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FENCHEL TRANSFORMS OF A CONVEX FUNCTIONAL

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Abstract: There are many applications that involve the minimization of a convex, linear growth function of a measure. For example, image restoration models, Plateau's problem and deformation of a thin plate (the plasticity problem) involve minimizing such functions. In order to understand the theory of these problems, we must understand how to give meaning $F(\mu)$, where μ is a vector-valued measure and F is a convex function with linear growth.

In this paper, we use the space of continuous, bounded functions to define the Fenchel transform of a function of measure. We then show that under this definition, the double Fenchel transform coincides with the definition given by Anzellotti and Giaquinta (1982) and used throughout the literature. The lower semi-continuity of the functional $\int F(\mu)$ is a direct result of properties of the Fenchel transform.

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1. Introduction

There are many application problems involving variational integrals of the form

$$\min \int_\Omega F(x,u,Du), \tag{1}$$

for open $\Omega \subset \mathbb{R}^n$, where $u = (u^1(x),\dots,u^N(x))$ is a vector-valued function and $F(x,u,p)$ is convex in p. For example, such minimization problems are used in image denoising and edge detection, modeling the deformation of a thin plate and determining a surface of minimal area with prescribed boundary conditions. In fact, Hilbert's 19-th and 20-th problems deal with these "regular problems in the calculus of variations," for $n = 2$ and $N = 1$; see Giaquinta [10] for a comprehensive overview of these types of problems as well as extensive references. In 1912, Bernstein [4] used the calculus of variations method to establish existence and regularity results for the 2-dimensional real-valued Dirichlet problem. Serrin [18] applied similar methods to extend these results for n dimensions. We wish to explore the minimization problem in the more general case, where $u \in BV(\Omega,\mathbb{R}^N)$; in this case Du is a Radon measure and we need give meaning to the variational integral (1).

For example, Giusti [12] considers a minimal surface area problem (Plateau's problem), where

$$F(p) := \sqrt{1+|p|^2}. \tag{2}$$

In this case,

$$\int_\Omega F(Du)\,dx = \int_\Omega \sqrt{1+|Du|^2}\,dx$$

is the area of the surface of the graph of u.

Anzellotti and Giaquinta [1, 2], Hardt and Kinderlehrer [13] and Zhou [20] study anti-planar shear of a thin plate (the plasticity problem). Here one seeks solutions to the minimization problem (1), where

$$F(p) := \begin{cases} \frac{1}{2\beta}|p|^2 & \text{if } |p| \le \beta, \\ |p| - \frac{\beta}{2} & \text{if } |p| > \beta, \end{cases} \tag{3}$$

for some threshold $\beta > 0$. In this case, $u : \Omega \to \mathbb{R}$ is the displacement of the plate.

Chambolle and Lions [5] proposed to recover an image, u, from an observed, noisy image, I, where the two are related by $I = u + \textit{noise}$ by the model

$$\min \int_\Omega F(Du) + \int_\Omega \frac{\lambda}{2}(u-I)^2\,dx,$$

for $F(p)$ in (3). The diffusion from this minimization model is strictly perpendicular to the gradient when $|Du| > \beta$, where edges are likely to be present, and isotropic when $|Du| \leq \beta$. Thus the model preserves edges and eliminates noise.

Additionally, the first two authors and Levine [6] considered a function of $q(x)$ growth, for $q(x) \geq 1$, which is used as a model for image denoising, enhancement and restoration. In this paper they proposed a new model for image restoration. The proposed model incorporates the strengths of the various types of diffusion arising from the minimization problem

$$\min \int_\Omega |Du|^p + \frac{\lambda}{2}(u - I)^2,$$

for $1 \leq p \leq 2$. In particular, they considered the minimization problem (1) with

$$F(x,p) := \begin{cases} \frac{1}{q(x)}\,|p|^{q(x)} & \text{for } |p| \leq \beta, \\ |p| - \frac{\beta q(x) - \beta^{q(x)}}{q(x)} & \text{for } |p| > \beta, \end{cases} \tag{4}$$

where $\beta > 0$ is fixed and $1 < \alpha \leq q(x) \leq 2$. One may choose

$$q(x) = 1 + \frac{1}{1 + k\,|\nabla G_\sigma * I(x)|^2},$$

where $G_\sigma(x) = \frac{1}{\sigma}\exp\big(-|x|^2/(4\sigma^2)\big)$ is the Gaussian filter and $k, \sigma > 0$ are fixed. In this case, the model utilizes a total variation approach when the gradient is large (thus preserving edges) and L^2 smoothing when the gradient is small (thus removing noise). Furthermore, it employs anisotropic diffusion ($1 < p < 2$) in regions which may be piecewise smooth or in which the difference between noise and edges is difficult to distinguish.

Models based on minimization of a convex, linear growth functional of measures have shown promising results in numerical implentation. The developement of PDE methods in image analysis is dependent upon answering fundemental mathematical questions. In particular, the meaning of such a functional of a measure and its first variation are not trivial to establish or understand. While a definition for a convex, linear growth functional has been given by Anzellotti and Giaquinta [1], and Giaquinta, Modica and Souček [11], we explore the motivation for this formulation and its relation to previous results in convex analysis.

The functional

$$\int_\Omega F(Du) \tag{5}$$

is well defined on the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^N)$. However, it has been shown that the minimization problem for plasticity (F as in (3)), [1, 13, 20], and for the image processing problem (F as in (4)), [6], has solutions $u \in BV(\Omega)$. This is consistent with our intuition, as both models should allow for discontinuities (where there is shear or edges, respectively). Since the solution is BV, its derivative, Du, is a Radon measure. Thus we need to understand the meaning of (5) in such a case. Indeed, the study of existence and partial regularity of solutions to these problems depends on how one defines

$$\int_\Omega F(x, u, Du),$$

for $u \in BV(\Omega)$.

We explore the meaning of $\int_\Omega F(m)$, for a bounded $\mathbb{R}^n$-valued measure. To such ends, Giaquinta, Modica and Souček [11] defined a function

$$\bar{F}(x, p_0, p) := F\left(x, \frac{p}{p_0}\right) p_0,$$

where $x \in \Omega$, $p_0 > 0$ and $p \in \mathbb{R}^N$, and remarked that $\bar{F}$ is continuous on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^N$, convex in (p_0, p) and homogeneous of degree 1 in (p_0, p). They then proved that

$$\lim_{p_0 \to 0^+} \bar{F}(x, p_0, p)$$

exists under appropriate conditions (given below). Choose a positive Radon measure μ so that the total variation $|Du|$ and the Lebesgue measure $\mathcal{L}^n$ are absolutely continuous with respect to μ. Denote the Radon-Nikodym derivatives of $\mathcal{L}^n$ and the vector-valued measure Du with respect to μ by

$$\frac{d\mathcal{L}^n}{d\mu} \quad \text{and} \quad \frac{dDu}{d\mu},$$

respectively. For every $u \in BV(\Omega)$, they defined

$$\int_\Omega F(Du) := \int_\Omega \bar{F}\left(x, \frac{d\mathcal{L}^n}{d\mu}, \frac{dDu}{d\mu}\right) d\mu.$$

By the homogeneity of $\bar{F}$, it follows that this definition is independent of our choice of μ. The authors then applied a result of Reshetnyak [17] to this definition to establish the lower semi-continuity of the integral.

Anzellotti and Giaquinta [1] applied this definition to the plasticity problem with F as in (3) and for $\beta = 1$. Here, the authors defined a function on $\mathbb{R} \times \mathbb{R}^n$ by

$$\bar{F}(t,p) := \begin{cases} F\left(\frac{p}{t}\right) t & t > 0, \\ \lim_{t \searrow 0} F\left(\frac{p}{t}\right) t & t = 0. \end{cases}$$

For any $\mathbb{R}^n$-valued measure m, consider the $\mathbb{R}^{n+1}$-valued measure $\alpha = (\alpha_0, m)$, where $\alpha_0 = \mathcal{L}^n$ is the Lebesgue measure in Ω. The authors then defined

$$\int_\Omega F(m) := \int_\Omega \bar{F}\left(\frac{d\alpha_0}{d\,|\alpha|}, \frac{dm}{d\,|\alpha|}\right) d\,|\alpha|\,, \tag{6}$$

where $|\alpha|$ is the total variation of α, and $\frac{d\alpha_0}{d|\alpha|}$ and $\frac{dm}{d|\alpha|}$ are the Radon-Nikodym derivatives. Anzellotti and Giaquinta [2, Lemma 1.1] used (6) to show that

$$\int_\Omega F(Du) = \int_\Omega F(\nabla u)\,dx + \int_\Omega |D^s u|\,, \tag{7}$$

for $u \in BV(\Omega)$. Here Du is decomposed into its absolutely continuous and singular parts with respect to Lebesgue measure, i.e.

$$Du = \nabla u\,dx + D^s u.$$

To see that (7) follows from (6), we decompose Ω with respect to the measures $\nabla u\,dx$ and $D^s u$. That is to say, since the measures are mutually singular, there exists a set $A \subset \Omega$ on which ∇u is not zero and $D^s u$ is identically zero. Thus on A, $Du = \nabla u\,dx$ and on the complement, $Du = D^s u$. Splitting the integral in (6) into the sum of integrals over A and $\Omega \setminus A$ gives (7).

In a more general setting, Anzellotti and Giaquinta [3] provided a unified approach to the partial regularity to solutions of the minimization problem (1) for a general convex function F with growth $m \geq 1$; that is to say, there are positive constants α and β so that

$$\alpha\,|p|^m \leq F(x,p) \leq \beta(1 + |p|^m). \tag{8}$$

In this case, the authors defined

$$\int_\Omega F(Du) = \int_\Omega F(\nabla u)\,dx + \int_\Omega F^\infty\left(\frac{D^s u}{|D^s u|}\right) |D^s u|\,, \tag{9}$$

where

$$F^\infty(p) := \lim_{t \to \infty} \frac{1}{t} F(tp).$$

Notice that this agrees with the definition taken above for the plasticity problem, as $F^\infty(p) = |p|$ in that case. In fact, one may use (6) and the techniques of [2] to establish (9) in the general case. The definition (9) is used by Demengel, Hardt, Kinderlehrer, Temam, Tonegawa and Zhou anong others throughout the literature in the study of existence and partial regularity of solutions to the minimization problem (3); see [8], [13], [14], [15], [16] and [20] for example.

While the definition (9) is sufficient for the study of functionals of this type, there are still unsettled questions. In particular, what motivates this definition, how does it relate to previous results and what is the relation to convex analysis? The purpose of this paper is to take a very different route to the definition of $F(m)$, where F is a convex function on $\mathbb{R}^n$ and m is a vector-valued measure. Our approach is to use Fenchel transforms to define $\int_\Omega F(m)$ in (9). Briefly, given a convex function F on $\mathbb{R}^n$, consider the convex functional on $L^1(\Omega, \mathbb{R}^n)$ defined by

$$f \mapsto \int_\Omega F(f)\,dx.$$

We wish to extend this functional to the space of bounded vector-valued measures, $\mathcal{M}$. Additionally, we would like this extension to be lower semi-continuous on $\mathcal{M}$ in the topology induced by the space of bounded, $C(\Omega, \mathbb{R}^n)$ functions, denoted by C_B. The Fenchel transform (or conjugate) of F on C_B is defined by

$$F^*(\phi) := \sup_{f \in L^1} \int_\Omega (f \cdot \phi - F(f))\,dx.$$

Since bounded vector-valued Radon measures are linear maps on C_B we can naturally repeat this procedure on the space of bounded vector-valued measures $\mathcal{M}$, and define

$$F^{**}(m) := \sup_{\phi \in C_B} \int_\Omega \phi\,dm - F^*(\phi).$$

Since $L^1(\Omega, \mathbb{R}^n)$ is a subspace of $\mathcal{M}$, F^{**} is an extenstion of F. In this paper we prove that the double Fenchel transform, $F^{**}(m)$ thus defined, is indeed given by the formula

$$F^{**}(m) = \int_\Omega F(m^a)\,dx + \int_\Omega F^\infty\left(\frac{dm^s}{d\,|m^s|}\right)\,d\,|m^s|\,.$$

This result justifies and shows the "naturalness" of the definition of $\int_\Omega F(m)$ in (9) in the context of convex analysis. One immediate consequence is the lower semi-continuity of this functional. Additionally, this technique may be applied to define convex functionals of objects more general than measures (e.g., certain types of operators). Along the way we also find very interesting properties of F^{**}. Just to name one:

$$F^{**}(m+n) = F^{**}(m) + F^{**}(n),$$

whenever the vector measures m and n are mutually singular.

2. Notation and Preliminaries

Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $f \in L^1(\Omega, \mathbb{R}^n)$ and let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous, non-negative, convex function with $F(0) = 0$, satisfying the linear growth condition

$$\alpha\,|p| - \gamma \le F(p) \le \beta\,|p| + \gamma,$$

for some $\alpha, \beta, \gamma > 0$. Denote

$$\mathcal{M} = \{\text{bounded } \mathbb{R}^n\text{-valued measures on } \Omega\}.$$

For any $m \in \mathcal{M}$, let $|m|$ denote the total variation. We may decompose m into its absolutely continous and singular parts with respect to Lebegues measure:

$$m = m^a + m^s.$$

Denote

$$C_B = \{\phi \in C(\Omega, \mathbb{R}^n) | \phi \text{ is bounded}\}.$$

Endow L^1 with the coarsest topology such that the mapping

$$L_\phi(f) = \int_\Omega f \cdot \phi \, dx$$

is continuous for all $\phi \in C_B$. Note that this topology separates points in L^1 as $\int f \cdot \phi = \int g \cdot \phi$, for all $\phi \in C_B$ if and only if $f = g$ $\mathcal{L}^n$-a.e. With this topology, $L^1(\Omega, \mathbb{R}^n)$ becomes a locally convex, Hausdorff, topological vector space. $\big(L^1(\Omega, \mathbb{R}^n), C_B\big)$ is a dual pair; see Conway [7] for the general theory of topological vector spaces.

Suppose F satisfies the conditions above. If $f \in L^1(\Omega, \mathbb{R}^n)$, the linear growth condition on F gives

$$\int_\Omega (\alpha\,|f| - \gamma)\; dx \le \int_\Omega F(f)\, dx \le \int_\Omega (\beta\,|f| + \gamma)\; dx;$$

i.e., $F(f) \in L^1$. The Fenchel transform F^* of F is defined on C_B by

$$F^*(\phi) := \sup_{f \in L^1} \int_\Omega f \cdot \phi \, dx - \int_\Omega F(f). \tag{10}$$

Observe that for fixed $f \in L^1$, the mapping L_ϕ is a continuous, affine map. Hence $F^*(\phi)$ is convex and lower semi-continuous on C_B. Thus the desired lower semi-continuity follows automatically from the Fenchel transform approach. See van Tiel [19] and Ekeland and Temam [9] for more on the general theory of Fenchel transforms.

We observe that

Proposition 1. *Let* $\phi \in C_B$. *If* $\|\phi\|_\infty > 1$, *then* $F^*(\phi) = \infty$.

Proof. Suppose that $\|\phi\|_\infty > 1$ and let $f = |\phi|^{k-2}\phi$, for $k \geq 2$. Since Ω is bounded, we have $f \in L^1$. Then

$$\begin{aligned} \int (f \cdot \phi - F(f)) \, dx &= \int \left(|\phi|^k - F(\phi \, |\phi|^{k-2}) \right) \, dx \\ &\geq \int |\phi|^k - (\beta \, |\phi|^{k-1} + \gamma) \, dx \to \infty \, , \quad \square \end{aligned}$$

as $k \to \infty$.

For $m \in \mathcal{M}$, define

$$F^{**}(m) := \sup_{\phi \in C_B} \int_\Omega \phi \, dm - F^*(\phi). \tag{11}$$

By Proposition 1, we may define

$$F^{**}(m) = \sup_{\substack{\phi \in C_B \\ |\phi| \leq 1}} \int_\Omega \phi \, dm - F^*(\phi). \tag{12}$$

If we endow C_B with the uniform norm, then $\mathcal{M} \subset C_B^*$. The coarsest topology on $\mathcal{M}$ so that the mapping $m \mapsto \int \phi \, dm$ is continuous is called the weak topology on $\mathcal{M}$. By (11), $F^{**}(m)$ is lower semi-continuous and convex on $\mathcal{M}$.

For each $f \in L^1$, $f \, dx \in \mathcal{M}$. It is easy to see that

Proposition 2. *Let* $f \in L^1(\Omega, \mathbb{R}^n)$. *Then*

$$F^{**}(f \, dx) \leq \int_\Omega F(f) \, dx.$$

Proof. By (12),

$$F^{**}(f\,dx) = \sup_{\substack{\phi \in C_B \\ |\phi| \leq 1}} \int_\Omega \phi \cdot f\,dx - F^*(\phi),$$

where $F^*(\phi) = \sup_{f \in L^1} \int (f \cdot \phi - F(f))\,dx$. For any $f \in L^1$ and every $\phi \in C_B$, we have

$$F^*(\phi) \geq \int (f \cdot \phi - F(f))\,dx,$$

so

$$\int F(f)\,dx \geq \int f \cdot \phi\,dx - F^*(\phi).$$

Thus taking the supremum, we have

$$\int F(f)\,dx \geq \sup_{\substack{\phi \in C_B \\ |\phi| \leq 1}} \int f \cdot \phi\,dx - F^*(\phi) = F^{**}(f\,dx). \qquad \square$$

Using the duality we started with, we can show that if F is convex and lower semi-continuous, then

$$F^{**}(f\,dx) = \int_\Omega F(f)\,dx,$$

for all $f \in L^1$.

Our goal is to derive an explicit formula for the double Fenchel transform F^{**}. This we hope will justify the definition given in (9).

3. Main Result

We wish to show

Theorem. *Let $m \in \mathcal{M}$ be a vector-valued measure. Decompose m into its absolutely continuous part, m^a, and its singular part, m^s, with respect to Lebesgue measure. Then*

$$F^{**}(m) = \int_\Omega F(m^a)\,dx + \int_\Omega F^\infty\left(\frac{dm^s}{d\,|m^s|}\right) d\,|m^s|\,,$$

where

$$F^\infty(p) = \lim_{t\to\infty} \frac{1}{t} F(tp).$$

Note that since F is convex and $F(0) = 0$, the limit in the definition of F^∞ exists.

Proposition 3. *Let $m, n \in \mathcal{M}$ be mutually singular, denoted $m \perp n$. Then*

$$F^{**}(m+n) = F^{**}(m) + F^{**}(n).$$

We present the proof as a series of claims:

Claim 1. *$F^{**}(m)$ is lower semi-continuous in the sense that if $m_k \rightharpoonup m$ weakly in $\mathcal{M}$ (i.e., $\int_\Omega \phi\, dm_k \to \int_\Omega \phi\, dm$, for all $\phi \in C_B$), then $F^{**}(m) \le \liminf F^{**}(m_k)$.*

Proof. For fixed $\phi \in C_B$, the map $m \mapsto \int_\Omega \phi\, dm - F^*(\phi)$ is continuous. Since $F^{**}(m)$ is defined to be the supremum of a family of such maps (for $|\phi| \le 1$), we conclude that F^{**} is lower semi-continuous. □

Claim 2. *For any set $K \subset \Omega$, we have*

$$F^{**}(m1_K) \le F^{**}(m) + |m1_{K^c}|(\Omega) = F^{**}(m) + |m|(K^c),$$

where we denote $m1_K = m|_K$.

Proof. Indeed, let $\langle m, \phi\rangle$ denote the pairing $\int_\Omega \phi\, dm$. We have

$$\langle m, \phi\rangle = \langle m1_K + m1_{K^c}, \phi\rangle = \langle m1_K, \phi\rangle + \langle m1_{K^c}, \phi\rangle.$$

Therefore, for $\phi \in C_B$ with $|\phi| \le 1$, we have

$$\begin{aligned}\langle m1_K, \phi\rangle - F^*(\phi)& \\ &= \langle m, \phi\rangle - F^*(\phi) - \langle m1_{K^c}, \phi\rangle \le \langle m, \phi\rangle - F^*(\phi) + |m1_{K^c}|(\Omega), \quad (13)\end{aligned}$$

as $|\phi| \le 1$. The claim follows immediately from (12). □

Claim 3. *Let $\phi \in C_B$. For $f \in L^1$, denote*

$$A = A(f) := \{x \in \Omega | \phi \cdot f \ne 0\}.$$

Then

$$F^*(\phi) := \sup_{f \in L^1} \int_\Omega \phi \cdot f - F(f)\, dx = \sup_{f \in L^1} \int_\Omega \phi \cdot (f1_{A(f)}) - F(f1_{A(f)})\, dx.$$

Proof. Indeed, since $F(0) = 0$ and $F(f) \geq 0$, we see that for $x \in A(f)$ we have $F(f) = F(f1_{A(f)})$ and for $x \notin A(f)$ we have $F(f1_{A(f)}) = F(0) = 0$. Thus $F(f) \geq F(f1_{A(f)})$. Therefore,

$$\int_\Omega \phi \cdot f - F(f)\,dx \leq \int_\Omega \phi \cdot (f1_{A(f)}) - F(f1_{A(f)})\,dx.$$

Thus

$$F^*(\phi) = \sup_{f\in L^1} \int_\Omega \phi \cdot f - F(f)\,dx \leq \sup_{f\in L^1} \int_\Omega \phi \cdot (f1_{A(f)}) - F(f1_{A(f)})\,dx.$$

On the other hand,

$$\begin{aligned}\sup_{f\in L^1} \int_\Omega \phi \cdot (f1_A) - F(f1_A)\,dx &\leq \sup_{f\in L^1_\phi} \int_\Omega f \cdot \phi - F(f)\,dx \\ &\leq \sup_{f\in L^1} \int_\Omega f \cdot \phi - F(f)\,dx = F^*(\phi),\end{aligned}$$

where $L^1_\phi = \{f \in L^1 | f \cdot \phi \neq 0\}$. □

Claim 4. *Let $\phi_1, \phi_2 \in C_B$. If $|\phi_1|\,|\phi_2| = 0$, then*

$$F^*(\phi_1 + \phi_2) = F^*(\phi_1) + F^*(\phi_2).$$

Proof. Let $A = A(f) := \{x \in \Omega | \phi_1 \cdot f \neq 0\}$ and $B = B(f) := \{x \in \Omega | \phi_2 \cdot f \neq 0\}$. Then for any $x \in \Omega$, we have

$$|(\phi_1 \cdot f)(\phi_2 \cdot f)(x)| \leq |\phi_1|\,|\phi_2|\,|f|^2 = 0.$$

Hence, for each $x \in \Omega$ we cannot have both $(\phi_1 \cdot f)(x) \neq 0$ and $(\phi_2 \cdot f)(x) \neq 0$. Therefore, $A(f) \cap B(f) = \emptyset$, for any $f \in L^1$.

Since $A(f)$ and $B(f)$ are disjoint for each $f \in L^1$, we have by Claim 3 that

$$\begin{aligned}F^*(\phi_1) + F^*(\phi_2) &= \sup_{f\in L^1} \int_\Omega \phi_1 \cdot (f1_A) - F(f1_A)\,dx \\ &+ \sup_{f\in L^1} \int_\Omega \phi_2 \cdot (f1_B) - F(f1_B)\,dx = \sup_{f\in L^1} \left[\int_\Omega \phi_1 \cdot (f1_A) - F(f1_A)\,dx\right. \\ &+ \left.\int_\Omega \phi_2 \cdot (f1_B) - F(f1_B)\,dx\right] = \sup_{f\in L^1} \int_\Omega \phi_1 \cdot (f1_A) + \phi_2 \cdot (f1_B) \\ &- (F(f1_A) + F(f1_B))\;dx = \sup_{f\in L^1} \int_\Omega (\phi_1 + \phi_2) \cdot (f1_{A\cup B}) - F(f1_{A\cup B})\,dx\end{aligned}$$

$$= F^*(\phi_1 + \phi_2),$$

as $\phi_1 \cdot (f1_{A\cup B}) = \phi_1 \cdot (f1_A + f1_B) = \phi_1 \cdot (f1_A) + \phi_1 \cdot (f1_B) = \phi_1 \cdot (f1_A)$. Similarly, $\phi_2 \cdot (f1_{A\cup B}) = \phi_2 \cdot (f1_B)$. Also note that since $A \cap B = \emptyset$ and $F(0) = 0$, that $F(f1_A) + F(f1_B) = F(f1_{A\cup B})$. Therefore, $F^*(\phi_1 + \phi_2) = F^*(\phi_1) + F^*(\phi_2)$ as desired. □

Claim 5. *Let $\phi \in C_B$. Suppose that $\rho \in C(\Omega, \mathbb{R})$ such that $0 \le \rho \le 1$. Then*

$$F^*(\rho\phi) \le F^*(\phi).$$

Proof. We have

$$F^*(\phi) = \sup_{f\in L^1} \int_\Omega \phi \cdot f - F(f)\, dx \ge \sup_{\substack{g=\rho f\\ f\in L^1}} \int_\Omega \phi \cdot g - F(g)\, dx$$
$$= \sup_{f\in L^1} \int_\Omega \phi \cdot (\rho f) - F(\rho f)\, dx \ge \sup_{f\in L^1} \int_\Omega (\rho\phi) \cdot f - F(f)\, dx = F^*(\rho\phi).$$

For the second inequality, we note that for $0 \le \alpha \le 1$, the convexity of F gives

$$F(\alpha x) \le (1-\alpha)F(0) + \alpha F(x) = 0 + \alpha F(x) \le F(x).$$ □

Claim 6. *Suppose there exists a compact set K such that $|m|(K^c) = 0$. Let U be an open set such that $K \subset U$. Then*

$$F^{**}(m) = \sup_{\substack{|\phi|\le 1\\ \mathrm{spt}(\phi)\subset U}} \int_\Omega \phi\, dm - F^*(\phi).$$

Proof. From (12) we have

$$F^{**}(m) := \sup_{|\phi|\le 1} \int_\Omega \phi\, dm - F^*(\phi) \ge \sup_{\substack{|\phi|\le 1\\ \mathrm{spt}(\phi)\subset U}} \int_\Omega \phi\, dm - F^*(\phi).$$

On the other hand, let $\rho \in C(\Omega, \mathbb{R})$ such that $0 \le \rho \le 1$, $\rho \equiv 1$ on K and $\rho = 0$ on U^c. Then from Claim 5, we have

$$\int_\Omega \phi\, dm - F^*(\phi) = \int_\Omega (\phi 1_K + \phi 1_{K^c})\, dm - F^*(\phi)$$
$$\le \int_\Omega (\phi 1_K + \phi 1_{K^c})\, dm - F^*(\rho\phi) = \int_\Omega \phi 1_K\, dm - F^*(\rho\phi)$$

$$= \int_\Omega \rho\phi \, dm - F^*(\rho\phi),$$

as $|m|(K^c) = 0$. Therefore, we have

$$F^{**}(m) = \sup_{|\phi|\leq 1} \int_\Omega \phi \, dm - F^*(\phi) \leq \sup_{|\phi|\leq 1} \int_\Omega \rho\phi \, dm - F^*(\rho\phi)$$
$$\leq \sup_{\substack{|\phi|\leq 1 \\ \mathrm{spt}(\phi)\subset U}} \int_\Omega \phi \, dm - F^*(\phi),$$

as $\{\rho\phi | \phi \in C_B\} \subset \{\psi \in C_B | \mathrm{spt}(\psi) \subset U\}$. Thus the claim holds. Observe that from the proof, we may also write

$$F^{**}(m) = \sup_{|\phi|\leq 1} \int_\Omega \rho\phi - F^*(\rho\phi). \qquad \square$$

Claim 7. *Let $m, n \in \mathcal{M}$. Suppose there exist disjoint, compact sets K and L so that $|m|(K^c) = 0$ and $|n|(L^c) = 0$. Then*

$$F^{**}(m+n) = F^{**}(m) + F^{**}(n).$$

Proof. Let U and V be disjoint, open sets such that $K \subset U$ and $L \subset V$. Let $\rho_1, \rho_2 \in C(\Omega, \mathbb{R})$ such that $0 \leq \rho_1, \rho_2 \leq 1$ with $\rho_1 \equiv 1$ on K and $\rho_1 = 0$ on U^c, and $\rho_2 \equiv 1$ on L and $\rho_2 = 0$ on V^c. By Claims 4, 5 and 6, we have

$$F^{**}(m) + F^{**}(n) = \sup_{\substack{|\phi|\leq 1 \\ \mathrm{spt}(\phi)\subset U}} \left[\int_\Omega \phi \, dm - F^*(\phi)\right] + \sup_{\substack{|\phi|\leq 1 \\ \mathrm{spt}(\phi)\subset V}} \left[\int_\Omega \phi \, dn - F^*(\phi)\right]$$
$$= \sup_{|\phi|\leq 1} \left[\int_\Omega \rho_1\phi \, dm - F^*(\rho_1\phi)\right] + \sup_{|\phi|\leq 1} \left[\int_\Omega \rho_2\phi \, dn - F^*(\rho_2\phi)\right]$$
$$= \sup_{|\phi|\leq 1} \left[\int_\Omega \rho_1\phi \, dm + \int_\Omega \rho_2\phi \, dn - \big(F^*(\rho_1\phi) + F^*(\rho_2\phi)\big)\right]$$
$$= \sup_{|\phi|\leq 1} \left[\int_\Omega \rho_1\phi \, dm + \int_\Omega \rho_2\phi \, dn - \big(F^*(\rho_1\phi + \rho_2\phi)\big)\right]$$
$$= \sup_{|\phi|\leq 1} \left[\int_\Omega \rho_1\phi + \rho_2\phi \, d(m+n) - \Big(F^*\big((\rho_1+\rho_2)\phi\big)\Big)\right]$$
$$= \sup_{\substack{|\phi|\leq 1 \\ \mathrm{spt}(\phi)\subset U\cup V}} \left[\int_\Omega \phi \, d(m+n) - F^*(\phi)\right] = F^{**}(m+n),$$

as desired. $\square$

Claim 8. *Suppose now that $m, n \in \mathcal{M}$ are mutually singular, denoted $m \perp n$. Then*

$$F^{**}(m+n) = F^{**}(m) + F^{**}(n).$$

Proof. For each $j \in \mathbb{N}$, we can choose compact disjoint sets K_j and L_j such that $|m|(L_j) = 0$, $|n|(K_j) = 0$, $|m|(K_j^c) \to 0$, $|n|(L_j^c) \to 0$ with $m1_{K_j} \rightharpoonup m$ and $n1_{L_j} \rightharpoonup n$, weakly in the sense of measure. By lower semi-continuity, we have

$$F^{**}(m) \leq \liminf F^{**}(m1_{K_j}).$$

On the other hand, from Claim 2 we see that

$$\lim F^{**}(m1_{K_j}) \leq \lim \left[F^{**}(m) + |m|(K_j^c)\right] = F^{**}(m).$$

Whence, $F^{**}(m) = \lim F^{**}(m1_{K_j})$. Similarly,

$$F^{**}(n) = \lim F^{**}(n1_{L_j}).$$

Furthermore, since $m1_{k_j} \rightharpoonup m$ and $n1_{L_j} \rightharpoonup n$, lower semi-continuity gives

$$F^{**}(m+n) \leq \liminf F^{**}(m1_{K_j} + n1_{L_j}).$$

Conversely, by Claim 2 we have

$$\begin{aligned} F^{**}(m1_{K_j} + n1_{L_j}) &= F^{**}\left((m+n)1_{K_j \cup L_j}\right) \\ &\leq F^{**}(m+n) + |m+n|\left((K_j \cup L_j)^c\right) \\ &\leq F^{**}(m+n) + |m|\left((K_j \cup L_j)^c\right) + |n|\left((K_j \cup L_j)^c\right). \end{aligned}$$

Letting $j \to \infty$, we have

$$\lim_{j\to\infty} F^{**}(m1_{K_j} + n1_{L_j}) \leq F^{**}(m+n).$$

Thus $F^{**}(m+n) = \lim F^{**}(m1_{K_j} + n1_{L_j})$.

Therefore, by Claim 7 we have

$$\begin{aligned} F^{**}(m+n) = \lim F^{**}(m1_{K_j} + n1_{L_j}) &= \lim \left[F^{**}(m1_{K_j}) + F^{**}(n1_{L_j})\right] \\ &= \lim F^{**}(m1_{K_j}) + \lim F^{**}(n1_{L_j}) = F^{**}(m) + F^{**}(n) \end{aligned}$$

as desired. □

Thus Proposition 3 is proved.

Let m be a vector-valued measure. Decompose m into its absolutely continuous part, m^a, and its singular part, m^s, with respect to Lebesgue measure. Since $m^a \perp m^s$, by Propositions 2 and 3 we have

$$F^{**}(m) = F^{**}(m^a + m^s) = F^{**}(m^a) + F^{**}(m^s) \\ \leq \int_\Omega F(m^a)\,dx + F^{**}(m^s). \quad (14)$$

Proposition 4. *Let $m \in \mathcal{M}$. Then*

$$F^{**}(m) \leq \int_\Omega F^\infty\left(\frac{dm}{d\,|m|}\right)\,d\,|m|\,.$$

Proof. Fix $x_0 \in \Omega$ and $\alpha > 0$ and let $\sigma : \Omega \to S^{n-1}$ be measurable. Consider the measure $\alpha\sigma(x_0)\delta_{x_0} = m$, where δ_{x_0} is the Dirac delta function. Then $|m| = \alpha\delta_{x_0}$ and $\frac{dm}{d|m|} = \sigma(x_0)$.

Let $|B(x,r)|$ denote the Lebesgue measure of the ball of radius r centered at x. Then we have

$$\frac{\alpha 1_{B(x_0,r)}}{|B(x_0,r)|}\sigma(x_0)\,dx \rightharpoonup \alpha\sigma(x_0)\delta_{x_0} = m$$

weakly in the sense of measures as $r \to 0$. Denote $B = B(x_0, r)$. From Claim 1 of Propostion 3 and Proposition 2, we have

$$\begin{aligned} F^{**}(m) &\leq \liminf_{r\to 0} F^{**}\left(\alpha\sigma(x_0)\frac{1_B}{|B|}\,dx\right) \\ &\leq \liminf_{r\to 0}\int_\Omega F\left(\alpha\sigma(x_0)\frac{1_B}{|B|}\right)\,dx = \liminf_{r\to 0}\int_B F\left(\alpha\sigma(x_0)\frac{1_B}{|B|}\right)\,dx \\ &= \liminf_{r\to 0} F\left(\alpha\sigma(x_0)\frac{1}{|B|}\right)|B| = F^\infty\big(\alpha\sigma(x_0)\big), \end{aligned}$$

by the definition of F^∞. Moreover, for $\alpha > 0$, we see that

$$\begin{aligned} F^\infty\big(\alpha\sigma(x_0)\big) &:= \lim_{t\to\infty}\frac{1}{t}F\big(t\alpha\sigma(x_0)\big) = \lim_{t\to\infty}\frac{\alpha}{t\alpha}F\big(t\alpha\sigma(x_0)\big) \\ &= \alpha F^\infty\big(\sigma(x_0)\big) = \alpha\int_\Omega F^\infty\big(\sigma(x)\big)\,d\delta_{x_0} = \int_\Omega F^\infty\big(\sigma(x)\big)\,d\,|m|\,. \end{aligned}$$

Thus we have

$$F^{**}(m) \leq \int_\Omega F^\infty\big(\sigma(x)\big)\,d\,|m|\,. \quad (15)$$

Now let $x_1, x_2, \ldots, x_k$ be a finite set of distinct points in Ω and $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$. For each i, denote $\sigma(x_i) = \sigma_i \in S^{n-1}$ Denote the measure $\alpha_i \sigma_i \delta_{x_i} = m_i$. We will refer to a measure of the form

$$m = \sum_{i=1}^{k} \alpha_i \sigma_i(x_i) \delta_{x_i} = \sum_{i=1}^{k} m_i$$

as a *simple measure*. Observe that $m_i \perp m_j$ for $i \neq j$. Thus by Propostion 3 and (15), we have

$$F^{**}(m) = \sum_{i=1}^{k} F^{**}(m_i) \leq \sum_{i=1}^{k} \int_\Omega F^\infty(\sigma_i)\, d|m_i| = \int_\Omega F^\infty \left(\frac{dm}{d|m|} \right) d|m|.$$

For the last equality, we need the fact that $d|m_i| = \alpha_i\, d\delta_{x_i}$ is a weighted point mass. In such a case, we have the desired additivity. Hence for a simple measure m, we have shown

$$F^{**}(m) \leq \int_\Omega F^\infty \left(\frac{dm}{d|m|} \right) d|m|.$$

We now extend this to a general measure:

Claim 9. *For any $m \in \mathcal{M}$, there is a sequence $\{m_i\}$ of simple measures such that $m_i \rightharpoonup m$ weakly and*

$$\int_\Omega F^\infty \left(\frac{dm}{d|m|} \right) d|m| = \lim_{i\to\infty} \int_\Omega F^\infty \left(\frac{dm_i}{d|m_i|} \right) d|m_i|.$$

As a limit of continuous functions,

$$F^\infty(p) = \lim_{t\to\infty} \frac{F(tp)}{t}$$

is measurable. Let $\sigma : \Omega \to S^{n-1}$ be a measurable function such that $\sigma\, d|m| = dm$. Then $F^\infty(\sigma)$ is measurable as well. Since F is convex and continuous and $\frac{1}{t}F(tp)$ is increasing in t, convex and continuous, it follows that $F^\infty(p)$ is lower semi-continuous and convex. Since F has linear growth, we have

$$0 \leq F^\infty(p) \leq \beta|p|.$$

By a standard result on convex functions, we conclude that F^∞ is indeed continuous. So by Dini's Theorem, $\frac{1}{t}F(tp)$ converges uniformly on compact sets.

Let $\phi_1, \phi_2, \ldots$ be a sequence in the unit ball of the Banach space $C(\bar{\Omega}, \mathbb{R}^n)$ with the sup norm. Then for any μ and any sequence $\mu_n \in \mathcal{M}$, we define $\mu_n \rightharpoonup \mu$ weakly if and only if

$$\lim_{n\to\infty} \langle \phi_j, \mu_n \rangle = \langle \phi_j, \mu \rangle,$$

for all j.

By Lusin's Theorem, for each k we may choose a compact set C_k such that $|m|(C_k^c) \leq \frac{1}{k}$, σ and $F^\infty(\sigma)$ are continuous on C_k, and $C_{k-1} \subset C_k$. Since σ, $F^\infty(\sigma)$ and ϕ_i $(1 \leq i \leq k)$ are continuous and thus uniformly continuous on C_k, we may choose δ_k such that

$$|\sigma(x) - \sigma(y)| + \left|F^\infty\big(\sigma(x)\big) - F^\infty\big(\sigma(y)\big)\right| + \sum_{i=1}^{k} |\phi_i(x)\cdot\sigma(x) - \phi_i(y)\cdot\sigma(y)| \leq \frac{1}{k},$$

whenever $x, y \in C_k$ with $|x - y| \leq \delta_k$. Split C_k into disjoint subsets $A_{i,k}$ $(1 \leq i \leq l_k)$ such that $\operatorname{diam}(A_{i,k}) \leq \delta_k$. Pick $x_{i,k} \in A_{i,k}$. Then the sequence of simple measures

$$m_k = \sum_{i=1}^{l_k} \sigma(x_{i,k})\,|m|\,(A_{i,k})\delta_{x_{i,k}}$$

satisfy the claim. That is to say, we have

$$\left|\int_\Omega F^\infty(\sigma)\,d\,|m| - \int_\Omega F^\infty\left(\frac{dm_k}{d\,|m_k|}\right)\,d\,|m_k|\right| \leq \frac{2\,\|F^\infty\|\,\|m\|}{k},$$

where $\|F^\infty\| = \sup_{|p|=1}|F^\infty(p)|$. Thus the claim holds and the proposition is proved. □

From (14) and the previous proposition, we have

$$F^{**}(m) \leq \int_\Omega F(m^a)\,dx + \int_\Omega F^\infty\left(\frac{dm^s}{d\,|m^s|}\right)\,d\,|m^s|\,. \tag{16}$$

Proposition 5. *Let $f \in L^1$. F^{**} is the largest convex functional on $\mathcal{M}$ such that*

$$F^{**}(f\,dx) \leq \int_\Omega F(f)\,dx.$$

Proof. We will show that F^{**} is the largest convex functional on $\mathcal{M}$ in the sense that if G is also a lower semi-continuous and convex functional on $\mathcal{M}$, and $G(f\,dx) \le \int_\Omega F(f)\,dx$, for all $f \in L^1$, then $G(m) \le F^{**}(m)$ for all $m \in \mathcal{M}$.

Indeed, let $m \in \mathcal{M}$ and suppose that G is lower semi-continuous and convex on $\mathcal{M}$ such that $G(f\,dx) \le \int_\Omega F(f)\,dx$. Let $\lambda \le G(m)$. Then by the Hahn-Banach Theorem there exists $\phi \in C_B$ and a number $\kappa > 0$ such that for all $n \in \mathcal{M}$,

$$G(n) \ge \int_\Omega \phi\,dn - \kappa, \quad \text{and} \quad \lambda < \int_\Omega \phi\,dm - \kappa.$$

In particular, for any $f \in L^1$, we see that

$$G(f\,dx) \ge \int_\Omega \phi \cdot f\,dx - \kappa.$$

However, $G(f\,dx) \le \int_\Omega F(f)\,dx$ by assumption. Therefore,

$$\int_\Omega F(f)\,dx \ge \int_\Omega \phi \cdot f\,dx - \kappa.$$

Thus, for all $f \in L^1$, we have

$$\int_\Omega \phi \cdot f\,dx - \int_\Omega F(f)\,dx \le \kappa.$$

Taking the supremum over $f \in L^1$, we conclude that $F^*(\phi) \le \kappa$. Thus for $\phi \in C_B$ chosen above and all $n \in \mathcal{M}$ we have

$$F^{**}(n) = \sup_{|\psi| \le 1} \int_\Omega \psi\,dn - F^*(\psi) \ge \int_\Omega \phi\,dn - F^*(\phi) \ge \int_\Omega \phi\,dn - \kappa.$$

In particular,

$$F^{**}(m) \ge \int_\Omega \phi\,dm - \kappa > \lambda.$$

Since $\lambda \le G(m)$ was chosen arbitrarily, it follows that

$$G(m) \le F^{**}(m).$$

In fact, if we suppose that $G(m) > F^{**}(m)$, we may choose λ such that $G(m) > \lambda \ge F^{**}(m)$, a contradiction. □

Finally, it remains to show that

Proposition 6. *Suppose that F is convex and $m \in \mathcal{M}$. Then*

$$F^{**}(m) = \int_\Omega F(m^a)\,dx + \int_\Omega F^\infty\left(\frac{dm^s}{d\,|m^s|}\right)\,d\,|m^s|\,.$$

Proof. By Proposition 5, we need only show that the right hand side is convex. Since F is convex, we know that F^∞ is convex as well. For the convenience of the reader, we show the convexity of the map

$$m \mapsto \int_\Omega F(m^a)\,dx + \int_\Omega F^\infty\left(\frac{dm^s}{d\,|m^s|}\right)\,d\,|m^s|\,. \tag{17}$$

The absolutely continuous (singular) parts of a sum of measures is the sum of their absolutely continuous (singular) parts. Thus we need only show the convexity of

$$m \mapsto \int_\Omega F^\infty\left(\frac{dm^s}{d\,|m^s|}\right)\,d\,|m^s|\,. \tag{18}$$

To this end, let $m, n \in \mathcal{M}$ be measures and let $0 \le t \le 1$; denote $s = 1 - t$. For brevity, denote

$$A = \frac{d(tm+sn)}{d\,|tm+sn|}\,, \qquad B = \frac{d\,|tm+sn|}{t\,d\,|m| + s\,d\,|n|}\,,$$
$$C = \frac{d\,|m|}{t\,d\,|m| + s\,d\,|n|}\,, \qquad D = \frac{d\,|n|}{t\,d\,|m| + s\,d\,|n|}\,.$$

Then

$$\begin{aligned}
\int_\Omega F^\infty\left(\frac{d(tm+sn)}{d\,|tm+sn|}\right)\,d\,|tm+sn| \\
&= \int_\Omega F^\infty\left(\frac{d(tm+sn)}{d\,|tm+sn|}\right) B\,(t\,d\,|m| + s\,d\,|n|) \\
&= \int_\Omega F^\infty\left(\frac{d(tm+sn)}{d\,|tm+sn|}B\right)(t\,d\,|m| + s\,d\,|n|)
\end{aligned}$$

(as $F^\infty(\alpha n) = \alpha F^\infty(n)$ for $\alpha > 0$)

$$= \int_\Omega F^\infty\left(\frac{t\,dm + s\,dn}{t\,d\,|m| + s\,d\,|n|}\right)(t\,d\,|m| + s\,d\,|n|)$$

(by convexity, we have)

$$\begin{aligned}
&\leq \int_\Omega \left[tF^\infty \left(\frac{dm}{t\,d\,|m| + s\,d\,|n|} \right) \right. \\
&\quad \left. + sF^\infty \left(\frac{dn}{t\,d\,|m| + s\,d\,|n|} \right) \right] (t\,d\,|m| + s\,d\,|n|) \\
&= t \int_\Omega F^\infty \left(\frac{dm}{d\,|m|} C \right) (t\,d\,|m| + s\,d\,|n|) \\
&\quad + s \int_\Omega F^\infty \left(\frac{dn}{d\,|n|} D \right) (t\,d\,|m| + s\,d\,|n|) \\
&= t \int_\Omega F^\infty \left(\frac{dm}{d\,|m|} \right) C (t\,d\,|m| + s\,d\,|n|) \\
&\quad + s \int_\Omega F^\infty \left(\frac{dn}{d\,|n|} \right) D (t\,d\,|m| + s\,d\,|n|) \\
&= t \int_\Omega F^\infty \left(\frac{dm}{d\,|m|} \right) d\,|m| + s \int_\Omega F^\infty \left(\frac{dn}{d\,|n|} \right) d\,|n|\,,
\end{aligned}$$

proving the map (18) is convex. □

The last proposition is precisely the result we have set out to establish, and thus the theorem is proved.

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