle = \tilde{N}_{X,j+1,h} |\gamma, s; \gamma', t\rangle \tag{27}$$

if sites j, h and $j + 1, h$ are occupied by x qubits. This corresponds to the observation that $2^{j-m} + 2^{j-m} = 2^{j+1-m}$.

One also has to extend the definition of $\tilde{N}_{X,j,h}$ to include cases where site j, h is unoccupied. Examples at either end of a string are $110.1 + 0.000001$ or $110.1 + 100000.0$ To accommodate this one defines an operator $Z_{X,m,j,h}$ by

$$Z_{X,m,j,h} = P_{X,occ,(j,h)} + \begin{cases} Z_{X,m,j-1,h} x_{0,j,h}^\dagger P_{X,unocc,(j,h)} & \text{if } j > m, \\ Z_{X,m,j+1,h} x_{0,j,h}^\dagger P_{X,unocc,(j,h)} & \text{if } j < m, \\ x_{0,m,h}^\dagger P_{X,unocc,(m,h)} & \text{if } j = m. \end{cases} \quad (28)$$

Note that, as defined, $Z_{X,m,j,h}$ is a many-one operator as it creates the same rational string states from states with different numbers of terminal 0s. To avoid this source of irreversibility one needs to first copy the state on which $Z_{X,m,j,h}$ will act. To this end a copying operator C is defined where

$$C|\gamma, (m, h), s; \gamma', t\rangle = |\gamma, (m, h), s; \gamma', t\rangle |\gamma, (m, h'), s; \gamma', t\rangle. \quad (29)$$

The state $|\gamma, (m, h'), s; \gamma', t\rangle$ is a copy of $|\gamma, (m, h), s; \gamma', t\rangle$ that is located along h' instead of h . One should note that, because of the No-Cloning Theorem, see [32], C does not copy a state that is a linear superposition of the basis states $|\gamma, s; \gamma', t\rangle$. Instead it creates an entangled superposition of pairs of basis states where each pair differs only in the value of the parameter h .

The action of $Z_{X,m,j,h'}$ on a pair of states $|\gamma, s; \gamma', t\rangle |\gamma_1 s'; \gamma'_1, t'\rangle$ does nothing if site j, h' is occupied by a type x qubit. otherwise it adds a string of x qubits along h' in state $|0\rangle$ that terminates at site j, h' . The addition is from u or u' to j if $j > m$, from l or l' to j if $j < m$, and just at m, h' if $j = m$.

The combination $\tilde{N}_{X,j,h} Z_{X,m,j,h}$ is quite useful. This follows from the observation that the state $c_{\gamma,(m,h)}^\dagger (a^\dagger)^{\bar{0}[(j,h),(m,h)]} a_{1,j+1,h}^\dagger |0\rangle$, corresponding to the number 2^{j-m} , can be expressed as $\tilde{N}_{R,j,h} Z_{R,m,j,h} c_{\gamma,(m,h)}^\dagger |0\rangle$. It can also be expressed as

$$\begin{aligned} c_{\gamma,(m,h)}^\dagger (a^\dagger)^{\bar{0}[(j,h),(m,h)]} a_{1,j-1,h}^\dagger |0\rangle &= c_{\gamma,(m,h)}^\dagger (\tilde{N}_{R,u,h} Z_{R,m,u,h})^{s(u)} \\ &\times (\tilde{N}_{R,u-1,h} Z_{R,m,u-1,h})^{s(u-1)} \dots (\tilde{N}_{R,l,h} Z_{R,m,l,h})^{s(l)} |0\rangle. \end{aligned} \quad (30)$$

Here $[(l, h), (u, h)]$ is the domain of s with $l = j + 1$, $u = m$, and $s(k, h) = 0$ for all $m \geq j \geq l + 1$ and $s(l, h) = 1$.

This can be extended to any state $|\gamma, s\rangle$ to give

$$\begin{aligned} |\gamma, s\rangle &= c_{\gamma,m,h}^\dagger (a^\dagger)^s |0\rangle = c_{\gamma,m,h}^\dagger (\tilde{N} Z_{R,[l,u],h})^s |0\rangle \\ &= c_{\gamma,m,h}^\dagger (\tilde{N}_{R,u,h} Z_{R,m,u,h})^{s(u,h)} \dots (\tilde{N}_{R,l,h} Z_{R,m,l,h})^{s(l,h)} |0\rangle. \end{aligned} \quad (31)$$

One has a similar expression for the imaginary component:

$$\begin{aligned} |\gamma', t\rangle &= d_{\gamma', m, h}^\dagger (b^\dagger)^t |0\rangle = d_{\gamma', m, h}^\dagger (\tilde{N} Z_{I, [l', u', h]})^s |0\rangle \\ &= d_{\gamma', m, h}^\dagger (\tilde{N}_{I, u', h} Z_{I, m, u', h})^{t(u', h)} \dots (\tilde{N}_{I, l', h} Z_{I, m, l', h})^{t(l', h)} |0\rangle. \end{aligned} \quad (32)$$

This can all be put together to define arithmetic addition and subtraction. One has for \tilde{O}_A , where $O = +$ or $O = -$,

$$\begin{aligned} \tilde{O}_A |\gamma, s; \gamma', t\rangle |\gamma_1 s'; \gamma_1', t'\rangle &= \tilde{O}'_A |\gamma, s; \gamma', t\rangle C |\gamma_1 s'; \gamma_1', t'\rangle \\ &= |\gamma_1 s'; \gamma_1', t'\rangle \tilde{O}'_A |\gamma, s; \gamma', t\rangle |\gamma_1 s'; \gamma_1', t'\rangle \\ &= |\gamma_1 s'; \gamma_1', t'\rangle |\gamma, s; \gamma', t\rangle |(\gamma, s; \gamma', t) O'_A (\gamma_1 s'; \gamma_1', t')\rangle. \end{aligned} \quad (33)$$

Here the state $|\gamma_1 s'; \gamma_1', t'\rangle$ is copied and \tilde{O}'_A acts on $|\gamma, s; \gamma', t\rangle$ and the produced copy state.

To define $|(\gamma, s; \gamma', t) O'_A (\gamma_1 s'; \gamma_1', t')\rangle$ it is easier to consider just the real component as the treatment for the imaginary component is the same for addition and subtraction. One also needs to consider separately the different signs of the string states.

For $\gamma = \gamma_1 = +$ or $\gamma = \gamma_1 = -$ and $O = +$,

$$|(\gamma, (m, h), s) +_A (\gamma_1, (m, h'') s')\rangle = c_{\gamma_1, (m, h'')}^\dagger (\tilde{N} Z_{R, [l, u], h''})^s (a^\dagger)^{s'} |0\rangle, \quad (34)$$

where, following equation (31),

$$(\tilde{N} Z_{R, [l, u], h''})^s = (\tilde{N}_{R, u, h''} Z_{R, m, u, h''})^{s(u, h)} \dots (\tilde{N}_{R, l, h''} Z_{R, m, l, h''})^{s(l, h)}. \quad (35)$$

Note that the powers $s(u, h) \dots s(l, h)$ are obtained from the qubit string along h but the NZ factors are applied to the copy string along h'' .

For $\gamma = -$ and $\gamma_1 = +$, there are two cases to consider. If

$$W_R |\gamma, (m, h), s\rangle \leq_A |\gamma_1, (m, h'), s'\rangle,$$

where

$$W_R = \sum_{m, h} c_{-, m, h}^\dagger c_{+, m, h} + c_{+, m, h}^\dagger c_{-, m, h} \quad (36)$$

is a sign change operator for the real component, then

$$|(\gamma, (m, h), s) +_A (\gamma_1, (m, h'') s')\rangle = c_{\gamma_1, (m, h'')}^\dagger (N^\dagger Z_{R, [l, u], h''})^s (a^\dagger)^{s'} |0\rangle \quad (37)$$

and

$$(N^\dagger Z_{R, [l, u], h''})^s = (\tilde{N}_{R, u, h''}^\dagger Z_{R, m, u, h''})^{s(u, h)} \dots (\tilde{N}_{R, l, h''}^\dagger Z_{R, m, l, h''})^{s(l, h)}. \quad (38)$$

This uses the observation that addition of a negative number is equivalent to subtraction of a positive one.

If

$$W_R|\gamma, (m, h), s\rangle \geq_A |\gamma_1, (m, h'), s'\rangle$$

then

$$\begin{aligned} |(\gamma, (m, h), s) +_A (\gamma_1, (m, h''), s')\rangle &= W_R|(\gamma_1, (m, h''), s') +_A (\gamma, (m, h), s)\rangle \\ &= W_R c_{\gamma, (m, h)}^\dagger (N_{R, [l, u], h}^\dagger Z_{R, [l, u], h})^{s'} (a^\dagger)^s |0\rangle. \end{aligned} \quad (39)$$

This expresses the fact that $A + (-B) = -(B + (-A))$.

This covers all the cases for addition because the case for $\gamma = +$ and $\gamma' = -$ is obtained from the results above. Also all the cases for subtraction are included because $A - B = A + (-B)$. This is shown in the above where $\tilde{N}_{R, j, h}^\dagger$, which corresponds to subtraction of 2^{j-m} , is used.

These results also extend to the imaginary part in an obvious way. $(\tilde{N} Z_{R, [l, u], h''})^s$ becomes

$$(\tilde{N} Z_{I, [l', u'], h''})^t = (\tilde{N}_{I, u', h''} Z_{I, m, u', h''})^{t(u', h)} \dots (\tilde{N}_{I, l', h''} Z_{I, m, l', h''})^{t(l', h)} \quad (40)$$

and W_R is replaced by

$$W_I = \sum_{m, h} (d_{+, m, h}^\dagger d_{-, m, h} + d_{-, m, h}^\dagger d_{+, m, h}). \quad (41)$$

Addition or subtraction of both the real and imaginary components is defined using the above definitions for all the cases that arise.

The definitions given have the axiomatic property that 0 is an additive identity. To see this one notes from equation (34) that if $s(j) = 0$ for all j in $[l, u]$ then

$$c_{\gamma_1, m, h''}^\dagger (\tilde{N}_{R, [l, u], h''} Z_{R, m, [l, u], h''})^s (a^\dagger)^{s'} |0\rangle = c_{\gamma_1, m, h''}^\dagger (a^\dagger)^{s'} |0\rangle. \quad (42)$$

If $s'(j) = 0$ for all j in $[l', u']$ then

$$c_{\gamma_1, m, h''}^\dagger (\tilde{N}_{R, [l, u], h''} Z_{R, m, [l, u], h''})^s (a^\dagger)^{s'} |0\rangle = c_{\gamma', m, h''}^\dagger (a^\dagger)^s |0\rangle. \quad (43)$$

This follows directly from equation (31).

Multiplication is more complex. Results given in more detail in [3] are summarized here. Based on equation (21), the state resulting from multiplication

can be represented as $|(\gamma, s; \gamma', t) \times_A (\gamma_1 s'; \gamma'_1, t')\rangle$. Following the usual rules of multiplication of complex numbers gives

$$|(\gamma, s; \gamma', t) \times_A \gamma_1 s'; \gamma'_1, t'\rangle = [|(\gamma, s \times_A (\gamma_1, s')) +_A ((\gamma', t) \times_A (\gamma'_1, t'))] ; +_A [((\gamma, s) \times_A (\gamma'_1, t')) +_A ((\gamma', t) \times_A (\gamma_1, s'))]. \quad (44)$$

This definition shows multiplication for complex string states defined by four component multiplications. Then the resulting real components are added together as are the imaginary components.

The four component multiplications are defined by

$$\begin{aligned} |(\gamma, s) \times_A (\gamma', s')\rangle &= c^\dagger_\gamma (a^\dagger)^s \times_A c^\dagger_{\gamma'} (a^\dagger)^{s'} |0\rangle = c^\dagger_{\gamma''} (a^\dagger)^{s''} |0\rangle, \\ &\quad \gamma'' = + \text{ if } \gamma = \gamma', \quad \gamma'' = - \text{ if } \gamma \neq \gamma', \\ |(\gamma', t) \times_A (\gamma'_1, t')\rangle &= d^\dagger_{\gamma'} (b^\dagger)^t \times_A d^\dagger_{\gamma'_1} (b^\dagger)^{t'} |0\rangle = |c^\dagger_{\gamma''} (a^\dagger)^{s_1} |0\rangle, \\ &\quad \gamma'' = + \text{ if } \gamma \neq \gamma', \quad \gamma'' = - \text{ if } \gamma = \gamma', \\ |(\gamma', t) \times_A (\gamma, s)\rangle &= d^\dagger_{\gamma'} (b^\dagger)^t \times_A c^\dagger_\gamma (a^\dagger)^s |0\rangle = |d^\dagger_{\gamma''} (b^\dagger)^{t''} |0\rangle, \\ &\quad \gamma'' = + \text{ if } \gamma = \gamma', \quad \gamma'' = - \text{ if } \gamma \neq \gamma', \\ |c^\dagger_\gamma (a^\dagger)^s \times_A d^\dagger_{\gamma'} (b^\dagger)^t |0\rangle &= |d^\dagger_{\gamma''} (b^\dagger)^{t''} |0\rangle, \\ &\quad \gamma'' = + \text{ if } \gamma = \gamma', \quad \gamma'' = - \text{ if } \gamma \neq \gamma'. \end{aligned} \quad (45)$$

Location labels have been left off c^\dagger and d^\dagger to save on space.

There are four multiplications to consider. However it is sufficient to examine only one as the others follow the same rules. A unitary shift operator T is useful here, where T satisfies the commutation rule

$$\begin{aligned} T a^\dagger_{i,j,h} &= a^\dagger_{i,j+1,h} T, & T a_{i,j,h} &= a_{i,j+1,h} T, \\ T b^\dagger_{i,j,h} &= b^\dagger_{i,j+1,h} T, & T b_{i,j,h} &= b_{i,j+1,h} T, \\ & & T |0\rangle &= |0\rangle. \end{aligned} \quad (46)$$

These equations show that T increases the value of j to $j + 1$. Conversely taking the adjoints of these equations shows that $T^\dagger = T^{-1}$ decreases the value of j to $j - 1$. The use of T and T^{-1} derive from the observation that their actions correspond to multiplying and dividing by 2. The state $(a^\dagger)^{s''} |0\rangle$ is defined from $(a^\dagger)^s |0\rangle$ and $(a^\dagger)^{s'} |0\rangle$ by

$$(a^\dagger)^{s''} |0\rangle = (T^{u-m} (a^\dagger)^{s'})^{s(u)} +_A ((T^{u-1-m} (a^\dagger)^{s'})^{s(u-1)} +_A (\dots +_A (T^{l-m} (a^\dagger)^{s'})^{s(l)})) |0\rangle. \quad (47)$$

Here

$$T^k (a^\dagger)^{s'} = (a^\dagger)^{s'_k} T^k, \quad (48)$$

where $s'_k(j+k) = s'(j)$ for $l \geq j \geq u$.

This can be expressed using the successor operators $\tilde{N}_{R,j,h} Z_{R,m,j,h}$ defined by equations (34)-(37) and (39). From equation (31) one can substitute $(\tilde{N} Z_{R,[l,u],h})^{s'}$ for $(a^\dagger)^{s'}$ and use the definition of addition, equation (34), to obtain

$$(a^\dagger)^{s''} |0\rangle = (T^{u-m} (\tilde{N} Z_{R,[l',u'],h})^{s'})^{s(u)} ((T^{u-1-m} (\tilde{N} Z_{R,[l',u'],h})^{s'})^{s(u-1)} \dots (T^{l-m} (\tilde{N} Z_{R,[l',u'],h})^{s'})^{s(l)}) |0\rangle. \quad (49)$$

This equation looks complex because it expresses the steps one goes through, using repeated additions to carry out multiplication. It is a quantum theoretical expression of the usual multiplication rule shown by the following simple example. Let s and s' be 0, 1 functions with respective integer interval domains $[l, u], [l', u']$ and with the ‘‘binal’’ point at m , where m is in both domains. Then $s \times s' = \sum_{j=l}^u s(j) (2^{j-m} \times s')$. Note that $s(j)$ appears as a factor here instead of an exponent.

Multiplication for the other three entries in equation (45) follows the same rules. Conversion of type b qubits to type a qubits and conversely can be expressed explicitly by the use of type change operators. This will not be done here as it adds nothing new.

As is the case for rational string numbers, the string states described here are not closed under division. However they are closed under division to any accuracy $|+, -\ell\rangle = a_{+,-\ell}^\dagger |0\rangle$. For any pair of states $|\gamma, s; \gamma', t\rangle \neq |0\rangle$ and $|\gamma_1 s'; \gamma'_1, t'\rangle$, the state $|(\gamma_1 s'; \gamma'_1, t' / \gamma, s; \gamma', t)_\ell\rangle$ is defined by

$$|(\gamma_1 s'; \gamma'_1, t' / \gamma, s; \gamma', t)_\ell\rangle = |(+, 1/\gamma, s; \gamma', t)_\ell \times_A (\gamma_1 s'; \gamma'_1, t')\rangle, \quad (50)$$

where the ℓ inverse state $|(+, 1/\gamma, s; \gamma', t)_\ell\rangle$ is defined by

$$\begin{aligned} & |(+, 1/\gamma, s; \gamma', t)_\ell\rangle \\ & = |(+, 1/[(\gamma s \times_A \gamma s) + (\gamma' t \times_A \gamma'' t)])_\ell \times_A (\gamma s, \gamma'' t)\rangle. \end{aligned} \quad (51)$$

Here $\gamma'' \neq \gamma'$ and

From equation (45) one sees that the denominator state $|(+, 1/[(\gamma s \times_A \gamma s) + (\gamma' t \times_A \gamma'' t)])_\ell\rangle$ of the ℓ inverse is a positive, real state. Thus to complete the definition of ℓ division it suffices to define the state that is the ℓ inverse of a positive, real state $|+, s\rangle \neq |+, 0\rangle$. To this end $|(+1/+, s)_\ell\rangle = |+, (a^\dagger)^{s'} |0\rangle$ is defined by two conditions:

$$|(+, 1) -_A (+, \bar{0}_{[m, m-\ell+1]} 1_{m-\ell})\rangle \leq_A |(+1/+, s)_\ell \times_A (+, s)\rangle \leq_A |+, 1\rangle, \quad (52)$$

and, if $|+, s''\rangle$ is another state such that $|(+1/+, s'')_\ell \times (+, s'')\rangle$ satisfies the above double inequality, then the smallest j value in s' , where $s'(j) \neq 0$ is larger than that in s'' .

The first condition states that $(1/s)_\ell \times s$ must lie between $1 - 2^{-\ell}$ and 1. The second condition states that $(1/s)_\ell$ is the largest number to satisfy the first condition.

As an example, assume $m = 0$ and let $|+, s\rangle = |+, a_{1,2}^\dagger a_{0,1}^\dagger a_{1,0}^\dagger |0\rangle$ and $\ell = 7$. Among many others, the states $|+, (a^\dagger)^{s_1} |0\rangle$ and $|+, (a^\dagger)^{s'_1} |0\rangle$, where

$$s_1(j) = 1 \text{ if } j = -3, -4, -7, -9, -10, -11$$

$$\text{and } s_1(j) = 0 \text{ for all other } j \text{ in } [-11, 0],$$

$$s'_1(j) = 1 \text{ if } j = -3, -4, -7, -8 \text{ and } s'_1(j) = 0 \text{ for all other } j \text{ in } [-8, 0],$$

both satisfy equation (52) for $\ell = 7$. Also a more restricted s'' with $s''(j) = 1$ if $j = -3, -4, -7$ does not satisfy the conditions. Based on this, the unique ℓ inverse state for $\ell = 7$ is $|+, (a^\dagger)^{s'_1} |0\rangle$ as -8 is greater than -11 .

In binary numbers the example says that 0.00110010111 and 0.00110011 are among the many results accurate to $0.0000001 = 2^{-7}$ of the division $1/101.0$ to an accuracy of 2^{-7} . The second condition says keep the largest one, which is 0.00110011. Additional details on the explicit construction of the state $|(+, 1/+, s)_\ell\rangle$ are given in [3].

So far, states of the form $|\gamma, s; \gamma', t\rangle$ and their superpositions have been defined,, along with the arithmetic relations $=_A, \leq_A$ and the operations $\tilde{+}_A, \tilde{-}_A, \tilde{\times}_A, \tilde{\div}_A, \ell$. The arithmetic relations and operations are defined on these states in order to show that these states do represent complex rational numbers. This follows from showing that the relations and operations satisfy the requisite axioms for complex natural numbers. Included are the commutativity and associativity of $\tilde{+}_A, \tilde{-}_A, \tilde{\times}_A$, the identity property of $|0\rangle$ and $|+, 1\rangle = c_{+,m}^\dagger a_{1,m}^\dagger |0\rangle$ for $\tilde{+}_A$ and $\tilde{\times}_A$, the distributivity of $\tilde{\times}_A$ over $\tilde{+}_A$, etc. These are discussed in [3].

3. Cauchy Sequences of States in \mathcal{F}

The rest of this paper is concerned with sequences of states that satisfy the Cauchy condition. Sequences of states are defined to be functions $\psi : \{1, 2, \dots\} \rightarrow \mathcal{F}$, where ψ_n is a state in \mathcal{F} . If the states ψ_n are basis states $|\gamma_n, s_n; \gamma'_n, t_n\rangle$ the sequence will be denoted by $\{|\gamma_n, s_n; \gamma'_n, t_n\rangle\}$ instead of the more general form

$\{\psi_n\}$. In the general case $\{\psi_n\}$ is a sequence of normalized states, where

$$\psi_n = \sum_{\gamma, s, \gamma', t} |\gamma, (m, h), s, \gamma', t\rangle \langle \gamma, (m, h), s, \gamma', t | \psi_n \rangle \quad (53)$$

and $\langle \psi_n | \psi_n \rangle = 1$ for each n . Note that the sum is over states of qubit strings of all finite lengths.

One condition implied by the definition of equation (53) is that the value of (m, h) is the same for each state in the sequence. This is also the case for the sequence $\{|\gamma_n, s_n; \gamma'_n, t_n\rangle\}$ of basis states. This is a locality condition that places the sign qubits at the same location in $I \times I$ for each n . One could relax this condition by including a sum over (m, h) in equation (53), but this will not be done here.

Sequences that satisfy the Cauchy conditions are of interest here as the goal (Section 5) is to show that (equivalence classes of) these Cauchy sequences are real or complex numbers. For the real numbers one must show that the equivalence classes are a complete, ordered field. For the complex numbers the required properties are those of a complete, algebraically closed, field, see [13], [18].

The reason that one studies Cauchy sequences of states instead of convergent sequences is that the definition of the Cauchy condition is based directly on properties of states in \mathcal{F} . Convergence is not used because the sequences themselves are not elements of \mathcal{F} . Thus convergence to a sequence has no meaning in \mathcal{F} .

3.1. The Cauchy Condition

A Cauchy sequence of rational numbers $\{x_n : n = 1, 2, \dots\}$ is a sequence that satisfies the Cauchy condition (see [13], [18])

$$\text{For each } \ell \text{ there is an } h \text{ such that for all } m, n > h \quad (54)$$

$$|x_n - x_m| < 2^{-\ell}.$$

This definition can also be applied to sequences $\{x_n = u_n + iv_n : n = 1, 2, \dots\}$ of complex rational numbers. In this case $|x_n - x_m| < 2^{-\ell}$ is expressed here as two separate conditions, $|u_n - u_m| < 2^{-\ell}$ and $|v_n - v_m| < 2^{-\ell}$, for the real and imaginary parts of the sequence numbers. It is possible to combine the two conditions into one, but this will not be done here.

The Cauchy condition can be applied to basis states of qubit strings. Let $\{|\gamma_n, s_n; \gamma'_n, t_n\rangle\}$ be a sequence of states of qubit strings. The Cauchy condition

for these states is:

$$\begin{aligned} &\text{For each } \ell \text{ there is an } h \text{ where for all } j, k > h \\ &|(|\gamma_j s_j -_A \gamma_k s_k|_A)\rangle <_A |+, -\ell\rangle \text{ and} \\ &|(|\gamma'_j t_j -_A \gamma'_k t_k|_A)\rangle <_A |+, -\ell\rangle. \end{aligned} \tag{55}$$

Here two separate conditions are used, one for the real component and one for the imaginary component.

Here $|(|\gamma_j s_j -_A \gamma_k s_k|_A)\rangle$ is the state that is the arithmetic absolute value of the state resulting from the arithmetic subtraction of $|\gamma_k, s_k\rangle$ from $|\gamma_j s_j\rangle$. The subscripts A are used to indicate that the operations are arithmetic and not the usual quantum theory operations.

The absolute value state $|(|(\gamma_k, s_k) -_A (\gamma_j, s_j)|_A)\rangle$ is represented in terms of AC operators by (equation (37))

$$\begin{aligned} &c^\dagger_{+, (m, h'')} (N^\dagger Z_{R, [l_k, u_k], h''})^{s_k} (a^\dagger)^{s_j} |0\rangle \text{ if } |+, s_j\rangle \geq_A |+, s_k\rangle, \\ &c^\dagger_{+, (m, h'')} (N^\dagger Z_{R, [l_j, u_j], h''})^{s_j} (a^\dagger)^{s_k} |0\rangle \text{ if } |+, s_k\rangle \geq_A |+, s_j\rangle, \end{aligned} \tag{56}$$

if $\gamma_k = \gamma_j$. If $\gamma_k \neq \gamma_j$, $|(|(\gamma_k, s_k) -_A (\gamma_j, s_j)|_A)\rangle$ is represented by

$$c^\dagger_{+, (m, h'')} (N^\dagger Z_{R, [l_k, u_k], h''})^{s_k} (a^\dagger)^{s_j} |0\rangle. \tag{57}$$

Similar relations with t replacing s hold for the imaginary component. In this case one is using $|\pm ix - \pm iy| = |i(\pm x - \pm y)| = |\pm x - \pm y|$.

Note that this definition of the Cauchy condition is local in the sense that it is defined for sequences at three locations $(m, h), (m, h')$ and (m, h'') . Each state in the sequence is a state of a qubit string at (m, h) (equation (53)). Since A subtraction is a binary operation, the sequence must be copied to another h location, (m, h') . The result of the subtraction is a third sequence at another h location, (m, h'') . Converting to the absolute value is simply a change in the state of the sign qubit(s) and does not create a new sequence. Finally, the state $|-, -\ell\rangle$ is at another location (m, h''') .

A global definition of the Cauchy condition can be given by summing over $(m, h), (m', h'), (m'', h''), (m''', h''')$ with the restriction that the h parameters are pairwise distinct. However this will not be done here. In the following these location labels will be suppressed.

The Cauchy condition can also be defined for sequences $\{\psi_n\}$, where ψ_n is a normalized state given by equation (53). These definitions make use of R and C as they are based on probabilities obtained from the expansion coefficients of the states. The coefficients are elements of C .

A sequence $\{\psi_n\}$ is defined to be a Cauchy sequence if the probability is unity that the Cauchy condition is true for the sequence. This probability is obtained by applying the conditions of equations (55) to the components of ψ_j, ψ_k and summing over all components that satisfy the conditions. To see this for equation (55), let $|\tilde{\text{Re}}|, |\tilde{\text{Im}}|$ be operators acting on the real and the imaginary parts of the states that are defined by

$$\begin{aligned} & |\tilde{\text{Re}}| |\tilde{\text{Im}}| |\gamma, s, \gamma_1, t\rangle |\gamma', s', \gamma'_1, t'\rangle \\ &= |\gamma, s, \gamma_1, t\rangle |\gamma', s', \gamma'_1, t'\rangle |(\gamma, s -_A \gamma', s'|_A)\rangle |(\gamma_1, t -_A \gamma'_1, t'|_A)\rangle. \end{aligned} \quad (58)$$

Here lattice locations of the qubit strings are suppressed to avoid notation clutter.

Let $\tilde{P}_{\text{Re} \leq \ell}, \tilde{P}_{\text{Im} \leq \ell}$ be projection operators for positive states being arithmetically less than $|+, -\ell\rangle$,

$$\begin{aligned} \tilde{P}_{\text{Re} \leq \ell} |+, s\rangle &= \begin{cases} |+, s\rangle & \text{if } |+, s\rangle \leq_A |+, -\ell\rangle, \\ 0 & \text{if } |+, s\rangle >_A |+, -\ell\rangle, \end{cases} \\ \tilde{P}_{\text{Im} \leq \ell} |+, t\rangle &= \begin{cases} |+, t\rangle & \text{if } |(+, t|_A)\rangle \leq_A |+, -\ell\rangle, \\ 0 & \text{if } |(+, t|_A)\rangle >_A |+, -\ell\rangle. \end{cases} \end{aligned} \quad (59)$$

Here the state $|(+, t|_A)\rangle = c_+^\dagger (a^\dagger)^t |0\rangle$ is a real rational state, so it can be directly compared with $|+, -\ell\rangle$.

Putting these results together and letting

$$\Theta_{\psi_j \psi_k} = |\tilde{\text{Im}}| |\tilde{\text{Re}}| \psi_j \psi_k \quad (60)$$

gives

$$\begin{aligned} P_{j,k,\ell}^{\{\psi_n\}} &= \langle \Theta_{\psi_j \psi_k} | \tilde{P}_{\text{Re} < \ell} \tilde{P}_{\text{Im} < \ell} | \Theta_{\psi_j \psi_k} \rangle \\ &= \sum_{\gamma, s, \gamma', t} \sum_{\gamma_1, s', \gamma'_1, t'} |\langle \gamma, s, \gamma', t | \psi_j \rangle|^2 |\langle \gamma_1, s', \gamma'_1, t' | \psi_k \rangle|^2 : \\ & |(\gamma, s -_A \gamma_1, s'|_A)\rangle \leq_A |+, -\ell\rangle \text{ and} \\ & |(\gamma', t -_A \gamma'_1, t'|_A)\rangle \leq_A |+, -\ell\rangle. \end{aligned} \quad (61)$$

The projection operator product $\tilde{P}_{\text{Re} < \ell} \tilde{P}_{\text{Im} < \ell}$ limits the sums to those component states that satisfy the Cauchy conditions for both the real and imaginary components. $P_{j,k,\ell}^{\{\psi_n\}}$ is the probability that $\{\psi_n\}$ satisfies these conditions at j and k , i.e. for ψ_j and ψ_k .

One now has to account for the quantifiers in the definition of the Cauchy condition. This is done in steps. The probability, $P_{h,\ell}^{\{\psi_n\}}$, that $\{\psi_n\}$ satisfies the conditions for all $j, k > h$ is given by

$$P_{h,\ell}^{\{\psi_n\}} = \liminf_{j,k > h} P_{j,k,\ell}^{\{\psi_n\}}. \quad (62)$$

The probability, $P_\ell^{\{\psi_n\}}$, that there exists an h such that the sequence $\{\psi_n\}$ satisfies the Cauchy conditions at ℓ for all $j, k > h$ is given by

$$P_\ell^{\{\psi_n\}} = \limsup_{h \rightarrow \infty} P_{h,\ell}^{\{\psi_n\}}. \tag{63}$$

Finally one includes all ℓ by

$$P^{\{\psi_n\}} = \liminf_{\ell \rightarrow \infty} P_\ell^{\{\psi_n\}}. \tag{64}$$

Putting these equations together gives

$$P^{\{\psi_n\}} = \liminf_{\ell \rightarrow \infty} \limsup_{h \rightarrow \infty} \liminf_{j,k > h} P_{j,k,\ell}^{\{\psi_n\}}, \tag{65}$$

where $P_{j,k,\ell}^{\{\psi_n\}}$ is given by equation (61).

One can use these equations to obtain necessary and sufficient conditions for the sequence $\{\psi_n\}$ to be Cauchy with probability $P^{\{\psi_n\}} = 1$. One condition is that $P_\ell^{\{\psi_n\}} = 1$ for each ℓ . This follows from equation (64) which shows that $P^{\{\psi_n\}}$ is the greatest lower bound of all the $P_\ell^{\{\psi_n\}}$. This condition is satisfied if, for each ℓ , $P_{h,\ell}^{\{\psi_n\}}$ either equals 1 for all h greater than some h_0 or approaches 1 asymptotically as $h \rightarrow \infty$. This follows from equation (63) which gives $P_\ell^{\{\psi_n\}}$ as the least upper bound of all the $P_{h,\ell}^{\{\psi_n\}}$. These two conditions can be combined to the single condition that for all ℓ , $P_{j,k,\ell}^{\{\psi_n\}} \rightarrow 1$ as $j, k \rightarrow \infty$ or $\liminf_{j,k \rightarrow \infty} P_{j,k,\ell}^{\{\psi_n\}} = 1$.

It is useful at this point to consider examples. Let s be a 0 – 1 valued function on the infinite integer interval $[0, -\infty]$ and let the sign qubits be at site $m = 0$ and in state $+$. Define each ψ_n in the sequence $\{\psi_n\}$ by

$$\psi_n = c_{+,0}^\dagger a_{s(0),0}^\dagger \cdots a_{s(-n+1),-n+1}^\dagger \frac{1}{\sqrt{2}} (a_{1,-n}^\dagger + a_{0,-n}^\dagger |0). \tag{66}$$

This is a simple example of a pure real (no imaginary component) state sequence that does not correspond to any classical complex rational number sequence. The observation that the probability is 1 that this sequence is Cauchy follows from the fact that for each ℓ , the probability $P_{j,k,\ell}^{\{\psi_n\}} = 1$ for all $j, k > \ell$. It follows that $P_\ell^{\{\psi_n\}} = 1$ for each ℓ and thus $P^{\{\psi_n\}} = 1$.

There are many simple examples of this type. For instance, one can include an imaginary component to ψ_n by letting t be a 0 – 1 valued function with the same domain as s , including a string of $d_{+,0}^\dagger b_{t(0),0}^\dagger \cdots b_{t(-n+1),-n+1}^\dagger$, of creation

operators, and replacing the superposition state at site $-n$ by a Bell state operator

$$B_{-n} = \frac{1}{\sqrt{2}}(a_{1,-n}^\dagger b_{1,-n}^\dagger + a_{0,-n}^\dagger b_{0,-n}^\dagger). \quad (67)$$

$\{\psi_n\}$ is still a Cauchy sequence even though the component states of the sequence are entangled.

There are also more complex examples of Cauchy sequences based on rational approximations to analytical functions. An example of this type is based on a rational approximation to a Gaussian function. Let s be a 0–1 valued function on the infinite interval domain $[u, -\infty]$ with the sign at $m = 0$. Define $|+, S(s', n)\rangle$ to be the n th Gaussian approximation to the state $|+, s'\rangle$. That is,

$$|+, S(s', n)\rangle =_A |(\exp[-\frac{|((+, s') -_A (+, s_{[u,-n]})|^2)}{|+, \sigma|^2}] \times_A |+, n\rangle]_n)_n\rangle. \quad (68)$$

Here $|((+, s') -_A (+, s_{[u,-n]})|^2\rangle$ is the state resulting from subtracting $|+, s_{[u,-n]}\rangle$ from $|+, s'\rangle$ and multiplying the result by itself. $|+, n\rangle$ is a natural number state, and $|+, \sigma\rangle$ is a positive rational state. No imaginary components are present. The subscripts “ n ” on $[-]$ and $(-)$ denote division to accuracy n and evaluation of the exponential to accuracy n , perhaps as an initial part of a power series expansion. Note that n appears both in the exponent and as accuracy subscripts.

Define the state ψ_n by

$$\psi_n = \sum_{s'}^n \frac{\langle +, S(s',) | \tilde{N} | +, S(s', n)\rangle}{M_n} |+, s'\rangle. \quad (69)$$

Here the matrix element is the \tilde{N} eigenvalue associated with the state $|+, S(s', n)\rangle$ and \tilde{N} is given by equations (5)-(8).

The superscript n on the summation means the sum is restricted to all s with a domain $[u + n, -n]$. The restriction to a finite domain is necessary because of the presence of states with arbitrary numbers of leading and trailing 0's. Without such a restriction it would be difficult, if not impossible to normalize ψ_n with some normalization factor, M_n .

The coefficients on the right side of equation (68) correspond to a rational approximation to accuracy n of a Gaussian distribution about $|+, s_{[u,-n]}\rangle$ (or about the eigenvalue $N(+, s_{[u,-n]})$). The presence of n in the numerator of the exponent ensures that the sequence $\{\psi_n\}$ is Cauchy. This follows from the observation that the standard deviation, $\sigma_n^2 = \sigma^2/n \rightarrow 0$ as $n \rightarrow \infty$.

4. Properties of Operations on Cauchy State Sequences

Cauchy state sequences inherit many properties of the complex rational string states. They also have some additional properties that are not possessed by the string states. Here the emphasis is on properties and operations needed to show that (equivalence classes of) Cauchy state sequences have the requisite properties of real and complex numbers. The basic relations are equality $=_X$ and an ordering $<_X$ for $X = R, I,$ and C . These refer to equality and ordering defined separately for the real and imaginary components and for both together. As is well known, $<_C$ is defined only on those complex state pairs where both the real and imaginary components have the same ordering relation.

The basic operations on Cauchy sequences are those of a field, namely, $\tilde{+}_X, \tilde{-}_X, \tilde{\times}_X,$ and $\tilde{\div}_X$. The definitions of these operators will follow those for the arithmetic operations in that their action on pairs of Cauchy sequence leaves the pairs and creates a third sequence of states. The actions of these operators on Cauchy sequences of rational string states, can be represented by

$$\begin{aligned} \tilde{O}_X\{|\gamma_n s_n; (\gamma_1)_{nt_n}\}\{|\gamma'_n s'_n; (\gamma'_1)_{nt'_n}\} \\ = \{|\gamma_n s_n; (\gamma_1)_{nt_n}\}\{|\gamma'_n s'_n; (\gamma'_1)_{nt'_n}\}\{|\gamma''_n s''_n; (\gamma''_1)_{nt''_n}\}, \end{aligned} \quad (70)$$

where O stands for $+, -, \times,$ and \div . Here $\{|\gamma''_n s''_n; (\gamma''_1)_{nt''_n}\}$ is the sequence resulting from carrying out the operation O .

The state sequence, $\{|\gamma''_n s''_n; (\gamma''_1)_{nt''_n}\}$, which is the result of carrying out the operation \tilde{O}_X , is defined by

$$\{|\gamma''_n s''_n; (\gamma''_1)_{nt''_n}\} = \{ |(\gamma_n s_n; (\gamma_1)_{nt_n}) O_A (\gamma'_n s'_n; (\gamma'_1)_{nt'_n}) \cdot \} \quad (71)$$

For each n the n -th element of this sequence is the state obtained by carrying out the arithmetic O_A operation on the n -th elements of the pair of input Cauchy sequences.

This definition is satisfactory for all operations except division as the string states are not closed under division. One way around this is to use a diagonal definition: The n -th element of $\{|\gamma''_n s''_n; (\gamma''_1)_{nt''_n}\}$ is defined by $|(\gamma_n s_n; (\gamma_1)_{nt_n}) \div_{A,n} (\gamma'_n s'_n; (\gamma'_1)_{nt'_n})$. More details will be given later.

Note that the definitions of both the properties and operations are global in that they apply to tuples of state sequences anywhere in $I \times I$. This is implicitly assumed although it could be made explicit by including the location parameters and summing over them with the restriction that no two sequences have the same h value.

In the following the definitions of the properties and operations are extended to sequences of linear superposition states. Also, proofs that the resulting state sequences satisfying equation (71) are Cauchy are provided.

4.1. The Properties $=_X$ and $<_X$ for $X = R, I, C$

A first step is to lift the properties $=_A$ and $<_A$ from states in \mathcal{F} to Cauchy sequences of these states. Two Cauchy sequences of real rational states, $\{|\gamma_n, s_n\rangle\}$ and $\{|\gamma'_n, s'_n\rangle\}$, are R equal,

$$\{|\gamma_n, s_n\rangle\} =_R \{|\gamma'_n, s'_n\rangle\} \quad (72)$$

if for all ℓ there is an h such that for all $j, k > h$

$$|(|\gamma_j s_j -_A \gamma'_k s'_k|_A\rangle) \leq_A |+, -\ell\rangle. \quad (73)$$

Cauchy sequences of complex rational states, $\{|\gamma_n, s_n, (\gamma_1)_n, t_n\rangle\}$ and $\{|\gamma'_n, s'_n, (\gamma'_1)_n, t'_n\rangle\}$, are C equal

$$\{|\gamma_n, s_n, (\gamma_1)_n, t_n\rangle\} =_C \{|\gamma'_n, s'_n, (\gamma'_1)_n, t'_n\rangle\} \quad (74)$$

if for all ℓ

$$\begin{aligned} |(|\gamma_j, s_j -_A \gamma'_k, s'_k|_A\rangle) &\leq_A |+, -\ell\rangle, \\ |(|(\gamma_1)_j, t_j -_A (\gamma'_1)_k, t'_k|_A\rangle) &\leq_A |+, -\ell\rangle \end{aligned} \quad (75)$$

for all $j, k > \text{some } h$. Note that $=_C$ is equivalent to both $=_R$ and $=_I$ holding for the real and imaginary parts. One can also have Cauchy sequences in which either one, but not both, of the real and imaginary parts are equal.

Extension of these definitions to sequences of states that are linear superpositions of rational string states is based on the results obtained so far. The Cauchy sequences, $\{\psi_n\}$ and $\{\psi'_n\}$, are C equal,

$$\{\psi_n\} =_C \{\psi'_n\} \quad (76)$$

if $\liminf_{j,k \rightarrow \infty} P_{j,k,\ell} = 1$ where the probability $P_{j,k,\ell}$ is given by equation (61) with ψ'_k replacing ψ_k .

As would be expected, there are many different Cauchy sequences that are C equal to a given sequence. It is also the case that for each Cauchy sequence $\{\psi_n\}$ there is a Cauchy sequence, $\{|\gamma_n, s_n, \gamma'_n, t_n\rangle\}$, of complex rational states that is C equal to $\{\psi_n\}$. The proof of this is first given for real states and then extended to complex states. For real states the proof requires finding a rational string state sequence $\{|\gamma_n, s_n\rangle\}$ where the probability is one that $\{|\gamma_n, s_n\rangle\} =_R \{\psi_n\}$.

The probability condition can be expressed by first defining $Q_{j,\ell,\gamma',s'}$ for any $|\gamma',s'\rangle$ by

$$Q_{j,\ell,\gamma',s'} = \sum_{\gamma,s} |\langle \gamma, s | \psi_j \rangle|^2 : (|\gamma, s -_A \gamma', s' |_A \rangle \prec_A |+, -\ell \rangle). \tag{77}$$

Comparison with equation (61), restricted to real rational states, shows that

$$P_{j,k,\ell} = \sum_{\gamma',s'} |\langle \gamma', s' | \psi_k \rangle|^2 Q_{j,\ell,\gamma',s'}. \tag{78}$$

For each j define $|\gamma_j, s_j \rangle$ to be the string state $|\gamma', s' \rangle$ that maximizes $Q_{j,\ell,\gamma',s'}$. Let Q_{j,ℓ,γ_j,s_j} be the maximum value. One sees immediately that

$$P_{j,k,\ell} \leq \sum_{\gamma',s'} |\langle \gamma', s' | \psi_k \rangle|^2 Q_{j,\ell,\gamma_j,s_j} = Q_{j,\ell,\gamma_j,s_j}. \tag{79}$$

Since $\{\psi_n\}$ is Cauchy, $P_{j,k,\ell} \rightarrow 1$ as $j, k, \ell \rightarrow \infty$ which gives $Q_{j,\ell,\gamma_j,s_j} \rightarrow 1$ as $j \rightarrow \infty$ for each ℓ . This completes the proof that $\{\psi_n\} =_R \{|\gamma_n s_n \rangle\}$.

The proof for complex states follows that already given. One must show that for any Cauchy sequence $\{\psi_n\}$, there is a Cauchy sequence $\{|\gamma_n, s_n, \gamma'_n, t_n \rangle\}$ where $\{|\gamma_n, s_n, \gamma'_n, t_n \rangle\} =_C \{\psi_n\}$. Following equation (77) one defines for any $|\gamma, s, \gamma', t \rangle$

$$Q_{j,\ell,\gamma,s,\gamma',t} = \sum_{\gamma,s,\gamma',t} |\langle \gamma, s, \gamma', t | \psi_j \rangle|^2 : \begin{aligned} & (|\gamma, s -_A \gamma_1, s'_1 |_A \rangle \prec_A |+, -\ell \rangle) \\ & \text{and } (|\gamma, t -_A \gamma'_1, t'_1 |_A \rangle \prec_A |+, -\ell \rangle). \end{aligned} \tag{80}$$

For each j define $|\gamma_j, s_j, \gamma'_j, t_j \rangle$ to be the complex rational string state that maximizes $Q_{j,\ell,\gamma,s,\gamma',t}$. Then equation (61) gives

$$P_{j,k,\ell} \leq \sum_{\gamma,s,\gamma',t} |\langle \gamma, s, \gamma', t | \psi_k \rangle|^2 Q_{j,\ell,\gamma_j,s_j,\gamma'_j,t_j} = Q_{j,\ell,\gamma_j,s_j,\gamma'_j,t_j}. \tag{81}$$

Since $\{\psi_n\}$ is Cauchy, $P_{j,k,\ell} \rightarrow 1$ as $j, k \rightarrow \infty$. This implies that $Q_{j,\ell,\gamma_j,s_j,\gamma'_j,t_j} \rightarrow 1$ as $j, k \rightarrow \infty$, which completes the proof.

Definitions of $\prec_R, \prec_I,$ and \prec_C on Cauchy sequences of rational states are based on the definition of \prec_A . The Cauchy sequence $\{|\gamma_n, s_n \rangle\}$ of real rational states is R less than $\{|\gamma'_n, s'_n \rangle\}$,

$$\{|\gamma_n, s_n \rangle\} \prec_R \{|\gamma'_n, s'_n \rangle\} \tag{82}$$

if for some ℓ and h

$$|(\gamma_j, s_j) +_A (+, -\ell)\rangle \langle_A |\gamma'_k, s'_k\rangle \quad (83)$$

for all $j, k > h$. This is based on the observation that two Cauchy sequences are not R equal if they are separated asymptotically by a finite gap, denoted by $(+, -\ell)$ in the state on the left.

A similar definition of \langle_C applies to Cauchy sequences of complex rational states.

$$\{|\gamma_n, s_n \gamma'_n t_n\rangle\} \langle_C \{ |(\gamma_1)_n, s'_n, (\gamma'_1)_n, t'_n\rangle\} \quad (84)$$

if both the real and imaginary component sequences are separated asymptotically by gaps. Of course \langle_C is only partially defined on these sequences as the real and imaginary parts of a sequence can have different order relations.

The ordering relations \langle_R , \langle_I , and \langle_C can be extended to Cauchy sequences of superposition states. One has

$$\{\psi_n\} \langle_R \{\psi'_n\} \quad (85)$$

with probability 1 if for some $|+, -\ell\rangle$, $\lim_{j,k \rightarrow \infty} Q_{R,j,k,\ell} = 1$, where

$$Q_{R,j,k,\ell} = \sum_{\gamma,s} \sum_{\gamma_1,s'} |\langle \gamma, s | \psi_j \rangle|^2 \times |\langle \gamma_1, s' | \psi'_k \rangle|^2 : |(\gamma, s) +_A (+, -\ell)\rangle \langle_A |\gamma_1, s'\rangle. \quad (86)$$

That is, the probability is 1 that the real parts of $\{\psi_n\}$ and $\{\psi'_n\}_n$ are separated asymptotically by a gap. Similar relations hold for \langle_I and \langle_C .

Sequence pairs $\{\psi_n\}$ and $\{\psi'_n\}$ that are Cauchy satisfy the following relations for \langle_X and $=_X$ for $X = R$ or $X = I$,

$$\begin{aligned} \{\psi_n\} \langle_X \{\psi'_n\} &\text{ true with probability 1, or} \\ \{\psi_n\} =_X \{\psi'_n\} &\text{ true with probability 1, or} \\ \{\psi'_n\} \langle_X \{\psi_n\} &\text{ true with probability 1.} \end{aligned} \quad (87)$$

One way to show this is to prove it for Cauchy sequences, $\{|\gamma_n, s_n, \gamma'_n, t_n\rangle\}$, of complex rational string states and use the fact that any Cauchy sequence $\{\psi_n\}$ is C equal to some such sequence.

Equation (87) does not hold in general for $X = C$. An example would be a pair of Cauchy sequences in which the real and imaginary parts satisfy different alternatives in the equation, such as $\{\psi_n\} \langle_R \{\psi'_n\}$ and $\{\psi_n\} \langle_I \{\psi'_n\}$.

4.2. Addition and Subtraction

As shown by equations (70) and (71), addition of two Cauchy sequences of rational states, $\{|\gamma_n, s_n, (\gamma_1)_n, t_n\rangle\}$ and $\{|\gamma'_n, s'_n, (\gamma'_1)_n, t'_n\rangle\}$, gives the state sequence $\{|\gamma_n, s_n +_A \gamma'_n, s'_n, (\gamma_1)_n, t_n +_A (\gamma'_1)_n, t'_n\rangle\}$. Proof that this sequence is Cauchy requires showing that for all ℓ there is an h such that

$$\begin{aligned} & |(|\gamma_j, s_j +_A \gamma'_j, s'_j\rangle -_A |\gamma_k, s_k +_A \gamma'_k, s'_k\rangle|_A) \langle +, -\ell \rangle \text{ and} \\ & |(|(\gamma_1)_j, t_j +_A (\gamma'_1)_j, t'_j\rangle -_A |(\gamma_1)_k, t_k +_A (\gamma'_1)_k, t'_k\rangle|_A) \langle +, -\ell \rangle \end{aligned} \tag{88}$$

for all $j, k > h$. Rearranging the terms in the left hand parts of the inequalities and using

$$\begin{aligned} & |(|\gamma_j, s_j - \gamma_k, s_k + \gamma'_j, s'_j - \gamma'_k, s'_k\rangle|) \\ & \quad <_A |(|\gamma_j, s_j - \gamma_k, s_k\rangle|) +_A |(|\gamma'_j, s'_j - \gamma'_k, s'_k\rangle|_A)|; \\ & |(|(\gamma_1)_j, t_j - (\gamma_1)_k, t_k +_A (\gamma'_1)_j, t'_j -_A (\gamma'_1)_k, t'_k\rangle|_A) \\ & \quad <_A |(|(\gamma_1)_j, t_j - (\gamma_1)_k, t_k\rangle|_A) +_A |(|(\gamma'_1)_j, t'_j -_A (\gamma'_1)_k, t'_k\rangle|_A)|, \end{aligned} \tag{89}$$

gives equation (88) with ℓ replaced by $\ell - 1$. This result uses the Cauchy property of the two sequences $\{|\gamma_n, s_n, (\gamma_1)_n, t_n\rangle\}$ and $\{|\gamma'_n, s'_n, (\gamma'_1)_n, t'_n\rangle\}$. Equation (34) was used to equate $|+, -\ell\rangle +_A |+, -\ell\rangle$ to $|+, -(\ell - 1)\rangle$.

Addition³ of two Cauchy sequences $\{\psi_n\}, \{\psi'_n\}$ gives the sequence of density operator states $\{\rho_{\psi_n + \psi'_n}\}$, where by equation (24)

$$\begin{aligned} \rho_{\psi_n + \psi'_n} = \sum_{\gamma, s, \gamma_1, t} \sum_{\gamma', s', \gamma'_1, t'} & |\langle \gamma, s, \gamma_1, t | \psi_n \rangle|^2 \\ & |\langle \gamma', s', \gamma'_1, t' | \psi'_n \rangle|^2 \rho_{(\gamma, s, \gamma_1, t) + (\gamma', s', \gamma'_1, t')}, \end{aligned} \tag{90}$$

and $\rho_{(\gamma, s, \gamma_1, t) + (\gamma', s', \gamma'_1, t')} = |(\gamma, s, \gamma_1, t) + (\gamma', s', \gamma'_1, t')\rangle \langle (\gamma, s, \gamma_1, t) + (\gamma', s', \gamma'_1, t')|$.

To show that $\{\rho_{\psi_n + \psi'_n}\}$ is Cauchy, let $Q'_{j,k,\ell}$ be the probability that $|\text{Re } \rho_j - \text{Re } \rho_k| <_N \rho_{+, -\ell}$ and $|\text{Im } \rho_j - \text{Im } \rho_k| <_N \rho_{+, -\ell}$, where $\rho_j = \rho_{\psi_j + \psi'_j}$, $\rho_k = \rho_{\psi_k + \psi'_k}$, and $\rho_{+, -\ell} = |+, -\ell\rangle \langle +, -\ell|$. This is given by

$$\begin{aligned} Q'_{j,k,\ell} = \sum_{\gamma_j, s_j, (\gamma_1)_j, t_j} \sum_{\gamma'_j, s'_j, (\gamma'_1)_j, t'_j} \sum_{\gamma_k, s_k, (\gamma_1)_k, t_k} \sum_{\gamma'_k, s'_k, (\gamma'_1)_k, t'_k} \\ \times | \langle \gamma_j, s_j, (\gamma_1)_j, t_j | \psi_j \rangle |^2 | \langle \gamma'_j, s'_j, (\gamma'_1)_j, t'_j | \psi'_j \rangle |^2 \\ \times | \langle \gamma_k, s_k, (\gamma_1)_k, t_k | \psi_k \rangle |^2 | \langle \gamma'_k, s'_k, (\gamma'_1)_k, t'_k | \psi'_k \rangle |^2 : \\ \rho_{|\gamma_j, s_j + \gamma'_j, s'_j - \gamma_k, s_k - \gamma'_k, s'_k|} <_A \rho_{+, -\ell} \\ \text{and } \rho_{|(\gamma_1)_j, t_j + (\gamma'_1)_j, t'_j - (\gamma_1)_k, t_k - (\gamma'_1)_k, t'_k|} <_A \rho_{+, -\ell}. \end{aligned} \tag{91}$$

³From now on the subscript A will not be used when it is clear that the relations are arithmetic.

The \langle_A condition stated for the density operators is equivalent to that given by equation (88) for string states.

Let $|\gamma_j, s_j, (\gamma_1)_j, t_j\rangle, |\gamma_k, s_k, (\gamma_1)_k, t_k\rangle$ and $|\gamma'_j, s'_j, (\gamma'_1)_j, t'_j\rangle, |\gamma'_k, s'_k, (\gamma'_1)_k, t'_k\rangle$ be four string states that satisfy the Cauchy conditions given in equation (61) for $P_{j,k,\ell}^{\{\psi_n\}}$ and $P_{j,k,\ell}^{\{\psi'_n\}}$. From equation (89) one sees that these states also satisfy the Cauchy conditions in equation (91) for $\ell - 1$. This gives the result that $Q'_{j,k,\ell}$ is related to $P_{j,k,\ell}^{\{\psi_n\}}$ and $P_{j,k,\ell}^{\{\psi'_n\}}$ by

$$Q'_{j,k,\ell-1} \geq P_{j,k,\ell}^{\psi} P_{j,k,\ell}^{\psi'} \quad (92)$$

Since the sequences $\{\psi_n\}$ and $\{\psi'_n\}$ are Cauchy, $P_{j,k,\ell}^{\{\psi_n\}} \rightarrow 1$ and $P_{j,k,\ell}^{\{\psi'_n\}} \rightarrow 1$ as $j, k \rightarrow \infty$ for any ℓ . It follows that $Q'_{j,k,\ell-1} \rightarrow 1$ as $j, k \rightarrow \infty$. It follows immediately from this that $\{\rho_{\psi_n +_A \psi'_n}\}$ is a Cauchy sequence.

Based on the results obtained so far, other properties of addition of Cauchy state sequences can be proved. These include commutativity, associativity, and any sequence $\{\psi_n\}$ which converges to 0 (or $\{\psi_n\} =_C \{|+, 0\rangle_n\}$, the constant 0 state sequence), is an additive identity, etc. The definition of subtraction, as the inverse of addition, is straightforward as

$$\begin{aligned} \{|\gamma_n, s_n, (\gamma_1)_n, t_n - \gamma'_n, s'_n, (\gamma'_1)_n, t'_n\rangle\} =_C \\ \{|\gamma_n, s_n, (\gamma_1)_n, t_n + \gamma''_n, s''_n, (\gamma''_1)_n, t''_n\rangle\}. \end{aligned} \quad (93)$$

Here $\gamma''_n \neq \gamma'_n$ and $(\gamma''_1)_n \neq (\gamma'_1)_n$.

4.3. Multiplication and Division

For multiplication it is useful to first consider sequences of real rational states and then extend the results to complex rational state sequences. The goal is to show that the product state sequence, $\{|\gamma_n s_n \times \gamma'_n s'_n\rangle\}$, of two Cauchy sequences, $\{|\gamma_n s_n\rangle\}$ and $\{|\gamma'_n s'_n\rangle\}$, is a Cauchy sequence. For all $j, k >$ some h ,

$$\begin{aligned} & |(|\gamma_j s_j \times \gamma'_j s'_j - \gamma_k s_k \times \gamma'_k s'_k|)| \\ & \leq_A |(|\gamma_j s_j -_A \gamma_k s_k|_A \times_A |\gamma'_j s'_j|_A +_A |\gamma_k s_k|_A \times_A |\gamma'_j s'_j -_A \gamma'_k s'_k|_A)| \\ & <_A |(|\gamma_j s_j -_A \gamma_k s_k|_A + |\gamma'_j s'_j -_A \gamma'_k s'_k|_A) \times_A (+, \ell_u)| \\ & <_A |+, -(\ell - \ell_u)|. \end{aligned} \quad (94)$$

Here $|+, \ell_u\rangle$ is an upper bound to $|(|\gamma'_j s'_j|_A)|$ and to $|(|\gamma_k s_k|_A)|$ for all j, k . Such a bound exists because $\{|\gamma_n s_n\rangle\}$ and $\{|\gamma'_n s'_n\rangle\}$ are Cauchy sequences. Since ℓ_u is fixed and is independent of ℓ , equation (94) shows that $|(|\gamma_j s_j \times \gamma'_j s'_j - \gamma_k s_k \times$

$\gamma'_k s'_k| \rangle \rightarrow |0\rangle$ as $j, k \rightarrow \infty$. This shows that the product sequence $\{|\gamma_n s_n \times \gamma'_n s'_n\rangle\}$ is Cauchy.

This result can be extended directly to products of Cauchy sequences of complex rational states. Let $\{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}$ and $\{|\gamma'_n s'_n, (\gamma'_1)_n t'_n\rangle\}$ be two Cauchy sequences. The product sequence, $\{\gamma_n s_n, (\gamma_1)_n t_n \times \gamma'_n s'_n, (\gamma'_1)_n t'_n\}$, with the real and imaginary parts separated, is given by $\{[(\gamma_n s_n \times \gamma'_n s'_n) + ((\gamma_1)_n t_n \times (\gamma'_1)_n t'_n)], [(\gamma_n s_n \times \gamma'_n s'_n) + (\gamma'_n s'_n \times (\gamma_1)_n t_n)]\}$. To save on notation let this sequence be represented by $\{|\eta_n v_n, \delta_n w_n\rangle\}$, where $\eta_n v_n$ is the real part and $\delta_n w_n$ is the imaginary part. To prove that the product sequence is Cauchy. it is sufficient to show that

$$\begin{aligned} |(\eta_j v_j - \eta_k v_k)| \rangle &<_A |+, -\ell'\rangle \\ |(\delta_j w_j - \delta_k w_k)| \rangle &<_A |+, -\ell'\rangle, \end{aligned} \tag{95}$$

then

$$|(\eta_j v_j + \delta_j w_j - \eta_k v_k - \delta_k w_k)| \rangle <_A |+, -(\ell' - 1)\rangle. \tag{96}$$

To prove equation (95) one has

$$\begin{aligned} |(\eta_j v_j - \eta_k v_k)| \rangle &\leq_A |(\gamma_j s_j \times \gamma'_j s'_j - \gamma_k s_k \times \gamma'_k s'_k)| \rangle \\ &+_A |((\gamma_1)_j t_j \times (\gamma'_1)_j t'_j - (\gamma_1)_k t_k \times (\gamma'_1)_k t'_k)| \rangle. \end{aligned} \tag{97}$$

Applying the argument used to verify equation (94) to each state gives

$$|(\eta_j v_j - \eta_k v_k)| \rangle <_A |+, -(\ell - 1 - \ell_u)\rangle. \tag{98}$$

Here $|+, \ell_u\rangle$ is an upper bound to $|(\gamma_j s_j)| \rangle, |(\gamma'_j s'_j)| \rangle, |((\gamma_1)_j t_j)| \rangle, |((\gamma'_1)_j t'_j)| \rangle$ for all j . Applying the same argument to $|(\delta_j w_j - \delta_k w_k)| \rangle$ and setting $\ell' = \ell - 1 - \ell_u$ finishes the proof.

For Cauchy sequences, $\{\psi_n\}$ and $\{\psi'_n\}$, of states that are linear superpositions of complex string states, the product states $\rho_{\psi_n \times \psi'_n}$ in the sequence of density operators are given by equation (90) with $\rho_{(\gamma,s,\gamma_1,t) \times (\gamma',s',\gamma'_1,t')}$ replacing $\rho_{(\gamma,s,\gamma_1,t) + (\gamma',s',\gamma'_1,t')}$ on the right hand side.

To prove that $\rho_{\psi_n \times \psi'_n}$ is Cauchy, it is convenient to first suppress the imaginary component and consider just the real string states. In this case the probability, $Q_{j,k,\ell}^\times$ that $|\rho_j - \rho_k|_N <_A \rho_{+,-\ell}$ is given by

$$\begin{aligned} Q_{j,k,\ell}^\times &= \sum_{\gamma_j s_j} \sum_{\gamma'_j s'_j} \sum_{\gamma_k s_k} \sum_{\gamma'_k s'_k} \\ &\times |\langle \gamma_j s_j | \psi_j \rangle|^2 |\langle \gamma'_j s'_j | \psi'_j \rangle|^2 |\langle \gamma_k s_k | \psi_k \rangle|^2 |\langle \gamma'_k s'_k | \psi'_k \rangle|^2 : \\ &\rho_{|\gamma_j s_j \times \gamma'_j s'_j - \gamma_k s_k \times \gamma'_k s'_k|} <_A \rho_{+,-\ell}. \end{aligned} \tag{99}$$

The condition on the density operators is equivalent to the condition $|(\gamma_j s_j \times \gamma'_j s'_j - \gamma_k s_k \times \gamma'_k s'_k)| \rangle <_A |+, -\ell\rangle$ for the pure states.

One would like to use the righthand inequality of equation (94) for the proof. However there is a problem in that the middle inequality does not hold because the states $|(|\gamma'_j s'_j|)\rangle$ and $|(|\gamma_k s_k|)\rangle$ have no arithmetic upper bound. However, because $\{\psi_n\}$ and $\{\psi'_n\}$ are Cauchy, there exists an ℓ_u such that the probabilities

$$\begin{aligned} P_{\ell_u}^{\psi_j} &= \sum_{\gamma'_j, s'_j} |\langle \gamma'_j, s'_j | \psi_j \rangle|^2 : |(|\gamma'_j s'_j|)\rangle \langle_A |+, \ell_u\rangle, \\ P_{\ell_u}^{\psi_k} &= \sum_{\gamma_k, s_k} |\langle \gamma_k, s_k | \psi_k \rangle|^2 : |(|\gamma_k s_k|)\rangle \langle_A |+, \ell_u\rangle \end{aligned} \quad (100)$$

converge to 1 as $j, k \rightarrow \infty$.

Let $P_{j,k,\ell',\ell_u}^{\{\psi_n\}}$ and $P_{j,k,\ell',\ell_u}^{\{\psi'_n\}}$ be defined by

$$\begin{aligned} P_{j,k,\ell',\ell_u}^{\{\psi_n\}} &= \sum_{\gamma_j s_j} \sum_{\gamma_k s_k} |\langle \gamma_j s_j | \psi_j \rangle|^2 |\langle \gamma_k s_k | \psi_k \rangle|^2 : \\ &\quad |(|\gamma_k s_k|)\rangle \langle_A |+, \ell_u\rangle \\ &\quad \text{and } |(|\gamma_j s_j - \gamma_k s_k|)\rangle \langle_A |+, -\ell'\rangle; \\ P_{j,k,\ell',\ell_u}^{\{\psi'_n\}} &= \sum_{\gamma'_j s'_j} \sum_{\gamma'_k s'_k} |\langle \gamma'_j s'_j | \psi'_j \rangle|^2 |\langle \gamma'_k s'_k | \psi'_k \rangle|^2 : \\ &\quad |(|\gamma'_k s'_k|)\rangle \langle_A |+, \ell_u\rangle \\ &\quad \text{and } |(|\gamma'_j s'_j - \gamma'_k s'_k|)\rangle \langle_A |+, -\ell'\rangle. \end{aligned} \quad (101)$$

The Cauchy conditions for $\{\psi_n\}$ and $\{\psi'_n\}$ give the result that for some ℓ_u ,

$$\begin{aligned} \lim_{j,k \rightarrow \infty} P_{j,k,\ell',\ell_u}^{\{\psi_n\}} &= 1, \\ \lim_{j,k \rightarrow \infty} P_{j,k,\ell',\ell_u}^{\{\psi'_n\}} &= 1 \end{aligned} \quad (102)$$

for each ℓ' .

Comparison of equation (101) with equation (99) and use of equation (94) gives the result that

$$Q_{j,k,\ell'-1-\ell_u}^\times \geq P_{j,k,\ell',\ell_u}^{\{\psi_n\}} P_{j,k,\ell',\ell_u}^{\{\psi'_n\}}. \quad (103)$$

One sees from equation (102) that $Q_{j,k,\ell'-1-\ell_u}^\times \rightarrow 1$ as $j, k \rightarrow \infty$. Since ℓ_u is fixed and ℓ' is any positive integer, it follows that $\rho_{\psi \times \psi'}$ is Cauchy.

Extension of this result to include multiplication of sequences of superposition states over complex string states is more involved. It will not be given as nothing new is added. Sums over γ, s , are expanded to sums over γ, s, γ_1, t and probabilities of the form $|\langle \gamma, s | \psi_n \rangle|^2$ become $|\langle \gamma, s, \gamma_1, t | \psi_n \rangle|^2$.

There are several well known properties that the definition of multiplication given here must satisfy. These include commutativity, distributivity over additivity, and the property that any sequence that is A equal to the constant

identity sequence, $\{|+, 1\rangle\}_c =_R \{c_{+,m}^\dagger a_{1,m}^\dagger |0\rangle\}_c$, is a multiplicative identity. The subscript c means that every element of the sequence is the same. Also if $\{\psi'_n\} =_R \{|+, 0\rangle\}_c =_R \{c_{+,m}^\dagger a_{0,m}^\dagger |0\rangle\}_c$, the constant zero sequence, then for any Cauchy $\{\psi_n\}$, $\{\rho_{\psi_n \times \psi'_n}\} =_C \{\rho_{|+,0}\}_c$.

Proofs of these properties for the Cauchy sequences follow the proofs of the Cauchy condition for the multiplicative and additive sequences. For each property there are conditions with associated probabilities of validity. One must show that the relevant probabilities approach 1 as the indices of the states in the sequences increase without bound. Alternatively one can prove the properties for Cauchy sequences $\{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}$ of complex string states and use the fact that any Cauchy sequence $\{\psi_n\}$ of superposition states is $=_C$ to some Cauchy sequence $\{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}$ to extend the properties to the $\{\psi_n\}$.

One property that should be discussed in more detail is the existence of a multiplicative inverse. Unlike the case for string states and their linear superpositions, Cauchy sequences of states have multiplicative inverses. To see this let $\{|\gamma_n s_n\rangle\}$ be a Cauchy sequence of real string states, where $\{|\gamma_n s_n\rangle\} \neq_R \{|+, 0\rangle\}_c$.

A sequence $\{|\gamma'_n s'_n\rangle\}$ inverse to $\{|\gamma_n s_n\rangle\}$ can be constructed by a diagonal process: For each ℓ let $|\gamma'_\ell s'_\ell\rangle$ be a state that satisfies

$$\begin{aligned} |(+, 1) - (+, -\ell)\rangle &\leq_A |\gamma_\ell s_\ell \times \gamma'_\ell s'_\ell\rangle \leq_A |(+, 1)\rangle \\ &\text{if } |\gamma_\ell s_\ell\rangle \neq_A |+, 0\rangle; \\ |\gamma'_\ell s'_\ell\rangle &=_A |+, 1\rangle \text{ if } |\gamma_\ell s_\ell\rangle =_A |+, 0\rangle. \end{aligned} \tag{104}$$

This definition is based on the previous description, equation (52), of the ℓ inverse for string states.

As noted before, for each ℓ and $|\gamma_\ell s_\ell\rangle \neq_A |+, 0\rangle$, there are many states $|\gamma'_\ell s'_\ell\rangle$ satisfying equation (104). Any one of them will suffice here. However a unique choice can be made by requiring that, of all states $|\gamma'' s''\rangle$ satisfying equation (104), $|\gamma'_\ell s'_\ell\rangle$ is the state, where the smallest j value for which $s'_\ell(j) = 1$ is larger than that for any other s'' . The example following equation (52) shows how this works.

It is clear from the definition that the product sequence $\{|\gamma_n s_n \times \gamma'_n s'_n\rangle\}$ is Cauchy and is R equal to the constant unit sequence $\{|+, 1\rangle\}_c$. The Cauchy property of $\{|\gamma'_n s'_n\rangle\}$ follows from that for $\{|\gamma_n s_n\rangle\}$.

The construction outlined above cannot be applied directly to find the inverse of a Cauchy sequence $\{\psi_n\}$ of linear superposition states as linear superposition states do not have ℓ inverses. In this case, one is interested in finding for any Cauchy $\{\psi_n\} \neq_C \{|+, 0\rangle\}_c$ a Cauchy state sequence $\{\psi'_n\}$ that satisfies

$$\{\psi_n \times \psi'_n\} =_R \{|+, 1\rangle\}_c. \tag{105}$$

The meaning of this equation can be expressed using equations (61). For each j, ℓ define the probability $P_{(\psi \times \psi')_{j, \ell}}$ by

$$P_{(\psi \times \psi')_{j, \ell}} = \sum_{\gamma, s} \sum_{\gamma', s'} |\langle \gamma, s | \psi_j \rangle|^2 |\langle \gamma', s' | \psi'_j \rangle|^2 : \quad (106)$$

$$|(|(\gamma s \times \gamma' s') - (+, 1)|) \rangle \langle_A |+, -\ell\rangle.$$

Equation (105) is satisfied with probability one if

$$\liminf_{\ell \rightarrow \infty} \limsup_{h \rightarrow \infty} \liminf_{j > h} P_{(\psi \times \psi')_{j, \ell}} = 1. \quad (107)$$

A necessary and sufficient condition that equation (107) is satisfied is that $P_{(\psi \times \psi')_{j, \ell}} \rightarrow 1$ as $j \rightarrow \infty$ for each ℓ .

The above gives the conditions to be satisfied by a Cauchy sequence $\{\psi'_n\}$ that is inverse to $\{\psi_n\}$ but it gives no clue as to how to construct such an inverse. One way to proceed is to use the substitution property of $=_R$ for the property of being an inverse. (Extension to other properties and operations is discussed in the next section.) Let $\{\psi_n\}$ be a Cauchy sequence of superpositions of real string states and $\{|\gamma_n s_n\rangle\}$ a Cauchy sequence, where $\{\psi_n\} =_R \{|\gamma_n s_n\rangle\}$. If $\{|\gamma'_n s'_n\rangle\}$ is a Cauchy sequence that is the inverse of $\{|\gamma_n s_n\rangle\}$, then $\{|\gamma'_n s'_n\rangle\}$, and any Cauchy sequence $\{\psi'_n\}$, where $\{\psi'_n\} =_R \{|\gamma'_n s'_n\rangle\}$, is the inverse of $\{\psi_n\}$.

Extension of the diagonal method to construct a Cauchy sequence that is the inverse of the complex Cauchy sequence $\{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}$ is more complex, but nothing new is required. From equation (52) one has, for each ℓ ,

$$|(+, 1) - (+, -\ell)\rangle \leq_A |(+1/\gamma, s, \gamma_1, t)_\ell \times_A (\gamma, s, \gamma_1, t)\rangle \leq_A |+, 1\rangle, \quad (108)$$

where

$$|(+, 1/\gamma, s, \gamma_1, t)_\ell \rangle =_A |(+s'')_\ell \times_A (\gamma s, \gamma'_1 t)\rangle. \quad (109)$$

Here $|(+s'')_\ell \rangle$ is given by equation (51) and $\gamma'_1 \neq \gamma_1$.

Proof of algebraic closure for the complex Cauchy sequences is limited to showing the existence of a Cauchy sequence whose square is N equal to the constant sequence $\{|-, 1\rangle\}_c$, (equivalent to a solution of $x^2 = -1$). This is trivial because the square of the constant sequence $\{|+, i1\rangle\}_c = \{d_{+,m}^\dagger b_{1,m}^\dagger |0\rangle\}_c$ equals $\{|-, 1\rangle\}_c$. Also the square of any Cauchy sequence $\{\psi_n\}$, that is C equal to $\{|+, i1\rangle\}_c$, is C equal to $\{|-, 1\rangle\}_c$.

4.4. Completeness

Another needed property of Cauchy sequences is that of completeness. This property is different from those discussed so far in that it deals with sets

or sequences of Cauchy sequences of states in \mathcal{F} . These have not been used so far in the development. To this end it is useful to use a double indexing $|\gamma_{n,m}s_{n,m}, (\gamma_1)_{n,m}t_{n,m}\rangle$ for complex string states. Here m is the sequence index and n labels the n -th component in the m -th sequence.

To save on notation let $|\gamma_{n,m}s_{n,m}, (\gamma_1)_{n,m}t_{n,m}\rangle$ be denoted by $|x_{n,m}\rangle$. Also let $\text{Re } |x_{n,m}\rangle = |\gamma_{n,m}s_{n,m}\rangle$ and $\text{Im } |x_{n,m}\rangle = |(\gamma_1)_{n,m}t_{n,m}\rangle$. From the indexing one sees that $\{\{|x_{n,m}\rangle\}_n\}_m$ denotes a double sequence of states, where $\{|x_{n,m}\rangle\}_n$ is the m -th sequence and $|x_{n,m}\rangle$ is the n -th state in the m -th sequence. For linear superposition states a similar representation of sequences of sequences is denoted by $\{\{\psi_{n,m}\}_n\}_m$.

The proof of completeness requires showing that every sequence of Cauchy sequences that is itself Cauchy, converges to a Cauchy sequence that is unique up to $=_C$. There are two Cauchy conditions to consider, one for each sequence in the sequence and one for the sequence of sequences. A sequence $\{\{|x_{n,m}\rangle\}_n\}_m$ of Cauchy sequences is itself Cauchy if:

$$\begin{aligned} &\text{For each } \ell \text{ there is an } h \text{ such that for all } j, k > h \\ &|\{\text{Re } |x_{n,j}\rangle\}_n -_R \{\text{Re } |x_{n,k}\rangle\}_n|_R <_R \{|+, -\ell\rangle\}_C \text{ and} \\ &|\{\text{Im } |x_{n,j}\rangle\}_n -_I \{\text{Im } |x_{n,k}\rangle\}_n|_R <_R \{|+, -\ell\rangle\}_C. \end{aligned} \tag{110}$$

Here $\{|+, -\ell\rangle\}_C$ is the constant sequence of states $|+, -\ell\rangle$, and $|\{\text{Re } |x_{n,j}\rangle\}_n -_R \{\text{Re } |x_{n,k}\rangle\}_n|_R$ and $|\{\text{Im } |x_{n,j}\rangle\}_n -_I \{\text{Im } |x_{n,k}\rangle\}_n|_R$ are the Cauchy state sequences that are the absolute values of the differences between the two real state Cauchy sequences $\{\text{Re } |x_{n,j}\rangle\}_n$ and $\{\text{Re } |x_{n,k}\rangle\}_n$ and the two imaginary state Cauchy sequences $\{\text{Im } |x_{n,j}\rangle\}_n$ and $\{\text{Im } |x_{n,k}\rangle\}_n$.

These two difference sequences are R equal to the two sequence of states that are absolute values of the differences of the real part and of the imaginary parts of the component states:⁴

$$\begin{aligned} &|\{\text{Re } |x_{n,j}\rangle\}_n -_R \{\text{Re } |x_{n,k}\rangle\}_n|_R =_R \{(|\text{Re } (x_{n,j}) -_R \text{Re } (x_{n,k})|_R)\}_n; \\ &|\{\text{Im } |x_{n,j}\rangle\}_n -_I \{\text{Im } |x_{n,k}\rangle\}_n|_R =_R \{(|\text{Im } (x_{n,j}) -_I \text{Im } (x_{n,k})|_R)\}_n. \end{aligned} \tag{111}$$

Here $\text{Re } (x_{n,j}) = \gamma_{n,j}s_{n,j}$ and $\text{Im } (x_{n,j}) = (\gamma_1)_{n,j}t_{n,j}$. Because of the substitution property of $=_R$, the righthand sequences in the above also satisfy the Cauchy conditions of equation (110). The subscript R on the absolute value of the difference of two imaginary state sequences accounts for the fact that absolute values of imaginary numbers are real.

⁴That is, the absolute value of the difference of two Cauchy sequences is R equal to the sequence whose elements are the absolute values of the difference of the individual sequence elements.

Convergence of a sequence $\{\{|x_{n,m}\rangle\}_m\}$ of Cauchy sequences to a sequence $\{|x'_n\rangle\}_n$ is expressed by

$$\begin{aligned} &\text{For each } \ell \text{ there is an } h \text{ such that for all } j > h \\ &|\{\text{Re } |x_{n,j}\rangle\}_n -_R \{\text{Re } |x'_n\rangle\}_n|_R <_R \{|+, -\ell\rangle\}_c \text{ and} \\ &|\{\text{Im } |x_{n,j}\rangle\}_n -_I \{\text{Im } |x'_n\rangle\}_n|_R <_R \{|+, -\ell\rangle\}_c. \end{aligned} \quad (112)$$

Here the double inequality for the real and imaginary parts can be replaced by a single inequality:

$$|\{|x_{n,j}\rangle\}_n -_C \{|x'_n\rangle\}_n|_R <_R \{|+, -\ell\rangle\}_c. \quad (113)$$

It should be emphasized that this definition of convergence is entirely different from those based on the usual properties of states and operators in \mathcal{F} . The latter include definitions based on norm convergence of state sequences or on definitions of statistical distance between states [31], [5], [17].

A complete proof of completeness will not be given here. Instead some salient aspects of a proof, which follows that in [13] for equivalence classes of Cauchy sequences of rational numbers, will be outlined.

To prove completeness one looks first at Cauchy sequences of constant sequences of string states $\{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_c$ for each n . The goal is to show that

$$\lim_{n \rightarrow \infty} \{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_c =_C \{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n. \quad (114)$$

To get this result, one notes that for any k ,

$$\begin{aligned} &\{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n -_C \{|\gamma_k s_k, (\gamma_1)_k t_k\rangle\}_c \\ &= _C \{|\gamma_n s_n, (\gamma_1)_n t_n -_C \gamma_k s_k, (\gamma_1)_k t_k\rangle\}_n. \end{aligned} \quad (115)$$

From the Cauchy conditions for the sequence of constant sequences,

$$\begin{aligned} &|\{\gamma_j s_j\}_c -_R \{\gamma_k s_k\}_c|_R <_R \{|+, -\ell\rangle\}_c \\ &|\{(\gamma_1)_j t_j\}_c -_I \{(\gamma_1)_k t_k\}_c|_R <_R \{|+, -\ell\rangle\}_c \\ &\text{for all } j, k, > h, \end{aligned} \quad (116)$$

one sees that both the real and imaginary components of the sequence on the right hand side of equation (115) are $<_A \{|+, -\ell\rangle\}$ for all $n, k > \text{some } h$. Equation (114) follows from this.

Similar arguments apply to more general Cauchy sequences of Cauchy sequences $\{\{|x_{n,f}\rangle\}_n\}_f$, where $|x_{j,f}\rangle \neq_A |x_{k,f}\rangle$ is possible. Here the goal is to show that

$$\lim_{f \rightarrow \infty} \{|x_{n,f}\rangle\}_n =_C \{|x_{n,n}\rangle\}_n, \quad (117)$$

where $\{|x_{n,n}\rangle\}_n$ is the desired limit $\{x'_n\}$ of equation (112).

One starts by noting that the Cauchy conditions for the sequence of sequences are

$$\begin{aligned} |\{\text{Re } |x_{n,f}\rangle\}_n - \{\text{Re } |x_{n,g}\rangle\}_n| <_R \{|+, -\ell\rangle\}_c \\ |\{\text{Im } |x_{n,f}\rangle\}_n - \{\text{Im } |x_{n,g}\rangle\}_n| <_R \{|+, -\ell\rangle\}_c \end{aligned} \tag{118}$$

for all $f, g > \text{some } h$.

Consider the sequence $|\{|x_{f,f}\rangle\}_c - \{|x_{n,f}\rangle\}_n| =_R \{|(|x_{f,f} - x_{n,f}|)\rangle\}_n$. Because each sequence, $\{|x_{n,f}\rangle\}_n$, is Cauchy,

$$\begin{aligned} \{|(|\text{Re } (x_{f,f} -_R x_{n,f})|_R)\rangle\}_n <_R \{|+, -\ell\rangle\}_c \\ \{|(|\text{Im } (x_{f,f} -_I x_{n,f})|_R)\rangle\}_n <_R \{|+, -\ell\rangle\}_c \end{aligned} \tag{119}$$

for $f > \text{some } h$.

From this it follows that $\{|x_{n,n}\rangle\}_n$ is Cauchy and, by equation (114),

$$\lim_{f \rightarrow \infty} \{|x_{f,f}\rangle\}_c =_C \{|x_{n,n}\rangle\}_n. \tag{120}$$

Equation (117) follows because, for sufficiently large f ,

$$\begin{aligned} &|\{|x_{n,n}\rangle\}_n -_C \{|x_{n,f}\rangle\}_n|_R \\ &<_R |\{|x_{n,n}\rangle\}_n -_C \{|x_{f,f}\rangle\}_c|_R +_R |\{|x_{f,f}\rangle\}_c -_C \{|x_{n,f}\rangle\}_n|_R. \end{aligned} \tag{121}$$

5. Representation of Real and Complex Numbers in Quantum Theory

5.1. Equivalence Classes of Cauchy Sequences

In the preceding it has been shown or made plausible that Cauchy sequences of states in \mathcal{F} have properties corresponding to those of real and complex numbers. If these properties can be lifted to equivalence classes of these sequences, then the sets of equivalence classes are real or complex numbers.

To this end let $R^{\{\psi_n\}}$ and $C^{\{\psi_n\}}$ denote the sets of equivalence classes of Cauchy sequences based on real and complex string state sequences. Two Cauchy sequences, $\{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n$ and $\{|\gamma'_n s'_n, (\gamma'_1)_n t'_n\rangle\}_n$, are equivalent or in the same equivalence class if and only if they are C equal:

$$\begin{aligned} \{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n &\equiv \{|\gamma'_n s'_n, (\gamma'_1)_n t'_n\rangle\}_n \\ &\leftrightarrow \{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n =_C \{|\gamma'_n s'_n, (\gamma'_1)_n t'_n\rangle\}_n. \end{aligned} \tag{122}$$

This definition extends to sequences of other types of states. Thus the Cauchy sequence $\{\psi_n\} \equiv \{\psi'_n\}_n$ or $\{\psi_n\} \equiv \{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n$ if and only if $\{\psi_n\} =_C \{\psi'_n\}_n$ or $\{\psi_n\} =_C \{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n$. Similar relations hold for Cauchy sequences of real and imaginary string states and their superpositions. Thus

$$\{|\gamma_n s_n\rangle\}_n \equiv \{|\gamma'_n s'_n\rangle\}_n \leftrightarrow \{|\gamma_n s_n\rangle\}_n =_R \{|\gamma'_n s'_n\rangle\}_n, \quad (123)$$

with a similar relation for $=_I$.

As was noted in the introduction, each equivalence class in $C^{\{\psi_n\}}$ and $R^{\{\psi_n\}}$ is larger than the corresponding class of classical Cauchy sequences of rational numbers. This occurs because Cauchy sequences of states $\{\psi_n\}$, where each ψ_n is a linear superposition of complex string states have no classical counterparts. However expansion of the notion of convergent sequences from classical rational string numbers to quantum states does not create more equivalence classes. This is an immediate consequence of the proof, equations (77)-(81), that for any Cauchy sequence $\{\psi_n\}$ of superposition states, there is a Cauchy sequence $\{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n$ of string states, where $\{\psi_n\} =_C \{|\gamma_n s_n, (\gamma_1)_n t_n\rangle\}_n$. A similar relation holds for Cauchy sequences of superpositions of real string states and of imaginary string states.

This result is satisfying because it is consistent with the requirement that $C^{\{\psi_n\}}$ and $R^{\{\psi_n\}}$ are isomorphic to the ground sets R and C that are the base for the Fock space \mathcal{F} . An isomorphism, M , from $C^{\{\psi_n\}}$ to C (and $R^{\{\psi_n\}}$ to R) can be easily constructed by use of the operator \tilde{N} defined in equation (5). This will not be done here as it is straightforward and adds nothing to the development.

It is often useful to let individual Cauchy sequences stand for equivalence classes. This practice is often done and does no harm here. It also makes various aspects easier to deal with as one can work with individual Cauchy sequences rather than with equivalence classes.

The proof that $R^{\{\psi_n\}}$ and $C^{\{\psi_n\}}$ are real and complex numbers requires that one lift the basic properties $=_X, \leq_X$ and operations $\tilde{+}_X, \tilde{-}_X$, and $\tilde{\times}_X, \tilde{\div}_X$ on Cauchy sequences up to the equivalence classes. To do this it is essential that the truth value of each property, and the result of each operation, is preserved under $=_R$ (for real states), $=_I$ (for imaginary states), and $=_C$ (for complex states) based substitution. For instance, if $\mathcal{P}(\{|x_n\rangle\}_n, \{|y_n\rangle\}_n)$ is true for $\{|x_n\rangle\}_n, \{|y_n\rangle\}_n$ and $\{\psi_n\} =_C \{|x_n\rangle\}_n$ and $\{\psi'_n\} =_C \{|y_n\rangle\}_n$, then $\mathcal{P}(\{\psi_n\}, \{\psi'_n\})$ should be true. If $\tilde{O}(\{|x_n\rangle\}_n, \{|y_n\rangle\}_n)$ represents an outcome sequence for the operation of \tilde{O} on the sequence pair $\{|x_n\rangle\}_n, \{|y_n\rangle\}_n$, then $\tilde{O}(\{|x_n\rangle\}_n, \{|y_n\rangle\}_n) =_C \tilde{O}(\{\psi_n\}, \{\psi'_n\})$ should be true.

An example of this invariance follows from the global definitions of the basic

operations and relations. If $\{|\gamma_n, s_n; \gamma'_n, t_n\rangle\}$ is a sequence of states at location (m, h) and $\tilde{T}\{|\gamma_n, s_n; \gamma'_n, t_n\rangle\}$ is a translation of the sequence to location (m, h') , then

$$\tilde{T}\{|\gamma_n, s_n; \gamma'_n, t_n\rangle\} =_X \{|\gamma_n, s_n; \gamma'_n, t_n\rangle\}. \tag{124}$$

Also the two sequences have exactly the same properties relative to $<_X$ and the basic operations.

This raises the question of determining which properties are preserved for $=_R$, $=_I$, or $=_C$ based substitution. At least one would expect that all mathematical properties and operations described and used in the theory of complex analysis would be included. One approach is to use the standard construction of terms and formulas in languages as described in mathematical logic, see [23], [22]. Here the set of operations would be the smallest set that contains $\tilde{+}$, $\tilde{\times}$, and their inverses, and is closed under all finite combinations of these operations and the taking of limits. The set of properties would be the smallest set that contains the basic relations $=_R$, $<_R$ (or $=_I$, $<_I$ or $=_C$, $<_C$) and operations, and is closed under the use of logical connectives and existential quantifiers.

Expansion of the above definitions to include more operations and properties may lead one to very difficult questions. These include defining the difference between physical and mathematical properties, and determining exactly what would be meant by requiring that “all” mathematical properties and operations should be included. These questions will not be dealt with here as they are outside the scope of this work.

6. Discussion

The goal of this paper has been to show that the sets, $R^{\{\psi_n\}}$, $C^{\{\psi_n\}}$, of equivalence classes of Cauchy sequences of real and complex string states and their superpositions satisfy the required properties of real and complex numbers. This was done with no reference to the properties of numbers in the underlying R and C that are the base of \mathcal{F} . This includes the definition of the Cauchy condition, and definitions and properties of the relations $=_R$, $=_I$, $=_C$, $<_R$, $<_I$, and $<_C$, and the addition, subtraction, multiplication, and division operations. For sequences of superposition states $\{\psi_n\}$ this is not possible because the coefficients of the ψ_n are complex numbers in C .

It was also noted that the equivalence classes of Cauchy sequences in $C^{\{\psi_n\}}$ and $R^{\{\psi_n\}}$ are larger than the corresponding equivalence classes in any classical R and C . This follows from the existence of Cauchy sequences of quantum states that have no classical equivalences. These states can be somewhat coun-

terintuitive. For example, following equations (66) and (67), let s be a 0 – 1 function with domain $[0, -\infty]$ and t be the constant 0 function on the same domain. Define ψ_n by

$$\psi_n = c_{+,m,h}^\dagger (a^\dagger)^{s[0,-n+1]} d_{+,m,h}^\dagger (b^\dagger)^{t[0,-n+1]} \times \left(\frac{1}{\sqrt{2}} (a_{1,-n}^\dagger b_{0,-n}^\dagger + a_{0,-n}^\dagger b_{1,n}^\dagger) |0\rangle \right), \quad (125)$$

where

$$\begin{aligned} (a^\dagger)^{s[0,-n+1]} &= a_{s(0),0}^\dagger a_{s(-1),-1}^\dagger \cdots a_{s(-n+1),-n+1}^\dagger \\ (b^\dagger)^{t[0,-n+1]} &= b_{t(0),0}^\dagger b_{t(-1),-1}^\dagger \cdots b_{t(-n+1),-n+1}^\dagger. \end{aligned} \quad (126)$$

Each state in this sequence is an entangled state of real and imaginary components. However, as a Cauchy sequence, $\{\psi_n\}$ is a real number with no imaginary component.

This, and other examples, show that the quantum equivalence classes are larger than the classical ones, however, no new classes are created. It follows that $R^{\{\psi_n\}}$ and $C^{\{\psi_n\}}$ are isomorphic to R and C . This shows that $C^{\{\psi_n\}}$ and $R^{\{\psi_n\}}$ are in every way just as good and valid a representation of real and complex numbers as are the original R and C over which the Fock space \mathcal{F} was constructed.

In this case there is no reason why one could not use $C^{\{\psi_n\}}$ and $R^{\{\psi_n\}}$ to be the base of physical theories such as QED, QCD, special and general relativity, string theory, etc. Also $(R^{\{\psi_n\}})^4$ is just as good a representation of a space time manifold as is R^4 . This raises all sorts of interesting open questions concerning the relations between states and properties of the physical systems described by the states in \mathcal{F} , and those described by the theories based on $R^{\{\psi_n\}}$ and $C^{\{\psi_n\}}$.

One interesting question is based on the observation that a Fock space \mathcal{F} of states of finite qubit strings, equipped with an associated Hamiltonian H and a discrete space time lattice \mathcal{L} of points in R^4 , can be used to describe the lattice quantum dynamics of the qubit strings. As noted \mathcal{F} is based on R and C . Let $\mathcal{F}^{\{\psi_n\}}$ be another Fock space based on the real and complex numbers in $R^{\{\psi_n\}}$ and $C^{\{\psi_n\}}$ which consist of Cauchy sequences of the string states or their linear superpositions in \mathcal{F} .

Now consider the situation, where $\mathcal{F}^{\{\psi_n\}}$ with some Hamiltonian H' and corresponding space time lattice $\mathcal{L}^{\{\psi_n\}}$ of points in $(R^{\{\psi_n\}})^4$ describes the quantum dynamics of the *same* physical systems whose quantum dynamics is described by \mathcal{F} , H , and \mathcal{L} . This is an interesting situation since the states in \mathcal{F} of the physical systems are states in the Cauchy sequences in $C^{\{\psi_n\}}$ on which

$\mathcal{F}^{\{\psi_n\}}$ and the space time $(R^{\{\psi_n\}})^4$ are based. Is this situation even possible? Do there exist physical systems whose quantum dynamics is described by both \mathcal{F}, \mathcal{L} in R^4 and H , and by $\mathcal{F}^{\{\psi_n\}}, \mathcal{L}^{\{\psi_n\}}$ in $(R^{\{\psi_n\}})^4$, and H' ? It is hoped to investigate this and related questions in future work.

This all suggests that this process can be iterated, leading to a hierarchy of Fock spaces over complex numbers as equivalence classes of Cauchy sequences of states that are based on the previous space and complex numbers in the iteration, see [4]. The existence of such a hierarchy suggests that it may be of interest to study the relationship between two neighboring spaces and states in the iteration.

Of particular interest is the relation between the original R and C , and $R^{\{\psi_n\}}$ and $C^{\{\psi_n\}}$. Here, as in other physical theories R and C is taken for granted, or as given, without thought as to what the structure is, if any, of the numbers in R and C . If they are equivalence classes of Cauchy sequences of complex rational numbers, then what are the rational numbers? Pursuing this line leads back to the natural numbers or nonnegative integers. Depending on ones point of view they can either be accepted as primary and unanalyzable, or one can ask what they are and how they relate to physical systems and quantities.⁵ This is part of the more general question of the foundational relation between mathematics and physics, see [1], [26].

It should be noted that for any function or property on R or C , there is a corresponding function or property on $R^{\{\psi_n\}}$ or $C^{\{\psi_n\}}$. For example, corresponding to a metric on R one has a metric on $R^{\{\psi_n\}}$ defined by

$$\begin{aligned}
 D(\{|\gamma_n s_n\rangle\}_n, \{|\gamma'_n s'_n\rangle\}_n) &= | \{|\gamma_n s_n\rangle\}_n -_R \{|\gamma'_n s'_n\rangle\}_n |_R \\
 &=_R \{ |(|\gamma_n s_n -_A \gamma'_n s'_n|_R) \rangle \}_n. \quad (127)
 \end{aligned}$$

The right hand term in the above is a Cauchy sequence whose elements are the states that are the absolute values of the differences between $|\gamma_n s_n\rangle$ and $|\gamma'_n s'_n\rangle$ for $n = 1, 2, \dots$. A similar map can be given for $C^{\{\psi_n\}}$ by replacing $|\gamma_n s_n\rangle$ with $|\gamma_n s_n, (\gamma_1)_n t_n\rangle$, etc. This shows that $R^{\{\psi_n\}}$ and $C^{\{\psi_n\}}$ (and R and C) are metric spaces, see [18], [19].

It is clear that there is much to do. Future work includes more examination of the iterative hierarchy noted above. Also the treatment should be expanded to include qukits for any base $k \geq 2$, not just $k = 2$.

⁵In the description given here natural number states have the form $|\alpha s\rangle = c_{+,m,h}^\dagger (a^\dagger)^s |0\rangle$, where $s(j, h) = 1 \rightarrow j \geq m$.

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