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THE CONCEPT OF A WEAK SHAPE TYPE

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Abstract: For every category pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{C}$ is a dense and full subcategory, an (abstract) weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$ is constructed. There exists a faithful functor, which keeps the objects fixed, of the (abstract) shape category $Sh_{(\mathcal{C},\mathcal{D})}$ to $Sh_{*(\mathcal{C},\mathcal{D})}$. The main benefit is that one may expect existence of a pair of $\mathcal{C}$-objects (especially, topological spaces) having the same weak shape type and different shape types. Further, the weak shape type is coarser than the recently introduced coarse shape type, because there also exists a functor of the (abstract) coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ to $Sh_{*(\mathcal{C},\mathcal{D})}$. It is interesting that, for metric compacta, both (coarse and weak shape) types coincide with the S^*-equivalence, which is strictly coarser than the shape type classification. An operative characterization of a weak shape isomorphism is established. Finally, it is proved that several important well known shape invariants are, actually, weak shape invariant properties.

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1. Introduction

Homotopy theory, which classifies topological spaces strictly coarser than their homeomorphisms, is practically almost inapplicable outside the subclass of locally nice spaces (polyhedra, CW-complexes, ANR's,...). Namely, for a pair of locally bad spaces there is no efficacious procedure to determine whether they are (not) of the same homotopy type. Since 1968, when K. Borsuk [1] had founded shape theory (a generalization of homotopy theory) for compact metric spaces (compacta lying in the Hilbert cube), the homotopy type of a locally nice compactum may be considered its shape type. A significant step forward was made by S. Mardešić and J. Segal [14] who founded the very effective inverse system approach to shape theory (for compact Hausdorff spaces). A few years after, shape theory was founded for all topological spaces, see [11] and [18].

Since 1976 a few new classifications of compacta have been considered. For instance, K. Borsuk [2] introduced the relations of quasi-affinity and quasi-equivalence, while S. Mardešić [12] introduced the S-equivalence relation between compacta. All of them are shape-type invariant relations. (One should note that quasi-equivalence is not transitive, [8]; nevertheless, it generates a useful equivalence relation). The corresponding classifications are strictly coarser than the shape-type classification (see [2], [9], [6]). Moreover, the quasi-equivalence and S-equivalence classifications coincide with the homotopy type classification on compact ANR's, compact polyhedra, finite CW-complexes, etc.

The mentioned relations, being defined only on the class of objects, were not supported by appropriate with them associated theories. In other words, it was not clear whether these relations are categorical. Furthermore, if such an equivalence relation admits a characterization in terms of morphisms in a category, there should exist a functor relating the shape category Sh and the new category.

On this line the first named author studied in [20] the Borsuk quasi-equivalence and quasi-affinity. He constructed a certain category and an appropriate "quasi-homotopy" relation on its morphism sets such that the object relations are characterized in the category framework. However, this morphism-equivalence relation is not compatible with the category composition, so there is no corresponding quotient category. Nevertheless, one can slightly strengthen these Borsuk relations up to the new equivalence relations on compacta, which admit characterizations in terms of the category isomorphisms (dominations), i.e. there exist the corresponding quotient category and functor.

With a similar purpose Mardešić and Uglešić [16] studied the S^*-equivalence, which is a "uniformization" of the Mardešić S-equivalence. They constructed a certain category on compacta such that its mutually isomorphic objects are S^*-equivalent and vice versa. They also obtained the appropriate functor of the shape category to the new category. We should mention that the S-equivalence as well as the S^*-equivalence of compacta is a rather useful notion. For instance, some important shape invariant classes of compacta (continua, movable compacta, n-shape connected compacta, FANR's, compacta having $Fd \leq n, \dots$) are also S- and S^*-invariant classes. Furthermore, it is a well known fact that the fibres of a shape fibration (see [3], [13]) over a continuum may have different shape (see [9], [6]). However, all of them are mutually S- and S^*-equivalent (see [12], [16]).

In a recent paper [21] the authors have constructed a certain sequence of categories $\mathcal{S}(n)$, $n \in \mathbb{N} \cup \{\omega\}$, on compacta, endowed with an appropriate sequence of functors. The corresponding category isomorphisms classify compacta strictly coarser than the shape isomorphisms do. In this paper we have proved that the classifications in all the categories $\mathcal{S}(n)$, $n \in \mathbb{N}$, coincide – it is the classification by the S^*-equivalence (Section 7, Remark 2). Therefore, for the main purpose, it suffices to consider the case $n = 1$. We hereby apply the same basic idea for the rather sophisticated construction of the weak shape category. Thus, a few words concerning the previous construction are needed. Since the relations that classify compacta have been considered, the main objects were the inverse sequences of compact ANR's. The smallest building material for a morphism was a ladder, which imitated a mapping of inverse sequences restricted to a segment. A countable collection of such ladders, subjected to some conditions, was called a hyperladder. The hyperladders were organized into a category (on the inverse sequences). Such a category admitted a natural equivalence (homotopy) relation, which yields the quotient category $\underline{\mathcal{S}}(1)$. Clearly, there also existed the corresponding category $\mathcal{S}(1)$ on compacta. The appropriate functor of the shape category to this one has arisen naturally.

In this paper a significant step forward is done. Namely, the previous "S^*-theory" for metric compacta is generalized to an arbitrary category and its infinite inverse systems in a way that a theory of (abstract) weak shape can be developed. To do this, the following two main obstacles had to be successfully overcome:

- the passage from the index set $\mathbb{N}$ (positive integers) to an arbitrary index set Λ (directed, ordered, infinite, cofinite, having no maximal element) – Sections 3 and 4.

- the *index set changing problem*, i.e. the independence of the abstract "Λ-weak shape" type of an object of the chosen index set Λ (for an expansion of the object) - Section 6 (Theorems 3 and 4).

As a summary of the main theoretical part, for every category pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{C}$ is dense and full, there exists an (abstract) weak shape theory modelled on the corresponding (abstract) weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$. The appropriate faithful functor $T : Sh_{(\mathcal{C},\mathcal{D})} \to Sh_{*(\mathcal{C},\mathcal{D})}$ naturally relates the (abstract) shape category and the new category. Hence, one may consider $Sh_{(\mathcal{C},\mathcal{D})}$ to be a subcategory of $Sh_{*(\mathcal{C},\mathcal{D})}$ obtained by enrichment of the morphism sets. The most important fact is the existence of weak shape isomorphisms, between some pairs of objects, when there is no shape isomorphism (Section 7, Corollary 6, and Section 8, Corollary 9).

In Section 7 the recently introduced coarse shape theory (see [10]), which is also an extension of the "S^*-theory", is also compared with this weak shape theory. A coarse shape morphism essentially depends on the set $\mathbb{N}$ of positive integers, while a weak shape morphism does not. We have constructed (Theorem 6). a functor of the (abstract) coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ to the weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$. However, there is no natural way (except for inverse sequences) to achieve the converse. Consequently, the weak shape classification is, generally, coarser than the coarse shape classification. It is interesting that, for metric compacta, they both (weak and coarse shape) coincide with the classification by the S^*-equivalence (Section 7, Corollary 6), which is strictly coarser than the shape classification, [16].

In Section 8 an analogue (Theorem 10) of the well known Morita Lemma for a weak shape isomorphism is established. The formulation is even more general because it does not involve a level representative. Furthermore, in the case of inverse sequences an operative characterization of "their" weak shape isomorphyness is given (Theorem 11). At the end, in Section 9, we have shown that some important shape invariant properties (connectedness, trivial shape, shape dimension $\leq n$, n-shape connectedness, movability, n-movability, Mittag-Leffler property, quasi-stability, strong movability) are, actually, weak shape invariant properties.

2. Preliminaries

For the sake of completeness, let us briefly recall the well known notions and main facts concerning a pro-category as well as an abstract shape category (see [15]). The category language follows [7].

Let $\mathcal{A}$ be a category. An *inverse system* in $\mathcal{A}$, denoted by $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, consists of a directed preordered set $(\Lambda, \leq)$, called the *index set*, of $\mathcal{A}$-objects X_λ for each $\lambda \in \Lambda$, called the *terms* of $\boldsymbol{X}$, and of $\mathcal{A}$-morphisms $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$

($p_{\lambda\lambda} = 1_{X_\lambda}$), for each related pair $\lambda \leq \lambda'$ in Λ, called the *bonding morphisms* of $\boldsymbol{X}$, such that

$$p_{\lambda\lambda'}p_{\lambda'\lambda''} = p_{\lambda\lambda''},$$

whenever $\lambda \leq \lambda' \leq \lambda''$. A *morphism of inverse systems* $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ consists of a function $f : M \to \Lambda$, called the *index function*, and of an $\mathcal{A}$-morphism $f_\mu : X_{f(\mu)} \to Y_\mu$ for each $\mu \in M$, such that, for every related pair $\mu \leq \mu'$, there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f(\mu')$, for which

$$f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}.$$

The *composition* of morphisms of inverse systems is defined as follows: Given any $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ and any $(g, g_\nu) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, then $(g, g_\nu)(f, f_\mu) = (h, h_\nu) : \boldsymbol{X} \to \boldsymbol{Z}$, where $h = fg : N \to \Lambda$ and $h_\nu = g_\nu f_{g(\nu)} : X_{h(\nu)} \to Z_\nu$. Finally, the *identity* morphism on an $\boldsymbol{X}$ is $(1_\Lambda, 1_{X_\lambda}) : \boldsymbol{X} \to \boldsymbol{X}$. In this way is obtained a category, denoted by *inv*-$\mathcal{A}$, whose objects are all inverse systems in $\mathcal{A}$ and whose morphisms are all morphisms of inverse systems described above.

Notice that, for every index set Λ, there exists a full subcategory $\mathcal{A}^\Lambda$ of *inv*-$\mathcal{A}$ determined by all inverse systems indexed by Λ. If $\Lambda = \mathbb{N}$, then $\mathcal{A}^\mathbb{N} \subseteq$ *inv*-$\mathcal{A}$ is the full subcategory of all inverse sequences in $\mathcal{A}$.

A morphism $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *equivalent* (congruent, "homotopic") *to* a morphism $(f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_\mu) \simeq (f', f'_\mu)$, provided each $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$, such that

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

This defines an equivalence relation on each set *inv*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. The equivalence class $[(f, f_\mu)]$ of (f, f_μ) is denoted by $\boldsymbol{f}$. Furthermore, the equivalence relation respects the composition in *inv*-$\mathcal{A}$, i.e. if $(f, f_\mu) \simeq (f', f'_\mu)$ and $(g, g_\nu) \simeq (g', g'_\nu)$, then $(g, g_\nu)(f, f_\mu) \simeq (g', g'_\nu)(f', f'_\mu)$, whenever these compositions are defined. Therefore, there exists the corresponding quotient category (*inv*-$\mathcal{A})/\simeq$, denoted by *pro*-$\mathcal{A}$ and called the *pro-category* for the category $\mathcal{A}$. Its objects are all inverse systems $\boldsymbol{X}$ in $\mathcal{A}$ and its morphisms are all equivalence classes $\boldsymbol{f}$ of morphisms (f, f_μ) of *inv*-$\mathcal{A}$.

In the case of a fixed index set Λ, the full subcategory $\mathcal{A}^\Lambda \subseteq$ *inv*-$\mathcal{A}$ yields the full quotient subcategory $\mathcal{A}^\Lambda/\simeq\, \equiv pro^\Lambda$-$\mathcal{A} \subseteq$ *pro*-$\mathcal{A}$. Especially, for $\Lambda = \mathbb{N}$ (positive integers), the full subcategory $\mathcal{A}^\mathbb{N}/\simeq\, \equiv pro^\mathbb{N}$-$\mathcal{A}$ of *pro*-$\mathcal{A}$, determined by all inverse sequences in $\mathcal{A}$, is usually called the *tow-category* of $\mathcal{A}$ and is denoted by *tow*-$\mathcal{A}$.

Recall that, if the index set M of an inverse system $\boldsymbol{Y}$ is *cofinite* (every $\mu \in M$ has at most finitely many predecessors), then every $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ admits a representative (f, f_μ), such that the index function $f : M \to \Lambda$ is increasing and, for every related pair $\mu \leq \mu'$,

$$f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}.$$

Such a representative is called a *special morphism* of inverse systems. A special morphism $(1_\Lambda, f_\lambda)$, belonging to a subcategory $\mathcal{A}^\Lambda$ and having the identity index function, is called a *level morphism*. Finally, recall that every inverse system $\boldsymbol{X}$ admits an isomorphic (in *pro*-$\mathcal{A}$) $\boldsymbol{X}'$ having a cofinite index set (the well known " Mardešić trick").

Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a (full) subcategory of $\mathcal{C}$. Given an $X \in Ob\mathcal{C}$, a $\mathcal{D}$- *expansion* of X is a morphism $\boldsymbol{p} = [(c, p_\lambda)] : X \to \boldsymbol{X}$ of *pro*-$\mathcal{C}$ ($X \equiv (X)$ is the rudimentary system and c is the constant function), where $\boldsymbol{X}$ belongs to *pro*-$\mathcal{D}$, such that, for every $\boldsymbol{Y}$ in *pro*-$\mathcal{D}$ and every $\boldsymbol{p}' : X \to \boldsymbol{Y}$ in *pro*-$\mathcal{C}$, there exists a unique morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ (in *pro*-$\mathcal{D}$) satisfying $\boldsymbol{f}\boldsymbol{p} = \boldsymbol{p}'$. A full subcategory $\mathcal{D}$ is said to be *dense* in $\mathcal{C}$ provided every $\mathcal{C}$-object X admits a $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$.

Every two $\mathcal{D}$-expansions of the same object are naturally isomorphic (as the objects of *pro*-$\mathcal{D}$, by a unique isomorphism), and every system which is isomorphic to a $\mathcal{D}$-expansion of an X is also a $\mathcal{D}$-expansion of X. A $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$ is characterized by the following two properties:

(E1) For every $P \in Ob(\mathcal{D})$ and every $g : X \to P$ in $\mathcal{C}$, there exist a $\lambda \in \Lambda$ and an $f : X_\lambda \to P$ in $\mathcal{D}$, such that $fp_\lambda = g$;

(E2) If $f, f' : X_\lambda \to P$ in $\mathcal{D}$ satisfy $fp_\lambda = f'p_\lambda$, then there exists a $\lambda' \geq \lambda$ such that $fp_{\lambda\lambda'} = f'p_{\lambda\lambda'}$.

Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{p}' : X \to \boldsymbol{X}'$ be $\mathcal{D}$-expansions of the same object X of $\mathcal{C}$, and let $\boldsymbol{q} : Y \to \boldsymbol{Y}$ and $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be $\mathcal{D}$-expansions of the same object Y of $\mathcal{C}$. Then there exist two natural isomorphisms $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$. A morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *pro-$\mathcal{D}$- equivalent to* a morphism $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}'$, denoted by $\boldsymbol{f} \sim \boldsymbol{f}'$, provided the following diagram in *pro*-$\mathcal{D}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{i}} & \boldsymbol{X}' \\ \boldsymbol{f} \downarrow & & \downarrow \boldsymbol{f}' \\ \boldsymbol{Y} & \xrightarrow[\boldsymbol{j}]{} & \boldsymbol{Y}' \end{array} .$$

It defines an equivalence relation on the appropriate subclass of $Mor($*pro*-$\mathcal{D})$. The equivalence class of an $\boldsymbol{f}$ is denoted by $\langle \boldsymbol{f} \rangle$. If $\boldsymbol{f} \sim \boldsymbol{f}'$ and $\boldsymbol{g} \sim \boldsymbol{g}'$, then $\boldsymbol{g}\boldsymbol{f} \sim \boldsymbol{g}'\boldsymbol{f}'$ whenever it is defined. Further, given $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{q}$, $\boldsymbol{q}'$ and $\boldsymbol{f}$, there exists a unique $\boldsymbol{f}'$ such that $\boldsymbol{f} \sim \boldsymbol{f}'$.

For a given category pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D}$ dense and full (convenient, not essential) subcategory of $\mathcal{C}$, one defines the (*abstract*) *shape category* $Sh_{(\mathcal{C},\mathcal{D})}$ *for* $(\mathcal{C}, \mathcal{D})$ as follows. The objects of $Sh_{(\mathcal{C},\mathcal{D})}$ are all the objects of $\mathcal{C}$. A morphism $F \in Sh_{(\mathcal{C},\mathcal{D})}(X,Y)$ is the *pro*-$\mathcal{D}$-equivalence class $\langle \boldsymbol{f} \rangle$ of a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, with respect to any choice of a pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$. In other words, a *shape morphism* $F : X \to Y$ is given by a representing diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ \boldsymbol{f} \downarrow & & \downarrow F \\ \boldsymbol{Y} & \xleftarrow[\boldsymbol{q}]{} & Y \end{array} .$$

The *composition* of an $F : X \to Y$, $F = \langle \boldsymbol{f} \rangle$ and a $G : Y \to Z$, $G = \langle \boldsymbol{g} \rangle$, is well defined by the representatives, i.e. $GF : X \to Z$, $GF = \langle \boldsymbol{gf} \rangle$. The *identity shape morphism* on an object X, $1_X : X \to X$, is the *pro*-$\mathcal{D}$-equivalence class $\langle \boldsymbol{1_X} \rangle$ of the identity morphism $\boldsymbol{1_X}$ in *pro*-$\mathcal{D}$. Since

$$Sh_{(\mathcal{C},\mathcal{D})}(X,Y) \approx pro - \mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$$

is a set, the shape category $Sh_{(\mathcal{C},\mathcal{D})}$ is well defined. One often says that *pro*-$\mathcal{D}$ is the *realizing* category for the shape category $Sh_{(\mathcal{C},\mathcal{D})}$.

For every $f : X \to Y$ in $\mathcal{C}$ and every pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$, there exists a unique $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro*-$\mathcal{D}$, such that the following diagram in *pro*-$\mathcal{C}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ \boldsymbol{f} \downarrow & & \downarrow f \\ \boldsymbol{Y} & \xleftarrow[\boldsymbol{q}]{} & Y \end{array} .$$

The same f and another pair of $\mathcal{D}$-expansions $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ yield a unique $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}'$ in *pro*-$\mathcal{D}$. Then, however, $\boldsymbol{f} \sim \boldsymbol{f}'$ in *pro*-$\mathcal{D}$ must hold. Thus, every morphism $f \in \mathcal{C}(X,Y)$ yields a unique *pro*-$\mathcal{D}$-equivalence class $\langle \boldsymbol{f} \rangle$, i.e. a unique shape morphism $F \in Sh_{(\mathcal{C},\mathcal{D})}(X,Y)$. If one defines $S(X) = X$, $X \in Ob\mathcal{C}$, and $S(f) = F = \langle \boldsymbol{f} \rangle$, $f \in Mor\mathcal{C}$, then

$$S : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$$

becomes a functor, called the (*abstract*) *shape functor*. Notice that a lot of "realizing" (abstract) shape functors $\underline{S} : \mathcal{C} \to pro\text{-}\mathcal{D}$ occurs. Namely, every choice of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, when X passes through $Ob\mathcal{C}$ (the axiom of choice!), yields such a functor.

The restriction of S to $\mathcal{D}$ into the full subcategory of $Sh_{(\mathcal{C},\mathcal{D})}$, determined by $Ob\mathcal{D}$, is a category isomorphism. Therefore, P and Q are isomorphic objects of $\mathcal{D}$ if and only if they are isomorphic in $Sh_{(\mathcal{C},\mathcal{D})}$, i.e. they are of the same shape. Finally, if $X \in Ob\mathcal{C}$ and $P \in Ob\mathcal{D}$, then every shape morphism $F : X \to P$ admits a unique morphism $f : X \to P$ in $\mathcal{C}$ such that $S(f) = F$. Therefore, the restriction function (of S)

$$S|\cdot : \mathcal{C}(X,P) \to Sh_{(\mathcal{C},\mathcal{D})}(X,P)$$

is a bijection.

The most interesting example of the above construction is $\mathcal{C} = HTop$ - the homotopy category of topological spaces and $\mathcal{D} = HPol$ – the homotopy category of polyhedra (or $\mathcal{D} = HANR$ – the homotopy category of ANR's for metric spaces, which yields the same theory, since $Ob(Pol)$ and $Ob(ANR)$ are homotopy equivalent classes). Namely, the (full) subcategory $HPol \subseteq HTop$ is dense in $HTop$, since every space X admits a $HPol$-expansion $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$, which is obtained by applying the homotopy functor to a polyhedral resolution $(p_\lambda) : X \to \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of X, [15]. In this case, one speaks about the (ordinary or standard) *shape category* $Sh_{(HTop,HPol)} \equiv Sh$ of topological spaces and (ordinary or standard) *shape functor* $S : Htop \to Sh$. Clearly, the realizing category for Sh is the pro-category *pro*-$HPol$ (or *pro*-$HANR$). The underlying theory is called the (ordinary or standard) *shape theory* for topological spaces.

Let $HcM \subseteq HTop$ denote the homotopy subcategory of compact metric spaces, and let $HcPol \subseteq HPol$ denote the homotopy subcategory of compact polyhedra. Since $HcPol \subseteq HcM$ is a "*sequentially*" dense subcategory (every compactum X admits a $HcPol$-expansion $\boldsymbol{p} = ([p_i]) : X \to \boldsymbol{X} = (X_i, [p_{ii'}], \mathbb{N})$, which is obtained by applying the homotopy functor to the limit $(p_i) : X \to \underline{X} = (X_i, p_{ii'})$ of an inverse *sequence* of compact polyhedra, $X = \lim \underline{X}$, [5] and [15], there exists the shape category of compacta, $Sh_{(HcM,HcPol)} \equiv Sh(cM)$, which is a full subcategory of Sh. Notice that the realizing category for $Sh(cM)$ is the tower-category *tow*-$HcPol$. Clearly, since the classes $Ob(cPol)$ and $Ob(cANR)$ (all compact ANR's for metric spaces) are homotopy equivalent, the tower-category *tow*-$HcANR$ may also serve as the realizing category for the shape category $Sh(cM)$.

3. The Reduced Pro-Category

The first step in our construction of an appropriate category is to point out that, for every (abstract) shape category $Sh_{(\mathcal{C},\mathcal{D})}$, the realizing category *pro*-$\mathcal{D}$ admits a reduction within $Mor(pro\text{-}\mathcal{D})$ to its subclass of all morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same index set. It suffices to recall the following well known fact (see [15], Theorem I.1.3):

Proposition 1. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{A}$, and let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism in pro-$\mathcal{A}$. Then there exist inverse systems $\boldsymbol{X}' = (X'_\nu, p'_{\nu\nu'}, N)$ and $\boldsymbol{Y}' = (Y'_\nu, q'_{\nu\nu'}, N)$ in $\mathcal{A}$, indexed by the same cofinite directed ordered set N, and there exist isomorphisms $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$, $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$ in pro-$\mathcal{A}$ and a morphism $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}'$ in pro-$\mathcal{A}$ such that the following diagram commutes*

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{i}} & \boldsymbol{X}' \\ \boldsymbol{f}\downarrow & & \downarrow \boldsymbol{f}' \\ \boldsymbol{Y} & \xrightarrow[\boldsymbol{j}]{} & \boldsymbol{Y}' \end{array} .$$

Moreover, $\boldsymbol{f}'$ admits a level representative $(1_N, f'_\nu)$ belonging to $\mathcal{A}^N(\boldsymbol{X}', \boldsymbol{Y}')$.

One may also assume, if needed, that the index set N is infinite having no maximal element. Let $inv^\sim$-$\mathcal{A}$ be the subcategory of inv-$\mathcal{A}$ such that $Ob(inv^\sim$-$\mathcal{A})$ consists of all inverse systems in $\mathcal{A}$ over infinite cofinite directed ordered index sets having no maximal elements, and if $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$, let

$$inv^\sim\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) = \begin{cases} \emptyset, & \Lambda \neq M\,, \\ inv\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}), & \Lambda = M\,. \end{cases} .$$

The category $inv^\sim$-$\mathcal{A}$ is said to be the *reduced inv-category* of the category $\mathcal{A}$. The corresponding *reduced pro-category* $pro^\sim$-$\mathcal{A}$ of $\mathcal{A}$ is the appropriate quotient category,

$$pro^\sim\text{-}\mathcal{A} \equiv inv^\sim\text{-}\mathcal{A}/\simeq .$$

Further, for a fixed infinite cofinite directed ordered set Λ, the appropriate quotient category

$$pro^\Lambda\text{-}\mathcal{A} \equiv \mathcal{A}^\Lambda/\simeq$$

is a full subcategory of $pro^\sim$-$\mathcal{A}$ as well as of pro-$\mathcal{A}$. In the case $\Lambda = \mathbb{N}$, one obtains the full subcategory $pro^{\mathbb{N}}$-$\mathcal{A} \equiv tow$-$\mathcal{A}$.

Let $\mathcal{D}$ be a dense subcategory of $\mathcal{C}$. According to Proposition 1, for every pair $X, Y \in Ob\mathcal{C}$ there exists a pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$ such that $\boldsymbol{X}, \boldsymbol{Y} \in Ob(pro^\sim$-$\mathcal{D})$ have the same index set (Λ). Let $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be another pair of such $\mathcal{D}$-expansions over the same index set (Λ'). A morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *$pro^\sim$-$\mathcal{D}$ equivalent to* a morphism

$\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}'$, denoted by $\boldsymbol{f} \sim \boldsymbol{f}'$, provided the following diagram in *pro*-$\mathcal{D}$ ($\boldsymbol{i}$ and $\boldsymbol{j}$ are the natural isomorphisms) commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{i}} & \boldsymbol{X}' \\ \boldsymbol{f}\downarrow & & \downarrow \boldsymbol{f}' \\ \boldsymbol{Y} & \xrightarrow[\boldsymbol{j}]{} & \boldsymbol{Y}' \end{array} .$$

One readily sees that it defines an equivalence relation on the appropriate subclass of $Mor(pro^{\sim}\text{-}\mathcal{D})$. The equivalence class of an $\boldsymbol{f}$ is again denoted by $\langle \boldsymbol{f} \rangle$. Further, if $\boldsymbol{f} \sim \boldsymbol{f}'$ and $\boldsymbol{g} \sim \boldsymbol{g}'$, then $\boldsymbol{g}\boldsymbol{f} \sim \boldsymbol{g}'\boldsymbol{f}'$ whenever it is defined. Also, as in the case of *pro*-$\mathcal{D}$, given $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{q}$, $\boldsymbol{q}'$ and $\boldsymbol{f}$, there exists a unique $\boldsymbol{f}'$ such that $\boldsymbol{f} \sim \boldsymbol{f}'$. Clearly, every (abstract) shape morphism $F \in Sh_{(\mathcal{C},\mathcal{D})}(X,Y)$ has a representative $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in the reduced pro-category $pro^{\sim}$-$\mathcal{D}$. In other words, every $F : X \to Y$ in $Sh_{(\mathcal{C},\mathcal{D})}(X,Y)$ admits a representing diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ \boldsymbol{f}\downarrow & & \downarrow F \\ \boldsymbol{Y} & \xleftarrow[\boldsymbol{q}]{} & Y \end{array} ,$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same index set. Each identity 1_X is represented in $pro^{\sim}$-$\mathcal{D}$ by an identity $\boldsymbol{1_X}$. However, of course, a representative in $pro^{\sim}$-$\mathcal{D}$ of the composition of shape morphisms is *not*, in general, the composition of the chosen representatives – which may not exist! Nevertheless, we still may say that the reduced pro-category $pro^{\sim}$-$\mathcal{D}$ can also serve as a realizing category for the (abstract) shape category $Sh_{(\mathcal{C},\mathcal{D})}$, because of

$$Sh_{(\mathcal{C},\mathcal{D})}(X,Y) \approx pro^{\sim}\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y}),$$

whenever the index sets of $\boldsymbol{X}$ and $\boldsymbol{Y}$ coincide.

4. The Ladders and Hyperladders

Let Λ be an infinite cofinite directed ordered set which does not have any maximal element. Then, for every related pair $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 \leq \lambda_2$, the corresponding segment in Λ,

$$[\lambda_1, \lambda_2] = \{\lambda \in \Lambda \mid \lambda_1 \leq \lambda \leq \lambda_2\}$$

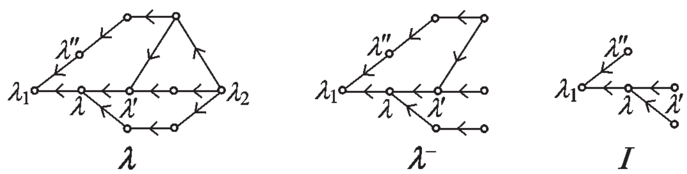
is a finite subset of Λ. Let us briefly denote $[\lambda_1, \lambda_2]$ by $\boldsymbol{\lambda}$. We shall often say that $\boldsymbol{\lambda}$ is *associated with* (λ_1, λ_2). Let $\boldsymbol{\Lambda}$ denote the set of all segments $\boldsymbol{\lambda}$ in

Λ. A *full chain* in a segment $\boldsymbol{\lambda}$ is every linearly ordered subset (chain) C of $\boldsymbol{\lambda}$, satisfying the following condition:

$$(\forall \lambda, \lambda' \in C)(\forall \lambda'' \in \boldsymbol{\lambda})\ \lambda \leq \lambda'' \leq \lambda' \Rightarrow \lambda'' \in C.$$

Notice that every noninitial element $\lambda \in \Lambda$ (the number of all its predecessors $|\lambda| > 0$) admits an immediate predecessor, denoted by λ^- (which, in general, is not unique), i.e. for every $\lambda \in \Lambda$ with $|\lambda| > 0$ there exists $\lambda^- \in \Lambda$ such that $\lambda^- < \lambda$ and, for every $\lambda' \in \Lambda$, if $\lambda^- \leq \lambda' \leq \lambda$ then $\lambda' = \lambda^-$ or $\lambda' = \lambda$. Clearly, every $\lambda \in \Lambda$ has at most finitely many immediate predecessors. An *initial subset* of the segment $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, associated with an ordered pair (λ_1, λ_2), is a subset $I \subseteq \boldsymbol{\lambda}$ satisfying the following condition: If $\lambda \in I$ belongs to a chain $C \subseteq \boldsymbol{\lambda}$, then every predecessor $\lambda' \in C$, $\lambda' \leq \lambda$, belongs to I. Thus, every initial subset I of a $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ is a (finite) union of full chains in $\boldsymbol{\lambda}$ starting with λ_1.

Obviously, the trivial subsets $\emptyset, \boldsymbol{\lambda} \subseteq \boldsymbol{\lambda}$ are initial. Further, the singleton $\{\lambda^1\} \subseteq \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \setminus \{\lambda_2\} \equiv \boldsymbol{\lambda}^- \subseteq \boldsymbol{\lambda}$ are initial in $\boldsymbol{\lambda}$. The pictures below illustrate a $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, its corresponding $\boldsymbol{\lambda}^-$ and a nontrivial initial subset $I \subseteq \boldsymbol{\lambda}$. The points denote the elements (λ, λ', ...), and the arrows – relations ($\lambda < \lambda'$, ...).



Now, given any category $\mathcal{A}$, we can generalize the definitions and main facts of [21], Section 2, from *tow-HcANR* to the reduced pro-category $pro^{\sim}$-$\mathcal{A}$. (The case $n = 1$ will satisfy our needs; see Remark 2 of Section 7 below.)

Definition 1. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda), \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in $\mathcal{A}$ such that $\Lambda = M$. Let $\mu_1, \mu_2 \in \Lambda$ such that $\mu_1 \leq \mu_2$, and let $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ be associated with (μ_1, μ_2). A *ladder* $f_{\boldsymbol{\mu}} = (f, f_\mu)$ of $\boldsymbol{X}$ to $\boldsymbol{Y}$ over $\boldsymbol{\mu}$, denoted by $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, consists of an increasing function

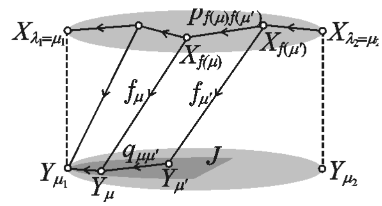
$$f : J \to \boldsymbol{\lambda} = \boldsymbol{\mu},$$

where J is an initial subset of $\boldsymbol{\mu}$, and of $\mathcal{A}$-morphisms

$$f_\mu : X_{f(\mu)} \to Y_\mu, \mu \in J,$$

such that

$$f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}$$



whenever $\mu \leq \mu'$. In the case $J = \emptyset$, $f_{\boldsymbol{\mu}}$ is said to be the *empty* ladder, because then there is no corresponding $\mathcal{A}$-morphism.

Clearly, $J \neq \emptyset$ if and only if $\mu_1 \in J$.

Example 1. Let $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be a special morphism of $\mathcal{A}^\Lambda$ with $f \geq 1_\Lambda$. Given a $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, let us choose, an initial subset $J \subseteq \boldsymbol{\mu}$ such that $f(J) \subseteq \boldsymbol{\mu}$. Then the restriction of (f, f_μ) to $\boldsymbol{X}|J$ and $\boldsymbol{Y}|J$ yields a certain (possibly empty) ladder $f_{\boldsymbol{\mu}} = (f, f_\mu)$ of $\boldsymbol{X}$ to $\boldsymbol{Y}$ over $\boldsymbol{\mu}$. In the case of the maximal subset J, the (" maximal") ladder $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$ over $\boldsymbol{\mu}$ is said to be *induced by* (f, f_μ). Notice that, for every $\boldsymbol{X}$ and every $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, the identity morphism $(1_\Lambda, 1_{X_\lambda}) : \boldsymbol{X} \to \boldsymbol{X}$ induces the *identity ladder* $1_{\boldsymbol{X\lambda}} : \boldsymbol{X} \to \boldsymbol{X}$ *over* $\boldsymbol{\lambda}$.

Let $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$ and $g_{\boldsymbol{\nu}} = (g, g_\nu) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, $\Lambda = M = N$, be ladders. Let J and K be the domains of f and g respectively. Then we *compose* $f_{\boldsymbol{\mu}}$ and $g_{\boldsymbol{\nu}}$ only in the case when $\boldsymbol{\mu} = \boldsymbol{\nu}$ by using the ordinary rule, i.e.

$$g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}} \equiv u_{\boldsymbol{\nu}} = (u, u_\nu),$$

such that $u = fg$, whenever it is defined on the appropriate $K' \subseteq K \subseteq \boldsymbol{\nu}$, and $u_\nu = g_\nu f_{g(\nu)}$, $\nu \in K'$. Clearly, $g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}} : \boldsymbol{X} \to \boldsymbol{Z}$ is a ladder of $\boldsymbol{X}$ to $\boldsymbol{Z}$ over $\boldsymbol{\nu}$. Notice that it is empty whenever $f_{\boldsymbol{\nu}}$ or $g_{\boldsymbol{\nu}}$ is empty, or $g(K) \cap J = \emptyset$. It is obvious that the composition of ladders is associative, and that

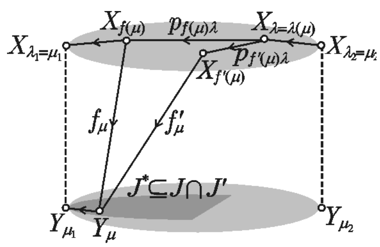
$$f_{\boldsymbol{\mu}} 1_{\boldsymbol{X\mu}} = f_{\boldsymbol{\mu}}, \text{ and } 1_{\boldsymbol{X\lambda}} g_{\boldsymbol{\lambda}} = g_{\boldsymbol{\lambda}}$$

for all ladders $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$ and $g_{\boldsymbol{\lambda}} : \boldsymbol{Z} \to \boldsymbol{X}$. Thus, for every category $\mathcal{A}$, every infinite cofinite directed ordered set Λ and every $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, there exists a certain category whose class of objects is $Ob(\mathcal{A}^\Lambda)$, and the sets of morphisms consist of all the corresponding ladders.

Let $f_{\boldsymbol{\mu}}, f'_{\boldsymbol{\mu}} = (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be a pair of ladders over the same $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, associated with a (μ_1, μ_2). Then $f_{\boldsymbol{\mu}}$ is said to be *equivalent* (*congruent*, "*homotopic*") *to* $f'_{\boldsymbol{\mu}}$ provided they both are empty or there exists an initial subset J^* of $\boldsymbol{\mu}$ such that $J^* \subseteq J \cap J'$ and

$$(\forall \mu \in J^*)(\exists \lambda = \lambda(\mu) \in \boldsymbol{\lambda} = \boldsymbol{\mu}, \lambda \geq f(\mu), f'(\mu))$$

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$



The equivalence (" homotopy") of ladders is an equivalence relation on the corresponding set. Namely, the relation is reflexive and symmetric by definition, while transitivity follows by the fact that the intersection of two initial subsets is an initial subset.

Definition 2. Let $\boldsymbol{X}, \boldsymbol{Y} \in Ob(inv^{\sim}\text{-}\mathcal{A})$ with $\Lambda = M$. A *hyperladder* of $\boldsymbol{X}$ to $\boldsymbol{Y}$ is a family $(f_{\boldsymbol{\mu}})$ of ladders $f_{\boldsymbol{\mu}}$ (of $\boldsymbol{X}$ to $\boldsymbol{Y}$), indexed by all $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, such that every ordered pair $\mu_1 \leq \mu'_1$ in Λ admit a $\lambda^1 \in \Lambda$, $\lambda^1 \geq \mu'_1$, such that, for every $\mu_2 \geq \lambda^1$, the ladder $f_{\boldsymbol{\mu}} = (f, f_\mu)$ of the family $(f_{\boldsymbol{\mu}})$, assigned to $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ which is associated with the chosen (μ_1, μ_2), fulfills the requirement that $\mu'_1 \in J$ (the domain of f) and $f(\mu'_1) \leq \lambda^1$.

Briefly, a family $(f_{\boldsymbol{\mu}})$ of ladders $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, is a hyperladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$, provided

$$(\forall \mu_1 \in \Lambda)(\forall \mu'_1 \geq \mu_1)(\exists \lambda^1 \geq \mu'_1)(\forall \mu_2 \geq \lambda^1),$$

the index function $f : J \to \boldsymbol{\lambda} = \boldsymbol{\mu}$ of the corresponding $f_{\boldsymbol{\mu}}$, associated with (μ_1, μ_2), fulfills the following two conditions:

$$\mu'_1 \in J \text{ and } f(\mu'_1) \leq \lambda^1.$$

Notice that, since f increases, the second condition implies $f(\mu) \leq \lambda^1$ for every $\mu \leq \mu'_1$.

The set of all hyperladders $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ is denoted by $\underline{L}(\boldsymbol{X}, \boldsymbol{Y})$.

Example 2. Let $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be a special morphism of $\mathcal{A}^\Lambda$ with $f \geq 1_\Lambda$. For every $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, let $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$ be the ladder induced by (f, f_μ) according to Example 1. Then, (f, f_μ) yields the family $(f_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$. Let us show that it is a hyperladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$. Given a $\mu_1 \in \Lambda$ and a $\mu'_1 \geq \mu_1$ put $\lambda^1 = f(\mu'_1)$, and let $\mu_2 \geq \lambda^1$. Let us consider the ladder $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, where

$\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ is associated with the chosen pair (μ_1, μ_2). By construction, $\mu_1' \in J$ and $f(\mu_1') \leq \lambda^1$ obviously hold. Such a hyperladder $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *induced by* a special morphism $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$. In particular, for every $\boldsymbol{X}$, the identity morphism $(1_\Lambda, 1_{X_\lambda})$ induces the hyperladder $(1_{\boldsymbol{X\lambda}}) : \boldsymbol{X} \to \boldsymbol{X}$, $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ (see also Example 1).

If $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g_{\boldsymbol{\nu}}) : \boldsymbol{Y} \to \boldsymbol{Z}$, $\boldsymbol{\nu} \in \boldsymbol{\Lambda}$, $\Lambda = M = N$, are hyperladders, then we *compose* them coordinatewise., i.e. by composing the corresponding ladders $f_{\boldsymbol{\mu}}$ and $g_{\boldsymbol{\nu}}$ whenever $\boldsymbol{\mu} = \boldsymbol{\nu}$. Hence,

$$(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}}) = (u_{\boldsymbol{\nu}}),$$

where $u_{\boldsymbol{\nu}} \equiv g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \boldsymbol{\Lambda}$.

Lemma 1. *Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in Ob(inv^{\sim}\text{-}\mathcal{A})$ have the same index set $\Lambda = M = N$. Then the composition of the corresponding hyperladders is well defined, i.e.*

$$(\forall (f_{\boldsymbol{\mu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Y}))(\forall (g_{\boldsymbol{\nu}}) \in \underline{L}(\boldsymbol{Y}, \boldsymbol{Z}))$$
$$(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}}) = (g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Z}).$$

Proof. Definition 2 allows the following procedure: For every $\nu_1 \in \Lambda$ and every $\nu_1' \geq \nu_1$ there exists a $\mu^1 \geq \nu_1'$, and for $\mu_1 = \nu_1$ and $\mu_1' = \mu^1 \geq \mu_1$ there exists a $\lambda^1 \geq \mu_1'$; choose any $\nu_2 = \mu_2 \geq \lambda^1$. Let $\boldsymbol{\nu}, \boldsymbol{\mu} \in \boldsymbol{\Lambda}$ be associated with the chosen pairs (ν_1, ν_2), (μ_1, μ_2) respectively. Clearly, $\boldsymbol{\nu} = \boldsymbol{\mu}$ because of $(\nu_1, \nu_2) = (\mu_1, \mu_2)$. Since $(g_{\boldsymbol{\nu}})$ is a hyperladder, $g_{\boldsymbol{\nu}}$ satisfies:

$$\nu_1' \in K \text{ and } g(\nu_1') \leq \mu^1.$$

Further, since $(f_{\boldsymbol{\mu}})$ is a hyperladder, $f_{\boldsymbol{\mu}}$ satisfies the conditions

$$\mu_1' \in J \text{ and } f(\mu_1') \leq \lambda^1.$$

Thus, the ladder $g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}} \in (g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}})$ satisfies (g and f are increasing) the conditions

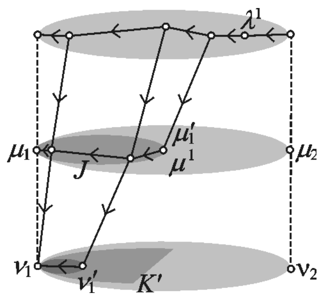
$$\nu_1' \in K' \subseteq K$$

and

$$fg(\nu_1') = f(g(\nu_1')) \leq f(\mu^1) = f(\mu_1') \leq \lambda^1.$$

Therefore, concerning the composition $(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}}) = (u_{\boldsymbol{\nu}})$, $u_{\boldsymbol{\nu}} \equiv g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \boldsymbol{\Lambda}$, we have established that

$$(\forall \nu_1 \in \Lambda)(\forall \nu_1' \geq \nu_1)(\exists \lambda^1 \geq \nu_1')(\forall \nu_2 \geq \lambda^1),$$



the index function $u = fg : K \to \boldsymbol{\nu}$ of the corresponding $u_{\boldsymbol{\nu}}$ satisfies $\nu_1' \in K'$ and $u(\nu_1') \leq \lambda^1$.

Hence, the family $(g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}})$, $\boldsymbol{\nu} \in \boldsymbol{\Lambda}$, fulfills both needed conditions for a hyperladder. Thus, $(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}}) = (g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Z})$, and the lemma is proved. □

Since the composition of ladders is associative, the composition of hyperladders is associative too. Notice that the family $(1_{\boldsymbol{X}\boldsymbol{\lambda}})$, $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, is the *identity hyperladder* on $\boldsymbol{X}$ (see Example 2). Indeed,

$$(f_{\boldsymbol{\mu}})(1_{\boldsymbol{X}\boldsymbol{\lambda}}) = (f_{\boldsymbol{\mu}} 1_{\boldsymbol{X}\boldsymbol{\mu}}) = (f_{\boldsymbol{\mu}}),$$
$$(1_{\boldsymbol{X}\boldsymbol{\lambda}})(g_{\boldsymbol{\lambda}}) = (1_{\boldsymbol{X}\boldsymbol{\lambda}} g_{\boldsymbol{\lambda}}) = (g_{\boldsymbol{\lambda}})$$

hold for all hyperladders $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g_{\boldsymbol{\nu}}) : \boldsymbol{Z} \to \boldsymbol{X}$, where $\Lambda = M = N$.

Therefore, for every category $\mathcal{A}$, there exists a certain category $inv_*^{\sim}$-$\mathcal{A}$ consisting of the object class $\mathrm{Ob}(inv_*^{\sim}\text{-}\mathcal{A}) = \mathrm{Ob}(inv^{\sim}\text{-}\mathcal{A})$ and of the morphism class $\mathrm{Mor}(inv_*^{\sim}\text{-}\mathcal{A})$ of all the morphism sets

$$inv_*^{\sim}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) = \begin{cases} \emptyset, & \Lambda \neq M\,, \\ \underline{L}(\boldsymbol{X}, \boldsymbol{Y}), & \Lambda = M\,. \end{cases}$$

Further, for a fixed Λ, there also exists the appropriate category inv_*^{Λ}-$\mathcal{A}$, which is a full subcategory of $inv_*^{\sim}$-$\mathcal{A}$.

In order to define a certain equivalence relation on every morphism set $\underline{L}(\boldsymbol{X}, \boldsymbol{Y})$, let us first recall the relation $f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, which means that either both ladders are empty or there exists an initial subset J^* of $\boldsymbol{\mu}$ such that $J^* \subseteq J \cap J'$ and

$$(\forall \mu \in J^*)(\exists \lambda = \lambda(\mu) \in \boldsymbol{\mu}, \lambda \geq f(\mu), f'(\mu))$$
$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

Definition 3. Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathrm{Ob}(inv^{\sim}\text{-}\mathcal{A})$ such that $\Lambda = M$, and let $(f_{\boldsymbol{\mu}}), (f'_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a pair of hyperladders. Then $(f_{\boldsymbol{\mu}})$ is said to be *equivalent* (*congruent*, *" homotopic"*) *to* $(f'_{\boldsymbol{\mu}})$, denoted by $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$, provided every related pair $\mu_1 \leq \mu'_1$ in Λ admits a $\lambda^1_* \geq \mu'_1$, such that, for every $\mu_2 \geq \lambda^1_*$, the corresponding ladders $f_{\boldsymbol{\mu}}$ and $f'_{\boldsymbol{\mu}}$ over $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ associated with the chosen pair (μ_1, μ_2), are equivalent, $f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}}$, and, in addition, the appropriate initial subset $J^* \subseteq J \cap J'$ contains μ'_1 and $\lambda = \lambda(\mu'_1) \leq \lambda^1_*$.

Briefly, $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$ provided

$$(\forall \mu_1 \in \Lambda)(\forall \mu'_1 \geq \mu_1)(\exists \lambda^1_* \geq \mu'_1)(\forall \mu_2 \geq \lambda^1_*)$$

the corresponding $f_{\boldsymbol{\mu}}$ and $f'_{\boldsymbol{\mu}}$ are equivalent, $f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}}$, such that $\mu'_1 \in J^* \subseteq J \cap J'$ and $\lambda = \lambda(\mu'_1) \leq \lambda^1_*$.

Obviously, by the last condition, for every $\mu \in J^*$, $\mu \leq \mu'_1$ implies $\lambda = \lambda(\mu) \leq \lambda^1_*$. Further, one may assume that, for the index λ^1_* in Definition 3 and the indices λ^1, λ'^1 (for $(f_{\boldsymbol{\mu}})$, $(f'_{\boldsymbol{\mu}})$ respectively) in Definition 2, $\lambda^1_* \geq \lambda^1, \lambda'^1$ holds.

Lemma 2. *The equivalence of hyperladders is a natural equivalence relation in the category* $inv^{\sim}_*\text{-}\mathcal{A}$.

Proof. It suffices to consider the case of all the hyperladders sets involved not empty. Since the equivalence of ladders is an equivalence relation, to prove that the equivalence of hyperladders is an equivalence relation, we only need to show that it is transitive. The verification is straightforward by means of $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$ and $(f'_{\boldsymbol{\mu}}) \simeq (f''_{\boldsymbol{\mu}})$. Namely, given a μ_1 and a $\mu'_1 \geq \mu_1$, one should choose a $\lambda^1_* \geq \lambda'^1_*, \lambda''^1_*$ and $J^* = J'^* \cap J''^*$.

Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in Ob(inv^{\sim}\text{-}\mathcal{A})$ such that $\Lambda = M = N$. Let $(f_{\boldsymbol{\mu}}), (f'_{\boldsymbol{\mu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Y})$ such that $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$, and let $(g_{\boldsymbol{\nu}}) \in \underline{L}(\boldsymbol{Y}, \boldsymbol{Z})$. Then Definitions 3 and 4 allow the following procedure: For every $\nu_1 \in \Lambda$ and every $\nu'_1 \geq \nu_1$ there exists a $\mu^1 \geq \nu'_1$, and for $\mu_1 = \nu_1$ and $\mu'_1 = \mu^1 \geq \mu_1$ there exists a $\lambda^1_* \geq \mu'_1$; let $\nu_2 = \mu_2 \geq \lambda^1_*$. Let $\boldsymbol{\nu}, \boldsymbol{\mu} \in \boldsymbol{\Lambda}$ be associated with the chosen pairs (ν_1, ν_2), (μ_1, μ_2) respectively. Then $\boldsymbol{\mu} = \boldsymbol{\nu}$ because of $(\mu_1, \mu_2) = (\nu_1, \nu_2)$. Consider the corresponding ladders $f_{\boldsymbol{\nu}}$, $f'_{\boldsymbol{\nu}}$ and $g_{\boldsymbol{\nu}}$. Since $(g_{\boldsymbol{\nu}})$ is a hyperladder,

$$g(\nu'_1) \leq \mu^1 = \mu'_1,$$

and because of $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$,

$$(\forall \mu \in J^*) f_\mu p_{f(\mu)\lambda(\mu)} = f'_\mu p_{f'(\mu)\lambda(\mu)},$$

where $\mu_1' \in J^*$ and $\lambda(\mu) \leq \lambda_*^1$ for $\mu \leq \mu_1'$. Thus,

$$(\exists K^* \subseteq K, \nu_1' \in K^*)(\forall \nu \in K^*)$$

$$g_\nu f_{g(\nu)} p_{f(g(\nu))\lambda(g(\nu))} = g_\nu f'_{g(\nu)} p_{f'(g(\nu))\lambda(g(\nu))}.$$

Therefore, concerning the compositions $(g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}})$ and $(g_{\boldsymbol{\nu}} f'_{\boldsymbol{\nu}})$, the following is fulfilled:

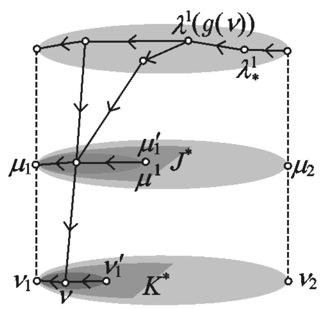
$$(\forall \nu_1 \in \Lambda)(\forall \nu_1' \geq \nu_1)(\exists \lambda_*^1 \geq \nu_1')(\forall \nu_2 \geq \lambda_*^1)$$

the equivalence relation

$$g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}} \simeq g_{\boldsymbol{\nu}} f'_{\boldsymbol{\nu}}$$

for the corresponding (composition of the) ladders holds, so that

$$\nu_1' \in K^* \quad \text{and} \quad \lambda = \lambda(\nu_1') \leq \lambda_*^1.$$



This means that $(g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}) \simeq (g_{\boldsymbol{\nu}} f'_{\boldsymbol{\nu}})$.

Let $(h_{\boldsymbol{\lambda}}) \in \underline{L}(\boldsymbol{W}, \boldsymbol{X})$, where the index set T of $\boldsymbol{W}$ equals Λ, and let $(f_{\boldsymbol{\mu}}), (f'_{\boldsymbol{\mu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Y})$ be such that $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$. Then Definitions 4 and 3 allow the following procedure: For every $\mu_1 \in \Lambda$ and every $\mu_1' \geq \mu_1$ there exists a $\lambda_*^1 \geq \mu_1'$, and for $\lambda_1 = \mu_1$ and $\lambda_1' = \lambda_*^1 \geq \lambda_1$ there exists a $\tau^1 \geq \lambda_1'$; let $\mu_2 = \lambda_2 \geq \tau^1$. Let $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ be associated with the chosen pairs (μ_1, μ_2), (λ_1, λ_2) respectively. Then $\boldsymbol{\lambda} = \boldsymbol{\mu}$ because of $(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$. Consider the ladders $h_{\boldsymbol{\mu}}$, $f_{\boldsymbol{\mu}}$ and $f'_{\boldsymbol{\mu}}$. By $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$,

$$(\forall \mu \in J^*)\ f_\mu p_{f(\mu)\lambda(\mu)} = f'_\mu p_{f'(\mu)\lambda(\mu)},$$

where $\mu_1' \in J^*$ and $\lambda(\mu) \leq \lambda_*^1$ for every $\mu \leq \mu_1'$. Since $(h_{\boldsymbol{\lambda}})$ is a hyperladder, $h(\lambda_1') \leq \tau^1$ holds. Thus,

$$(\forall \mu \in J^*)\ f_\mu h_{f(\mu)} u_{h(f(\mu))\tau^1} = f'_\mu h_{f'(\mu)} u_{h(f'(\mu))\tau^1}.$$

Therefore, concerning the compositions $(f_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}})$ and $(f'_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}})$, the following is fulfilled:

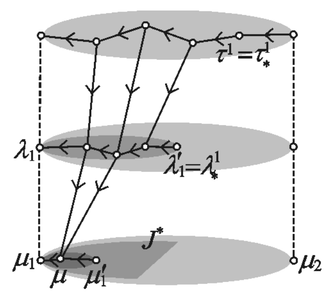
$$(\forall \mu_1 \in \Lambda)(\forall \mu_1' \geq \mu_1)(\exists \tau_*^1 = \tau^1 \geq \mu_1')(\forall \mu_2 \geq \tau_*^1)$$

the equivalence relation

$$f_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}}$$

for the corresponding (composition of the) ladders holds, and that

$$\mu_1'^* \in J^* \text{ and } \tau = \tau(\mu_1') \equiv h(\lambda(\mu_1')) \leq \tau_*^1.$$



This means that $(f_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}})$, and the lemma is proved. □

The equivalence class $[(f_{\boldsymbol{\mu}})]$ of an $(f_{\boldsymbol{\mu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Y})$ is denoted by $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$. The obtained results are summarized in the next theorem.

Theorem 1. *For every category $\mathcal{A}$, there exists a quotient category ("$*$ -reduced pro-category")*

$$pro_*^{\sim}\text{-}\mathcal{A} \equiv (inv_*^{\sim}\text{-}\mathcal{A})/ \simeq$$

consisting of the object class $\mathrm{Ob}(pro_*^{\sim}\text{-}\mathcal{A}) = \mathrm{Ob}(inv^{\sim}\text{-}\mathcal{A})$ *and of the morphism class* $\mathrm{Mor}(pro_*^{\sim}\text{-}\mathcal{A})$ *of all the sets*

$$pro_*^{\sim}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) = \begin{cases} \emptyset, & \Lambda \neq M\,, \\ inv_*^{\sim}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})/\simeq\,, & \Lambda = M\,. \end{cases}$$

with composition defined by

$$\boldsymbol{g}_* \boldsymbol{f}_* = [(g_{\boldsymbol{\nu}})][(f_{\boldsymbol{\mu}})] = [(g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}})],$$

and with the identity $(pro_*^{\sim}\text{-}\mathcal{A})$*-morphism* $\mathbf{1}_{\boldsymbol{X}*} = [(1_{\boldsymbol{X\lambda}})]$ *on each object* $\boldsymbol{X} \in \mathrm{Ob}(pro_*^{\sim}\text{-}\mathcal{A})$.

Further, for a fixed index set Λ, there exists the corresponding quotient category

$$pro_*^{\Lambda}\text{-}\mathcal{A} = (inv_*^{\Lambda}\text{-}\mathcal{A})/ \simeq,$$

which is a full subcategory of $pro_*^{\sim}\text{-}\mathcal{A}$.

Let us now establish a functorial relationship between $pro^{\sim}$-$\mathcal{A}$ and $pro^{\sim}_{*}$-$\mathcal{A}$.

Lemma 3. *(i) Let $\boldsymbol{X}, \boldsymbol{Y} \in Ob(pro^{\Lambda}$-$\mathcal{A})$ and let $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be special morphisms of $\mathcal{A}^{\Lambda}$ with $f, f' \geq 1_\Lambda$. If $(f, f_\mu) \simeq (f', f'_\mu)$, then the induced hyperladders $(f_{\boldsymbol{\mu}}), (f'_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ are equivalent, $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$.*

(ii) Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in Ob(pro^{\Lambda}$-$\mathcal{A})$, and let $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_\nu) : \boldsymbol{Y} \to \boldsymbol{Z}$ be special morphisms of $\mathcal{A}^{\Lambda}$ with $f, g \geq 1_\Lambda$. If $(f_{\boldsymbol{\mu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Y})$, $(g_{\boldsymbol{\nu}}) \in \underline{L}(\boldsymbol{Y}, \boldsymbol{Z})$ and $(h_{\boldsymbol{\nu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Z})$ are the hyperladders induced by (f, f_μ), (g, g_ν) and $(g, g_\nu)(f, f_\mu) = (gf, g_\nu f_{g(\nu)})$ respectively, then $(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}}) = (h_{\boldsymbol{\nu}})$.

Proof. Recall that the equivalence relation $(f, f_\mu) \sim (f', f'_\mu)$ in $\mathcal{A}^{\Lambda}$, $M = \Lambda$, means that

$$(\forall \mu \in \Lambda)(\exists \lambda \geq f(\mu), f'(\mu)) \; f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

Hence (see Examples 1 and 2), one can easily choose the appropriate indices and obtain a desired $\boldsymbol{\mu}$ which confirms that $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$. This proves claim (i). Further, if (f, f_μ) and (g, g_ν) are special morphisms with $f, g \geq 1_\Lambda$, $N = M = \Lambda$, then the induced hyperladders $(f_{\boldsymbol{\mu}})$, $(g_{\boldsymbol{\nu}})$ and $(h_{\boldsymbol{\nu}})$ obviously satisfy $h_{\boldsymbol{\nu}} = g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}$ for every $\boldsymbol{\nu} \in \boldsymbol{\Lambda}$. Thus, claim (ii) follows. □

Theorem 2. *There exists a faithful functor*

$$\underline{T} : pro^{\sim}\text{-}\mathcal{A} \to pro^{\sim}_{*}\text{-}\mathcal{A}$$

keeping the objects fixed. Further, for every Λ, there exists a faithful functor

$$\underline{T}^{\Lambda} : pro^{\Lambda}\text{-}\mathcal{A} \to pro^{\Lambda}_{*}\text{-}\mathcal{A},$$

which keeps the objects fixed.

Proof. It suffices to prove the case of a fixed index set. For every $\boldsymbol{X} \in$ Ob$(pro^{\Lambda}$-$\mathcal{A})$, put $\underline{T}^{\Lambda}(\boldsymbol{X}) = \boldsymbol{X}$. For every $\boldsymbol{f} \in pro^{\Lambda}$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, put $\underline{T}^{\Lambda}(\boldsymbol{f}) = \boldsymbol{f}_* \equiv [(f_{\boldsymbol{\mu}})] \in pro^{\Lambda}_{*}$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, where $(f_{\boldsymbol{\mu}}) \in \underline{L}(\boldsymbol{X}, \boldsymbol{Y})$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, $M = \Lambda$, is the hyperladder induced by a special representative $(f, f_\mu) \in \boldsymbol{f}$ (see Examples 1 and 2). By Lemma 3 (i), the correspondence $\boldsymbol{f} \mapsto \underline{T}^{\Lambda}(\boldsymbol{f}) = \boldsymbol{f}_*$ is well defined. Namely, $(f, f_\mu) \simeq (f', f'_\mu)$ implies $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$. Further, $\underline{T}^{\Lambda}(\boldsymbol{1_X}) = [(1_{\boldsymbol{X\lambda}})] = \boldsymbol{1}_{\boldsymbol{X}*}$, and, by Lemma 3 (ii),

$$\underline{T}^{\Lambda}(\boldsymbol{g}\boldsymbol{f}) = \underline{T}^{\Lambda}([(g, g_\nu)(f, f_\mu)]) = [(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}})] = [(g_{\boldsymbol{\nu}})][(f_{\boldsymbol{\mu}})] = \underline{T}^{\Lambda}(\boldsymbol{g})\underline{T}^{\Lambda}(\boldsymbol{f}).$$

Hence, $\underline{T}^{\Lambda} : pro^{\Lambda}$-$\mathcal{A} \to pro^{\Lambda}_{*}$-$\mathcal{A}$ is a functor. Further, if $\underline{T}^{\Lambda}(\boldsymbol{f}) = \boldsymbol{f}_* \equiv [(f_{\boldsymbol{\mu}})] = [(f'_{\boldsymbol{\mu}})] \equiv \boldsymbol{f}'_* = \underline{T}^{\Lambda}(\boldsymbol{f}')$, then $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$. Since $(f_{\boldsymbol{\mu}})$ and $(f'_{\boldsymbol{\mu}})$ are induced by representatives (f, f_μ) and (f', f'_μ) of $\boldsymbol{f}$ and $\boldsymbol{f}'$ respectively, one readily sees that $(f, f_\mu) \simeq (f', f'_\mu)$, i.e. $\boldsymbol{f} = \boldsymbol{f}'$. Thus, the functor $\underline{T}^{\Lambda}$ is faithful. □

Remark 1. Since the functor $\underline{T}$ ($\underline{T}^\Lambda$) is faithful and keeps the objects fixed, one may treat $pro^\sim$-$\mathcal{A}$ (pro^Λ-$\mathcal{A}$) as a subcategory of $pro_*^\sim$-$\mathcal{A}$ (pro_*^Λ-$\mathcal{A}$). Therefore, the following "inclusion"-functors diagram commutes:

$$\begin{array}{ccc} pro^\Lambda\text{-}\mathcal{A} & \longrightarrow & pro_*^\Lambda\text{-}\mathcal{A} \\ \downarrow & & \downarrow \\ pro^\sim\text{-}\mathcal{A} & \longrightarrow & pro_*^\sim\text{-}\mathcal{A} \end{array} .$$

5. The Λ-Weak Shape Category

Let $(\mathcal{C},\mathcal{D})$ be a category pair such that $\mathcal{D}\subseteq\mathcal{C}$ is a dense and full (convenient, not essential assumption) subcategory. Similarly to the corresponding abstract shape category $Sh_{(\mathcal{C},\mathcal{D})}$ having $pro^\sim$-$\mathcal{D}$ as a realizing category, one can construct, for every Λ, the abstract Λ *-weak shape category* $Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}$, having pro_*^Λ-$\mathcal{D}$ as a realizing category.

Given an index set Λ, let $\mathcal{C}_\Lambda$ denote the full subcategory of $\mathcal{C}$ determined by all the $\mathcal{C}$-objects admitting $\mathcal{D}$-expansions over Λ. The following fact is an immediate consequence of Theorem 2 and the existence of a realizing (abstract) shape functor $\underline{S}:\mathcal{C}\to pro$-$\mathcal{D}$.

Corollary 1. *For every infinite cofinite directed ordered set Λ having no maximal element, there exists a functor*

$$\underline{S}_*^\Lambda:\mathcal{C}_\Lambda\to pro_*^\Lambda\text{-}\mathcal{D},$$

such that $\underline{T}^\Lambda\underline{S}^\Lambda=\underline{S}_*^\Lambda$, where $\underline{S}^\Lambda:\mathcal{C}_\Lambda\to pro^\Lambda$-$\mathcal{D}$ is the restriction of a realizing (abstract) shape functor $\underline{S}:\mathcal{C}\to pro$-$\mathcal{D}$.

$$\begin{array}{ccc} & \mathcal{C}_\Lambda & \\ \underline{S}_*^\Lambda \swarrow & & \searrow \underline{S}^\Lambda \\ pro_*^\Lambda\text{-}\mathcal{D} & \underset{\underline{T}^\Lambda}{\longleftarrow} & pro^\Lambda\text{-}\mathcal{D} \end{array} .$$

Proof. Let $\underline{S}^\Lambda:\mathcal{C}_\Lambda\to pro^\Lambda$-$\mathcal{D}$ be the restriction of an appropriate realizing shape functor $\underline{S}:\mathcal{C}\to pro$-$\mathcal{D}$. Given an $X\in Ob\mathcal{C}_\Lambda$, put $\underline{S}_*^\Lambda(X)=\underline{S}^\Lambda(X)=\boldsymbol{X}$. Clearly, this includes a choice of a $\mathcal{D}$-expansion $\boldsymbol{p}:X\to\boldsymbol{X}$ indexed by Λ. Further, if $f\in\mathcal{C}_\Lambda(X,Y)$, put $\underline{S}_*^\Lambda(f)=\boldsymbol{f}_*=[(f_{\boldsymbol{\mu}})]$, where $(f_{\boldsymbol{\mu}})\in\underline{L}(\boldsymbol{X},\boldsymbol{Y})$ is the hyperladder induced by a special representative of the morphism $\boldsymbol{f}=\underline{S}^\Lambda(f):\boldsymbol{X}\to\boldsymbol{Y}$. The conclusion follows by Theorem 2. □

Given an index set Λ, let us now define the (abstract) Λ-*weak shape category* $Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$. The objects of $Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$ are all those objects of $\mathcal{C}$ which admit $\mathcal{D}$-expansions over Λ, i.e.

$$Ob(Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}) = Ob\mathcal{C}_{\Lambda},$$

while the morphisms of $Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$ are certain equivalence classes of morphisms of pro^{Λ}_{*}-$\mathcal{D}$ defined as follows. Given a pair $X, Y \in Ob\mathcal{C}_{\Lambda}$, let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be $\mathcal{D}$-expansions such that both $\boldsymbol{X}, \boldsymbol{Y}$ have the same index set Λ. Let $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be another pair of $\mathcal{D}$-expansions over Λ. A morphism $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$ of pro^{Λ}_{*}-$\mathcal{D}$ is said to be *equivalent to* a morphism $\boldsymbol{f}'_* : \boldsymbol{X}' \to \boldsymbol{Y}'$ of pro^{Λ}_{*}-$\mathcal{D}$, denoted by $\boldsymbol{f}_* \sim \boldsymbol{f}'_*$, provided the following diagram in pro^{Λ}_{*}-$\mathcal{D}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{i}_*} & \boldsymbol{X}' \\ \boldsymbol{f}_* \downarrow & & \downarrow \boldsymbol{f}'_* \\ \boldsymbol{Y} & \xrightarrow[\boldsymbol{j}_*]{} & \boldsymbol{Y}' \end{array} .$$

(hereby, $\boldsymbol{i}_* = \underline{T}^{\Lambda}(\boldsymbol{i})$ and $\boldsymbol{j}_* = \underline{T}^{\Lambda}(\boldsymbol{j})$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the natural isomorphisms in pro^{Λ}-$\mathcal{D}$; see Section 2). One readily sees that it defines an equivalence relation on the appropriate subclass of $Mor(pro^{\Lambda}_{*}$-$\mathcal{D})$. The equivalence class of an $\boldsymbol{f}_*$ is denoted by $\langle \boldsymbol{f}_* \rangle$. Further, if $\boldsymbol{f}_* \sim \boldsymbol{f}'_*$ and $\boldsymbol{g}_* \sim \boldsymbol{g}'_*$, then $\boldsymbol{g}_*\boldsymbol{f}_* \sim \boldsymbol{g}'_*\boldsymbol{f}'_*$ whenever these are defined. Further, as in the case of *pro*-$\mathcal{D}$, given $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{q}$, $\boldsymbol{q}'$ and $\boldsymbol{f}_*$, there exists a unique $\boldsymbol{f}'_*$ such that $\boldsymbol{f}_* \sim \boldsymbol{f}'_*$.

Given any $X, Y \in Ob\mathcal{C}_{\Lambda}$, we define an (abstract) Λ-*weak shape morphism*

$$F^{\Lambda}_{*} : X \to Y \text{ of } Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$$

to be the equivalence class $\langle \boldsymbol{f}_* \rangle$ of a morphism $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$ of pro^{Λ}_{*}-$\mathcal{D}$. In other words, every $F^{\Lambda}_{*} : X \to Y$ in $Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$ admits a (formal representing) diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ \boldsymbol{f}_* \downarrow & & \downarrow F^{\Lambda}_{*} \\ \boldsymbol{Y} & \xleftarrow[\boldsymbol{q}]{} & Y \end{array} ,$$

where both $\boldsymbol{X}$ and $\boldsymbol{Y}$ are indexed over Λ. For every $X \in Ob\mathcal{C}_{\Lambda}$, the identity 1^{Λ}_{X*} is represented by an identity $\boldsymbol{1}_{\boldsymbol{X}*}$ in pro^{Λ}_{*}-$\mathcal{D}$. Finally, because of

$$Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}(X, Y) \approx pro^{\Lambda}_{*}\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y}),$$

where $\boldsymbol{X}$, $\boldsymbol{Y}$ is any appropriate pair of expansions of X, Y respectively, we may say that pro^{Λ}_{*}-$\mathcal{D}$ is a *realizing* category for the Λ-weak shape category $Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$.

For every $f : X \to Y$ in $\mathcal{C}_\Lambda$ and every pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$ over Λ, there exists a unique $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in pro^Λ-$\mathcal{D}$, such that the following diagram in pro-$\mathcal{C}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \stackrel{\boldsymbol{p}}{\longleftarrow} & X \\ \boldsymbol{f}\downarrow & & \downarrow f \\ \boldsymbol{Y} & \underset{\boldsymbol{q}}{\longleftarrow} & Y \end{array} .$$

The same f and another pair of $\mathcal{D}$-expansions $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ over Λ yield a unique $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}'$ in pro^Λ-$\mathcal{D}$. Then, $\boldsymbol{f} \sim \boldsymbol{f}'$ in pro^Λ-$\mathcal{D}$ holds. Thus, $\underline{T}^\Lambda(\boldsymbol{f}) \equiv \boldsymbol{f}_* \sim \boldsymbol{f}'_* \equiv \underline{T}^\Lambda(\boldsymbol{f}')$ in pro^Λ_*-$\mathcal{D}$ (see Theorem 2). Consequently, every morphism $f \in \mathcal{C}_\Lambda(X,Y)$ yields a unique (pro^Λ_*-$\mathcal{D}$)-equivalence class $\langle \boldsymbol{f}_* \rangle$, i.e. a unique Λ-weak shape morphism $F^\Lambda_* \in Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}(X,Y)$. If one defines $S^\Lambda_*(X) = X$, $X \in Ob\mathcal{C}_\Lambda$, and $S^\Lambda_*(f) = F^\Lambda_* \equiv \langle \boldsymbol{f}_* \rangle$, where $\boldsymbol{f}_* = \underline{T}^\Lambda(\boldsymbol{f})$ and $\boldsymbol{f}$ is induced by $f \in Mor\mathcal{C}_\Lambda$, then

$$S^\Lambda_* : \mathcal{C}_\Lambda \to Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}$$

becomes a functor, called the (*abstract*) Λ-*weak shape functor*. Let us denote by

$$S^\Lambda : \mathcal{C}_\Lambda \to Sh^\Lambda_{(\mathcal{C},\mathcal{D})} = Sh_{(\mathcal{C}_\Lambda,\mathcal{D})}$$

the corresponding (abstract) shape functor. (Obviously, it is the restriction of the abstract shape functor $S : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$.) Then, by construction, the functor S^Λ_* factorizes through S^Λ, i.e. there exists a functor

$$T^\Lambda : Sh^\Lambda_{(\mathcal{C},\mathcal{D})} \to Sh^\Lambda_{*(\mathcal{C},\mathcal{D})},$$

which keeps the objects fixed, such that $S^\Lambda_* = T^\Lambda S^\Lambda$.

$$\begin{array}{ccccc} & & \mathcal{C}_\Lambda & & \\ & S^\Lambda \swarrow & & \searrow S^\Lambda_* & \\ Sh^\Lambda_{(\mathcal{C},\mathcal{D})} & & \underset{T^\Lambda}{\longrightarrow} & & Sh^\Lambda_{*(\mathcal{C},\mathcal{D})} \end{array}$$

Since $\underline{T}^\Lambda$ is faithful keeping the objects fixed, so is T^Λ, and thus, we may treat it to be the inclusion functor, i.e. $Sh^\Lambda_{(\mathcal{C},\mathcal{D})} \subseteq Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}$. Let us clarify this statement in some details. Let $F^\Lambda, F'^\Lambda : X \to Y$ be a pair of shape morphisms of $Sh^\Lambda_{(\mathcal{C},\mathcal{D})}$ such that $T^\Lambda(F^\Lambda) \equiv F^\Lambda_* = F'^\Lambda_* \equiv T^\Lambda(F'^\Lambda)$ in $Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}$. Then $F^\Lambda = \langle \boldsymbol{f} \rangle$ and $F'^\Lambda = \langle \boldsymbol{f}' \rangle$, and thus, $\langle \boldsymbol{f}_* \rangle = F^\Lambda_* = F'^\Lambda_* = \langle \boldsymbol{f}'_* \rangle$, where $\boldsymbol{f}_* = \underline{T}^\Lambda(\boldsymbol{f})$ and

$\boldsymbol{f}'_* = \underline{T}^\Lambda(\boldsymbol{f}')$. Therefore, $\langle \underline{T}^\Lambda(\boldsymbol{f})\rangle = \langle \underline{T}^\Lambda(\boldsymbol{f}')\rangle$, i.e. $\underline{T}^\Lambda(\boldsymbol{f}) \sim \underline{T}^\Lambda(\boldsymbol{f}')$ in pro^Λ_*-$\mathcal{D}$, which means that the diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\underline{T}^\Lambda(\boldsymbol{i})} & \boldsymbol{X}' \\ \underline{T}^\Lambda(\boldsymbol{f}) \downarrow & & \downarrow \underline{T}^\Lambda(\boldsymbol{f}') \\ \boldsymbol{Y} & \xrightarrow[\underline{T}^\Lambda(\boldsymbol{j})]{} & \boldsymbol{Y}' \end{array}$$

in pro^Λ_*-$\mathcal{D}$ commutes. Since the functor $\underline{T}^\Lambda$ is faithful, the diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{i}} & \boldsymbol{X}' \\ \boldsymbol{f} \downarrow & & \downarrow \boldsymbol{f}' \\ \boldsymbol{Y} & \xrightarrow[\boldsymbol{j}]{} & \boldsymbol{Y}' \end{array}$$

in pro^Λ-$\mathcal{D}$ commutes. This means that $\boldsymbol{f} \sim \boldsymbol{f}'$, i.e. $F^\Lambda = \langle \boldsymbol{f}\rangle = \langle \boldsymbol{f}'\rangle = F'^\Lambda$.

By analogy, any such functor $\underline{S}^\Lambda_* : \mathcal{C}_\Lambda \to pro^\Lambda_*$-$\mathcal{D}$ may be called the *realizing* (abstract) Λ-weak shape functor for $(\mathcal{C}_\Lambda, \mathcal{D})$. Obviously, its existence follows, in general, by the axiom of choice. However, the (abstract) Λ-weak shape functor $S^\Lambda_* : \mathcal{C}_\Lambda \to Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}$ does not depend on it.

6. The Weak Shape Category and Related Functors

The above constructed Λ-weak shape category $Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}$ and the corresponding functors, i.e. the corresponding "theory" of the abstract "Λ-weak shape (type)", has at this level a significant defect. Namely, for each Λ, the Λ-weak shape type of a $\mathcal{C}$-object X (which admits a $\mathcal{D}$-expansion over Λ) formally depends on the chosen index set Λ of a $\mathcal{D}$-expansion of X. However, one (naturally) expects that this dependence vanishes, i.e. if a $\mathcal{C}$-object X admits a $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$ over a Λ and a $\mathcal{D}$-expansion $\boldsymbol{p}' : X \to \boldsymbol{X}'$ over a Λ', then one expects that the Λ- and Λ'-weak shape type of X coincide. In other words, there should be a unique (abstract) "weak shape type" of X, that does not depend on any chosen index set Λ for a $\mathcal{D}$-expansion $\boldsymbol{X}$ of X. This leads to the *index set changing problem.* Recall that our index sets are ordered, directed, cofinite, infinite and do not have any maximal element. The index set changing problem is affirmatively solved, in general, by Theorems 3 and 4 below.

Theorem 3. *Let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\mu}, \Lambda)$, $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\mu}, \Lambda)$, $\boldsymbol{X}' = (X'_{\lambda'}, p'_{\lambda'\mu'}, \Lambda')$ *and* $\boldsymbol{Y}' = (Y'_{\lambda'}, q'_{\lambda'\mu'}, \Lambda')$ *be inverse systems of a* $pro^\sim$-$\mathcal{A}$ *such that*

$\boldsymbol{X} \cong \boldsymbol{X}'$ and $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in pro-$\mathcal{A}$. Then, for every pair of isomorphisms $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$, $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$ in pro-$\mathcal{A}$, there exists a bijection

$$h \equiv h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}} : pro_*^{\Lambda}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \to pro_*^{\Lambda'}\text{-}\mathcal{A}(\boldsymbol{X}', \boldsymbol{Y}'),$$
$$\boldsymbol{f}_* \mapsto h(\boldsymbol{f}_*) \equiv \boldsymbol{f}'_*,$$

which preserves isomorphisms. Further, if $\boldsymbol{Y} = \boldsymbol{X}$ and $\boldsymbol{Y}' = \boldsymbol{X}'$, then $h_{\boldsymbol{i}}^{\boldsymbol{i}^{-1}}$ preserves the identity, i.e. $h(\boldsymbol{1}_{\boldsymbol{X}*}) = \boldsymbol{1}_{\boldsymbol{X}'*}$. If, in addition, an isomorphism $\boldsymbol{k} : \boldsymbol{Z}(\Lambda) \to \boldsymbol{Z}'(\Lambda')$ of pro-$\mathcal{A}$ is given, then $h_{\boldsymbol{k}}^{\boldsymbol{i}^{-1}} = h_{\boldsymbol{k}}^{\boldsymbol{j}^{-1}} h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}$.

First of all, we want to prove the next auxiliary lemma.

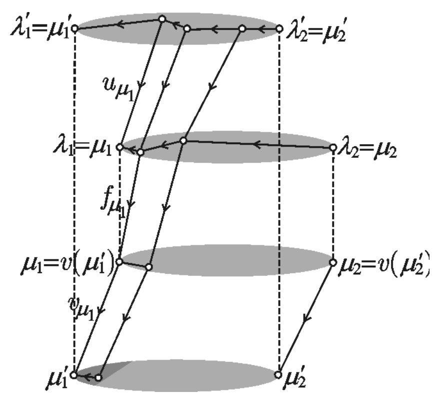
Lemma 4. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\mu}, \Lambda)$, $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\mu}, \Lambda)$, $\boldsymbol{X}' = (X'_{\lambda'}, p'_{\lambda'\mu'}, \Lambda')$ and $\boldsymbol{Y}' = (Y'_{\lambda'}, q'_{\lambda'\mu'}, \Lambda')$ be inverse systems in a category $\mathcal{A}$. Let $(u, u_\mu) : \boldsymbol{X}' \to \boldsymbol{X}$ and $(v, v_{\mu'}) : \boldsymbol{Y} \to \boldsymbol{Y}'$ be special morphisms of inv-$\mathcal{A}$ such that the index functions u and v are unbounded and $uv \geq 1_{\Lambda'}$. Then the morphisms (u, u_μ), $(v, v_{\mu'})$ and each hyperladder $(f_{\boldsymbol{\mu}}) \in inv_*^{\Lambda}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) = \underline{L}(\boldsymbol{X}, \boldsymbol{Y})$ yield a hyperladder $(f'_{\boldsymbol{\mu}'}) \in inv_*^{\Lambda'}\text{-}\mathcal{A}(\boldsymbol{X}', \boldsymbol{Y}')$, formally written as $(f'_{\boldsymbol{\mu}'}) =$ "$(v, v_{\mu'})(f_{\boldsymbol{\mu}})(u, u_\mu)$".*

Moreover, if $(f_{\boldsymbol{\mu}}) \simeq (g_{\boldsymbol{\mu}})$, then $(f'_{\boldsymbol{\mu}'}) \simeq (g'_{\boldsymbol{\mu}'})$.

Proof. Let $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a hyperladder. Given a special pair $(u, u_\mu) : \boldsymbol{X}' \to \boldsymbol{X}$ and $(v, v_{\mu'}) : \boldsymbol{Y} \to \boldsymbol{Y}'$, we have to construct a hyperladder $(f'_{\boldsymbol{\mu}'}) : \boldsymbol{X}' \to \boldsymbol{Y}'$ associated with $(f_{\boldsymbol{\mu}})$. We assume that $u : \Lambda \to \Lambda'$ and $v : \Lambda' \to \Lambda$ are unbounded, and that $uv \geq 1_{\Lambda'}$, i.e. $uv(\mu') \geq \mu'$, for every $\mu' \in \Lambda'$. Recall that u and v are increasing because the given morphisms are special. The idea is to " compose" (u, u_μ), $(f_{\boldsymbol{\mu}})$ and $(v, v_{\mu'})$.

$$\begin{array}{ccc} \boldsymbol{X} & \overset{(u,u_\mu)}{\longleftarrow} & \boldsymbol{X}' \\ (f_{\boldsymbol{\mu}}) \downarrow & & \downarrow (f'_{\boldsymbol{\mu}'}) \\ \boldsymbol{Y} & \underset{(v,v_{\mu'})}{\longrightarrow} & \boldsymbol{Y}' \end{array} .$$

To do this, given a $\boldsymbol{\mu}' \in \boldsymbol{\Lambda}'$, associated with a (μ'_1, μ'_2), we shall construct a corresponding ladder $f'_{\boldsymbol{\mu}'} : \boldsymbol{X}' \to \boldsymbol{Y}'$. Put $\mu_1 = v(\mu'_1)$ and $\mu_2 = v(\mu'_2)$. Let $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ be associated with (μ_1, μ_2), and let $f_{\boldsymbol{\mu}}$ be the corresponding ladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$. We now define the ladder $f'_{\boldsymbol{\mu}'} : \boldsymbol{X}' \to \boldsymbol{Y}'$ by means of $f_{\boldsymbol{\mu}}$ and the restrictions of (u, u_μ) to the appropriate subset of $\boldsymbol{\mu}$ and of $(v, v_{\mu''})$ to the appropriate subset of $\boldsymbol{\mu}'$. More precisely, the index function $f' : J' \to \boldsymbol{\lambda}' = \boldsymbol{\mu}'$, $J' \subseteq \boldsymbol{\mu}'$, of $f'_{\boldsymbol{\mu}'}$ is defined by the composition ufv, where $f : J \to \boldsymbol{\lambda} = \boldsymbol{\mu}$, $J \subseteq \boldsymbol{\mu}$, is the index function of $f_{\boldsymbol{\mu}}$. Notice that $f'(\mu'_1) \equiv ufv(\mu'_1) \geq \lambda'_1 \equiv \mu'_1$ holds by



our assumptions and by $f(\mu_1) \geq \lambda_1 \equiv \mu_1$. However, the initial subset J' could be empty, and thus, $f'_{\boldsymbol{\mu}'}$ could be the empty ladder.

Obviously, such an $f'_{\boldsymbol{\mu}'}$ is empty if and only if $f'(\mu'_1) \notin \boldsymbol{\mu}'$, i.e. if and only if $f'(\mu'_1) \not\leq \lambda'_2 \equiv \mu'_2$. Consider the family $(f'_{\boldsymbol{\mu}'})$, $\boldsymbol{\mu}' \in \boldsymbol{\Lambda}'$, of all such ladders. Let us prove that $(f'_{\boldsymbol{\mu}'}) : \boldsymbol{X}' \to \boldsymbol{Y}'$ is a hyperladder. Let $\mu'_1 \in \Lambda'$ and let $\mu''_1 \geq \mu'_1$. Since $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, is a hyperladder, for $\mu_1 = v(\mu'_1)$ and $v(\mu''_1) \geq \mu_1 \in \Lambda$, there exists a $\lambda^1 \in \Lambda$, $\lambda^1 \geq v(\mu''_1)$, having the appropriate property whenever a $\mu_2 \geq \lambda^1$ is chosen. Notice that

$$u(\lambda^1) \geq uv(\mu''_1) \geq uv(\mu'_1) \geq \lambda'_1 \equiv \mu'_1 \text{ and}$$
$$u(\lambda^1) \geq uv(\mu''_1) \geq \mu''_1 \geq \lambda'_1.$$

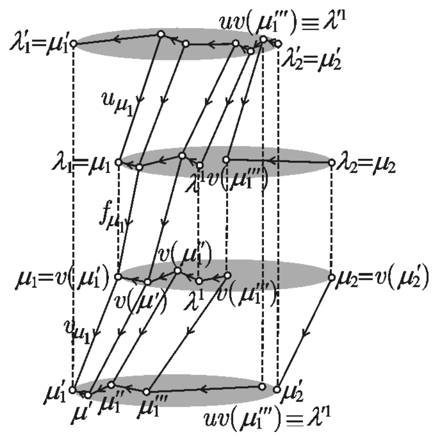
Choose a $\mu'''_1 \in \Lambda'$ such that $\mu'''_1 \geq \mu''_1$ and $v(\mu'''_1) \geq \lambda^1$. Thus, $uv(\mu'''_1) \geq u(\lambda^1)$. Put $\lambda'^1 \equiv uv(\mu'''_1)$. Notice that $\lambda'^1 \geq \mu'''_1 \geq \mu''_1$. Let $\mu'_2 \in \Lambda'$, $\mu'_2 \geq \lambda'^1$. Then the correponding ladder $f'_{\boldsymbol{\mu}'}$ over $\boldsymbol{\mu}' \in \boldsymbol{\Lambda}'$, associated with (μ'_1, μ'_2), satisfies the desired conditions with respect to the pair $\mu'_1 \leq \mu''_1$ and λ'^1.

Indeed, the ladder $f_{\boldsymbol{\mu}}$ over $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, associated with $(\mu_1, \mu_2 \equiv v(\mu'_2))$, fulfills the desired conditions with respect to the pair $\mu_1 \equiv v(\mu'_1) \leq v(\mu''_1)$ and $\lambda^1 \geq v(\mu''_1)$ because of $\mu_2 \geq \lambda^1$. This implies

$$v(\mu'_1) \equiv \mu_1 \leq f(\mu_1) \equiv fv(\mu'_1) \text{ and } fv(\mu''_1) \leq \lambda^1.$$

Consequently, since (u, u_μ) and $(v, v_{\mu'})$ are special and $uv \geq 1_{\Lambda'}$, the ladder $f'_{\boldsymbol{\mu}'}$ fulfills the desired conditions with respect to the pair $\mu'_1 \leq \mu''_1$ and $\lambda'^1 \geq \mu''_1$. More precisely,

$$\mu'_1 \leq uv(\mu'_1) \leq \lambda'_1 \equiv ufv(\mu'_1) \leq ufv(\mu''_1) \leq \lambda'^1 \leq \lambda'_2 \equiv \mu'_2$$



implies that the index function $f' = ufv : J' \to \boldsymbol{\lambda}' = \boldsymbol{\mu}'$ of $f'_{\boldsymbol{\mu}'}$ is defined (at least) for all $\mu' \in \boldsymbol{\mu}'$ such that $\mu'_1 \leq \mu' \leq \mu''_1$, and moreover, $f'(\mu''_1) \leq \lambda'^1$.

Let a hyperladder $(g_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ be given such that $(g_{\boldsymbol{\mu}}) \simeq (f_{\boldsymbol{\mu}})$, and let $(g'_{\boldsymbol{\mu}'}) : \boldsymbol{X}' \to \boldsymbol{Y}'$ be constructed in the same way by means of (u, u_μ), $(g_{\boldsymbol{\mu}})$ and $(v, v_{\mu'})$, i.e. $(g'_{\boldsymbol{\mu}'}) =$ "$(v, v_{\mu'})(g_{\boldsymbol{\mu}})(u, u_\mu)$". Then a straightforward verification shows that $(g'_{\boldsymbol{\mu}'}) \simeq (f'_{\boldsymbol{\mu}'})$. (One has to choose a desired index λ^1_* and then a desired index λ'^1_* so that, for every $\mu'_2 \geq \lambda'^1_*$, the relation $f_{\boldsymbol{\mu}} \simeq g_{\boldsymbol{\mu}}$ holds. Then, $f'_{\boldsymbol{\mu}'} \simeq g'_{\boldsymbol{\mu}'}$ follows by construction.) □

Proof of Theorem 3. Let $\boldsymbol{f}_* = [(f_{\boldsymbol{\mu}})] : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of the category pro^{Λ}_*-$\mathcal{A}$. Let $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$ be a pair of isomorphisms in pro-$\mathcal{A}$. We have to construct a morphism $\boldsymbol{f}'_* = [(f'_{\boldsymbol{\mu}'})] : \boldsymbol{X}' \to \boldsymbol{Y}'$ of the category $pro^{\Lambda'}_*$-$\mathcal{A}$, which should be associated with $\boldsymbol{f}_*$. The idea is to " compose" $\boldsymbol{i}^{-1}$, $\boldsymbol{f}_*$ and $\boldsymbol{j}$ in the diagram below.

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{i}^{-1}} & \boldsymbol{X}' \\ \boldsymbol{f}_* \downarrow & & \downarrow \boldsymbol{f}'_* \\ \boldsymbol{Y} & \xrightarrow[\boldsymbol{j}]{} & \boldsymbol{Y}' \end{array} .$$

Given a $\boldsymbol{\mu}' \in \boldsymbol{\Lambda}'$, associated with a (μ'_1, μ'_2), we shall define the corresponding ladder $f'_{\boldsymbol{\mu}'} : \boldsymbol{X}' \to \boldsymbol{Y}'$ by means of a representative $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ of $\boldsymbol{f}_*$ in $inv^{\sim}_*$-$\mathcal{A}$ and a pair of special representatives $(u, u_\mu) : \boldsymbol{X}' \to \boldsymbol{X}$ of $\boldsymbol{i}^{-1}$ and $(j, j_{\mu'}) : \boldsymbol{Y} \to \boldsymbol{Y}'$ of $\boldsymbol{j}$ in inv-$\mathcal{A}$, according to Lemma 4, i.e. u nad j shuould be unbounded, and $uj(\mu') \geq \mu'$ for every $\mu' \in \Lambda'$. To achieve this, one has to shift inductively the starting index functions u and j – if it is necessary (see Lemmata I. 1.1. and I. 1.2 of [15]). More precisely, one first chooses a pair of special representatives (u, u_μ), $(j, j_{\mu'})$ such that the index functions $j : \Lambda' \to \Lambda$ and $u : \Lambda \to \Lambda'$ are unbounded (Λ and Λ' are unbounded cofinite directed sets!). Since Λ' is directed, for every $\mu' \in \Lambda'$ there exists a $\lambda' \in \Lambda'$ such that $\lambda' \geq \mu', uj(\mu')$. Now one may define a $u' : \Lambda \to \Lambda'$ by putting

$$u'(\mu) = \begin{cases} \lambda', & \lambda' \geq \mu', u(\mu),\, \mu = j(\mu') \in j[\Lambda'], \\ u(\mu), & \mu \in \Lambda \setminus j[\Lambda']. \end{cases}$$

Notice that u' is unbounded and $u'j \geq 1_{\Lambda'}$. Indeed, both claims follow by the fact that $u'j(\mu') = \lambda' \geq \mu'$, for every $\mu' \in \Lambda'$. Further, $u' \geq u$, since each $\mu \in j[\Lambda']$ is some $j(\mu')$. Finally, one chooses a special representative (u'', u''_μ) of $\boldsymbol{i}^{-1}$ such that $u'' \geq u'$ (Λ is cofinite and Λ' is directed). Since $u'' \geq u'$, u'' is unbounded and $u''j \geq u'j \geq 1_{\Lambda'}$. too. So one can obtain a desired pair (u'', u''_μ), $(j, j_{\mu'})$ representing $\boldsymbol{i}^{-1}$, $\boldsymbol{j}$ respectively. By Lemma 4 and its proof, the correspondence $\boldsymbol{f}_* \mapsto \boldsymbol{f}'_* =$ "$\boldsymbol{j}\boldsymbol{f}_*\boldsymbol{i}^{-1}$" is well defined. It defines a function (depending on $\boldsymbol{i}^{-1}$ and $\boldsymbol{j}$)

$$h : pro_*^{\Lambda}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \to pro_*^{\Lambda'}\text{-}\mathcal{A}(\boldsymbol{X}', \boldsymbol{Y}'),\ h(\boldsymbol{f}_*) = \boldsymbol{f}'_*.$$

To see that h is a bijection, one should notice that the same construction (by using the pair $\boldsymbol{i}$, $\boldsymbol{j}^{-1}$) yields the function

$$h' : pro_*^{\Lambda'}\text{-}\mathcal{A}(\boldsymbol{X}', \boldsymbol{Y}') \to pro_*^{\Lambda}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}),\ h'(\boldsymbol{f}'_*) = \boldsymbol{f}_*,$$

such that

$$h'(h(\boldsymbol{f}_*)) = \boldsymbol{f}_* \text{ and } h(h'(\boldsymbol{f}'_*)) = \boldsymbol{f}'_*.$$

Namely, by our construction,

$$(f_{\boldsymbol{\mu}}) \mapsto (f'_{\boldsymbol{\mu}'}) \mapsto (f''_{\boldsymbol{\mu}}) \simeq (f_{\boldsymbol{\mu}})$$

holds. Therefore, the inverse $h' \equiv h^{-1}$ of $h = h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}$ is $h_{\boldsymbol{j}^{-1}}^{\boldsymbol{i}}$. Moreover, if $\boldsymbol{X} \cong \boldsymbol{Y}$ in pro_*^{Λ}-$\mathcal{A}$ and $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$ is an isomorphism of pro_*^{Λ}-$\mathcal{A}$, then $\boldsymbol{f}'_* = h(\boldsymbol{f}_*) =$ "$\boldsymbol{j}\boldsymbol{f}_*\boldsymbol{i}^{-1}$" $: \boldsymbol{X}' \to \boldsymbol{Y}'$ is an isomorphism of $pro_*^{\Lambda'}$-$\mathcal{A}$. Namely, $\boldsymbol{i}$ and $\boldsymbol{j}$ are the isomorphisms of *pro*-$\mathcal{A}$, and an identity morphism of $pro_*^{\sim}$-$\mathcal{A}$ is induced by the corresponding identity morphism of *inv*-$\mathcal{A}$. More precisely, one readily sees that $(h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}(\boldsymbol{f}_*))^{-1} = h_{\boldsymbol{i}}^{\boldsymbol{j}^{-1}}(\boldsymbol{f}_*^{-1})$, where $h_{\boldsymbol{i}}^{\boldsymbol{j}^{-1}} : pro_*^{\Lambda}\text{-}\mathcal{A}(\boldsymbol{Y}, \boldsymbol{X}) \to pro_*^{\Lambda'}\text{-}\mathcal{A}(\boldsymbol{Y}', \boldsymbol{X}')$ is defined via $\boldsymbol{j}^{-1}$ and $\boldsymbol{i}$. The same argument implies that $h_{\boldsymbol{i}}^{\boldsymbol{i}^{-1}}(\boldsymbol{1}_{\boldsymbol{X}*}) =$ "$\boldsymbol{i}\boldsymbol{1}_{\boldsymbol{X}*}\boldsymbol{i}^{-1}$" $= \boldsymbol{1}_{\boldsymbol{X}'*}$. Finally, let $\boldsymbol{k} : \boldsymbol{Z} \to \boldsymbol{Z}'$ be an isomorphism of *pro*-$\mathcal{A}$, where $\boldsymbol{Z}$, $\boldsymbol{Z}'$ are indexed by Λ, Λ' respectively, and let a morphism $\boldsymbol{g}_* : \boldsymbol{Y} \to \boldsymbol{Z}$ of pro_*^{Λ}-$\mathcal{A}$ be given. Let (v, v_μ), $(k, k_{\mu'})$ be a pair of representatives of $\boldsymbol{j}^{-1}$, $\boldsymbol{k}$ respectively, such that v and k are increasing and $vk \geq 1_{\Lambda'}$. By following the first part of this proof, one can also achieve that $uk \geq 1_{\Lambda'}$. Let $(g_{\boldsymbol{\nu}}) \in \boldsymbol{g}_*$, and let

$$(g'_{\boldsymbol{\nu}'}) = \text{“}(k, k_{\mu'})(g_{\boldsymbol{\nu}})(v, v_\mu)\text{”} : \boldsymbol{Y}' \to \boldsymbol{Z}'$$

be the hyperladder existing by Lemma 4 Further, let $(h_{\boldsymbol{\nu}}) \equiv (g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}) = (g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}})$, and let

$$(h'_{\boldsymbol{\nu}'}) = \text{“}(k, k_{\mu'})(h_{\boldsymbol{\nu}})(u, u_\mu)\text{”} : \boldsymbol{X}' \to \boldsymbol{Z}'$$

be obtained in the same manner. Then a straightforward verification shows that

$$(h'_{\boldsymbol{\nu}'}) \simeq (g'_{\boldsymbol{\nu}'} f'_{\boldsymbol{\nu}'}) = (g'_{\boldsymbol{\nu}'})(f'_{\boldsymbol{\mu}'}).$$

In other words, the composition $(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}}) = (g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}) \equiv (h_{\boldsymbol{\nu}})$ yields, by Lemma 4, a hyperladder $(h'_{\boldsymbol{\nu}'})$ which is equivalent (homotopic) to the composition $(g'_{\boldsymbol{\nu}'})(f'_{\boldsymbol{\mu}'}) = (g'_{\boldsymbol{\nu}'} f'_{\boldsymbol{\nu}'})$. Consequently, shortly writing,

$$\begin{aligned} h_{\boldsymbol{k}}^{\boldsymbol{i}^{-1}}(\boldsymbol{g}_* \boldsymbol{f}_*) &= \text{“}\boldsymbol{k}(\boldsymbol{g}_* \boldsymbol{f}_*)\boldsymbol{i}^{-1}\text{”} = \text{“}\boldsymbol{k}(\boldsymbol{g}_* \boldsymbol{j}^{-1} \boldsymbol{j} \boldsymbol{f}_*)\boldsymbol{i}^{-1}\text{”} \\ &= \text{“}(\boldsymbol{k}\boldsymbol{g}_* \boldsymbol{j}^{-1})(\boldsymbol{j} \boldsymbol{f}_*)\boldsymbol{i}^{-1})\text{”} = h_{\boldsymbol{k}}^{\boldsymbol{j}^{-1}}(\boldsymbol{g}_*) h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}(\boldsymbol{f}_*). \quad \square \end{aligned}$$

In the case of expansions, Theorem 3 admits the following significant improvement.

Theorem 4. *Let $(\mathcal{C}, \mathcal{D})$ be a category pair such that $\mathcal{D} \subseteq \mathcal{C}$ is dense and full. If $X, Y \in Ob\mathcal{C}$ admit $\mathcal{D}$-expansions indexed by Λ and by Λ', then there exists a bijection*

$$\widetilde{h} \equiv \widetilde{h}_Y^X : Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}(X, Y) \to Sh^{\Lambda'}_{*(\mathcal{C},\mathcal{D})}(X, Y), \ \widetilde{h}(F^{\Lambda}_*) = F^{\Lambda'}_*$$

which preserves isomorphisms. If $Y = X$, then $\widetilde{h}$ also preserves the identity. Moreover, the family of all such bijections induces a functor

$$H^{\Lambda,\Lambda'} : Sh^{\Lambda}_{*(\mathcal{C}',\mathcal{D})} \to Sh^{\Lambda'}_{*(\mathcal{C}',\mathcal{D})},$$

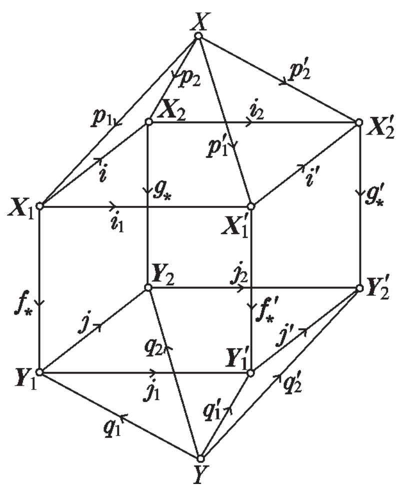
which is a category isomorphism keeping the objects fixed, whenever $\mathcal{C}' \subseteq \mathcal{C}$ is the full subcategory determined by all the objects admitting $\mathcal{D}$-expansions over Λ and over Λ'.

Proof. Let $(\mathcal{C}, \mathcal{D})$ be a category pair such that $\mathcal{D} \subseteq \mathcal{C}$ is dense. Let $X, Y \in Ob\mathcal{C}$. Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be $\mathcal{D}$-expansions such that both $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same (ordered, cofinite and infinite without a maximal element) index set Λ, and let $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be another pair of $\mathcal{D}$-expansions such that both $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ have the same index set Λ'. Then there exist unique (with respect to $\boldsymbol{p}$, $\boldsymbol{p}'$ and $\boldsymbol{q}$, $\boldsymbol{q}'$) isomorphisms $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$ such that $\boldsymbol{p}' = \boldsymbol{i}\boldsymbol{p}$ and $\boldsymbol{q}' = \boldsymbol{j}\boldsymbol{q}$. By Theorem 3, there exists the bijection

$$h \equiv h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}} : pro_*^{\Lambda}\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y}) \to pro_*^{\Lambda'}\text{-}\mathcal{D}(\boldsymbol{X}', \boldsymbol{Y}'),$$
$$h(\boldsymbol{f}_*) \equiv \boldsymbol{f}'_* = \text{“}\boldsymbol{j}\boldsymbol{f}_*\boldsymbol{i}^{-1}\text{”}.$$

defined by means of these $\boldsymbol{i}$ and $\boldsymbol{j}$.

Let $\boldsymbol{p}_{1,2} : X \to \boldsymbol{X}_{1,2}$ and $\boldsymbol{q}_{1,2} : Y \to \boldsymbol{Y}_{1,2}$ be $\mathcal{D}$-expansions over the same Λ, and let $\boldsymbol{p}'_{1,2} : X \to \boldsymbol{X}'_{1,2}$ and $\boldsymbol{q}'_{1,2} : Y \to \boldsymbol{Y}'_{1,2}$ be $\mathcal{D}$-expansions over the same Λ'. Let $\boldsymbol{i}_{1,2} : \boldsymbol{X}_{1,2} \to \boldsymbol{X}'_{1,2}$, $\boldsymbol{j}_{1,2} : \boldsymbol{Y}_{1,2} \to \boldsymbol{Y}'_{1,2}$, $\boldsymbol{i} : \boldsymbol{X}_1 \to \boldsymbol{X}_2$, $\boldsymbol{i}' : \boldsymbol{X}'_1 \to \boldsymbol{X}'_2$, $\boldsymbol{j} : \boldsymbol{Y}_1 \to \boldsymbol{Y}_2$ and $\boldsymbol{j}' : \boldsymbol{Y}'_1 \to \boldsymbol{Y}'_2$ be the corresponding unique isomorphisms. Let $\boldsymbol{f}_* \in pro_*^{\Lambda}\text{-}\mathcal{D}(\boldsymbol{X}_1, \boldsymbol{Y}_1)$ and $\boldsymbol{g}_* \in pro_*^{\Lambda}\text{-}\mathcal{D}(\boldsymbol{X}_2, \boldsymbol{Y}_2)$, and let $\boldsymbol{f}'_* = h_1(\boldsymbol{f}_*) \in pro_*^{\Lambda'}\text{-}\mathcal{D}(\boldsymbol{X}'_1, \boldsymbol{Y}'_1)$ and $\boldsymbol{g}'_* = h_2(\boldsymbol{g}_*) \in pro_*^{\Lambda}\text{-}\mathcal{D}(\boldsymbol{X}_2, \boldsymbol{Y}_2)$, where $h_{1,2}$ are the corresponding bijections, i.e. $\boldsymbol{f}'_* = \text{“}\boldsymbol{j}_1\boldsymbol{f}_*\boldsymbol{i}_1^{-1}\text{”}$ and $\boldsymbol{g}'_* = \text{“}\boldsymbol{j}_2\boldsymbol{g}_*\boldsymbol{i}_2^{-1}\text{”}$.



We are to prove that $\boldsymbol{f}_* \sim \boldsymbol{g}_*$ implies $\boldsymbol{f}'_* \sim \boldsymbol{g}'_*$. Recall that $\boldsymbol{f}_* \sim \boldsymbol{g}_*$ means $\boldsymbol{g}_* = \boldsymbol{j}_*\boldsymbol{f}_*\boldsymbol{i}_*^{-1}$, where $\boldsymbol{i}_* = \underline{T}(\boldsymbol{i})$, $\boldsymbol{j}_* = \underline{T}(\boldsymbol{j})$ and $\underline{T} \equiv \underline{T}^{\Lambda}$ is the appropriate functor (see Theorem 2). Notice that $\boldsymbol{i}' = \boldsymbol{i}_2\boldsymbol{i}\boldsymbol{i}_1^{-1}$ and $\boldsymbol{j}' = \boldsymbol{j}_2\boldsymbol{j}\boldsymbol{j}_1^{-1}$. Put $\boldsymbol{i}'_* = \underline{T}'(\boldsymbol{i}')$ and $\boldsymbol{j}'_* = \underline{T}'(\boldsymbol{j}')$, where $\underline{T}' \equiv \underline{T}^{\Lambda'}$. Then, shortly writing,

$$\boldsymbol{g}'_* = \text{“}\boldsymbol{j}_2\boldsymbol{g}_*\boldsymbol{i}_2^{-1}\text{”} = \text{“}\boldsymbol{j}_2\boldsymbol{j}_*\boldsymbol{f}_*\boldsymbol{i}_*^{-1}\boldsymbol{i}_2^{-1}\text{”} = \text{“}\boldsymbol{j}'_*\boldsymbol{j}_1\boldsymbol{f}_*\boldsymbol{i}_1^{-1}\boldsymbol{i}'^{-1}_*\text{”} = \text{“}\boldsymbol{j}'_*\boldsymbol{f}'_*\boldsymbol{i}'^{-1}_*\text{”},$$

which is the genuine composition $\boldsymbol{j}'_*\boldsymbol{f}'_*\boldsymbol{i}'^{-1}_*$. More precisely, by the last statement of Theorem 3,

$$\boldsymbol{g}'_* = h_{\boldsymbol{j}_2}^{\boldsymbol{i}_2^{-1}}(\boldsymbol{g}_*) = h_{\boldsymbol{j}_2}^{\boldsymbol{i}_2^{-1}}(\boldsymbol{j}_*\boldsymbol{f}_*\boldsymbol{i}_*^{-1}) = h_{\boldsymbol{j}_2}^{\boldsymbol{j}_1^{-1}}(\boldsymbol{j}_*)h_{\boldsymbol{j}_1}^{\boldsymbol{i}_1^{-1}}(\boldsymbol{f}_*)h_{\boldsymbol{i}_1}^{\boldsymbol{i}_2^{-1}}(\boldsymbol{i}_*^{-1})$$

$$= h_{\boldsymbol{j}_2}^{\boldsymbol{j}_1^{-1}}(\boldsymbol{j}_*) h_{\boldsymbol{j}_1}^{\boldsymbol{i}_1^{-1}}(\boldsymbol{f}_*)(h_{\boldsymbol{i}_2}^{\boldsymbol{i}_1^{-1}}(\boldsymbol{i}_*))^{-1} = \boldsymbol{j}'_* \boldsymbol{f}'_* \boldsymbol{i}'^{-1}_*.$$

Thus, $\boldsymbol{f}'_* \sim \boldsymbol{g}'_*$. Therefore, the family of all appropriate bijections $h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}$ defines a function

$$\widetilde{h} \equiv \widetilde{h}_Y^X : Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}(X,Y) \to Sh^{\Lambda'}_{*(\mathcal{C},\mathcal{D})}(X,Y),$$
$$\widetilde{h}(F_*^{\Lambda}) = F_*^{\Lambda'},$$

where $F_*^{\Lambda'} = \langle \boldsymbol{f}'_* \rangle$, $\boldsymbol{f}'_* = h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}(\boldsymbol{f}_*)$, $\langle \boldsymbol{f}_* \rangle = F_*^{\Lambda}$ and $h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}$ is the bijection determined by the chosen pairs of $\mathcal{D}$-expansions $\boldsymbol{p}, \boldsymbol{p}' : X \to \boldsymbol{X}, \boldsymbol{X}'$, $\boldsymbol{q}, \boldsymbol{q}' : Y \to \boldsymbol{Y}, \boldsymbol{Y}'$. The above proof assures that $\widetilde{h}$ does not depend on the chosen expansions, i.e. on the chosen " representative" $h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}$. The function $\widetilde{h}$ is a bijection that preserves isomorphisms, because every $h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}$ is such a function. In addition, $\widetilde{h}(1_{X*}^{\Lambda}) = 1_{X*}^{\Lambda'}$ because, by Theorem 3, $h_{\boldsymbol{i}}^{\boldsymbol{i}^{-1}}(\boldsymbol{1}_{\boldsymbol{X}*}) = \boldsymbol{1}_{\boldsymbol{X}^*}$. Finally, let us define

$$H^{\Lambda,\Lambda'} : Sh^{\Lambda}_{*(\mathcal{C}',\mathcal{D})} \to Sh^{\Lambda'}_{*(\mathcal{C}',\mathcal{D})}$$

by putting

$$H^{\Lambda,\Lambda'}(X) = X, \ X \in Ob\mathcal{C}',$$

and, for a morphism $F_*^{\Lambda} : X \to Y$ of $Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$, $X, Y \in Ob\mathcal{C}'$, by putting

$$H^{\Lambda,\Lambda'}(F_*^{\Lambda}) = \widetilde{h}(F_*^{\Lambda}) \equiv F_*^{\Lambda'}.$$

It is left to prove that

$$\widetilde{h}_Z^X(G_*^{\Lambda} F_*^{\Lambda}) = \widetilde{h}_Z^Y(G_*^{\Lambda}) \widetilde{h}_Y^X(F_*^{\Lambda}).$$

It suffices to verify the corresponding condition for the representatives, i.e. that

$$h_{\boldsymbol{k}}^{\boldsymbol{j}^{-1}}(\boldsymbol{g}_*) h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}}(\boldsymbol{f}_*) = h_{\boldsymbol{k}}^{\boldsymbol{i}^{-1}}(\boldsymbol{g}_* \boldsymbol{f}_*).$$

However, this follows by the last statement of Theorem 3. □

Theorem 4 allows us to define a natural equivalence relation on the appropriate subclass of $Mor(pro_*^{\sim}\text{-}\mathcal{D})$ as follows. Let $(\mathcal{C}, \mathcal{D})$ be a category pair such that $\mathcal{D} \subseteq \mathcal{C}$ is dense and full. Let $X, Y \in Ob\mathcal{C}$. Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be $\mathcal{D}$-expansions over the same Λ, and let $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be $\mathcal{D}$-expansions over the same Λ'. A morphism $\boldsymbol{f}_* \in pro_*^{\Lambda}\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$ is said to be *equivalent to* a morphism $\boldsymbol{f}'_* \in pro_*^{\Lambda'}\text{-}\mathcal{D}(\boldsymbol{X}', \boldsymbol{Y}')$, denoted by $\boldsymbol{f}_* \approx \boldsymbol{f}'_*$, provided $\boldsymbol{f}'_* = h(\boldsymbol{f}_*)$ (= "$\boldsymbol{j}\boldsymbol{f}_*\boldsymbol{i}^{-1}$"), where h is the appropriate bijection determined by the expansions $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{q}$ and $\boldsymbol{q}'$. One readily sees that $\approx$ is an equivalence relation. (Transitivity follows by the fact that the composition of two appropriate bijections is the corresponding bijection, i.e. $h_{\boldsymbol{j}'}^{\boldsymbol{i}'^{-1}} h_{\boldsymbol{j}}^{\boldsymbol{i}^{-1}} = h_{\boldsymbol{j}'\boldsymbol{j}}^{(\boldsymbol{i}'\boldsymbol{i})^{-1}}$.) The equivalence class of an $\boldsymbol{f}_*$ is denoted by $\lfloor \boldsymbol{f}_* \rfloor$. Furthermore, if $\boldsymbol{r} : Z \to \boldsymbol{Z}$ is a $\mathcal{D}$-expansion over Λ and $\boldsymbol{r}' : Z \to \boldsymbol{Z}'$ is a $\mathcal{D}$-expansion over Λ', and if $\boldsymbol{g}_* \in pro_*^{\Lambda}$-$\mathcal{D}(\boldsymbol{Y}, \boldsymbol{Z})$, $\boldsymbol{g}'_* \in pro_*^{\Lambda'}$-$\mathcal{D}(\boldsymbol{Y}', \boldsymbol{Z}')$ and (in addition to $\boldsymbol{f}_* \approx \boldsymbol{f}'_*$) $\boldsymbol{g}_* \approx \boldsymbol{g}'_*$, then $\boldsymbol{g}_* \boldsymbol{f}_* \approx \boldsymbol{g}'_* \boldsymbol{f}'_*$ by Theorem 4. Thus, we have established the following corollary.

Corollary 2. *Let $(\mathcal{C}, \mathcal{D})$ be a category pair such that $\mathcal{D} \subseteq \mathcal{C}$ is dense and full. Let $\mathcal{M}_*$ be the class which consists of all Λ-weak shape morphism classes $Mor(Sh_{*(\mathcal{C},\mathcal{D})}^{\Lambda})$. Then the equivalence relation $\boldsymbol{f}_* \approx \boldsymbol{f}'_*$ induces an equivalence relation $F_*^{\Lambda} = \langle \boldsymbol{f}_* \rangle \approx \langle \boldsymbol{f}'_* \rangle = F_*^{\Lambda'}$ on $\mathcal{M}_*$, which is compatible with the compositions in the corresponding categories. The equivalence class of an F_*^{Λ}, denoted by $\lfloor F_*^{\Lambda} \rfloor \equiv F_*$, does not depend on any particular index set.*

Consequently, for every category pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{C}$ is dense and full, we are now able to define the abstract *weak shape category* $Sh_{*(\mathcal{C},\mathcal{D})}$. The *objects* of $Sh_{*(\mathcal{C},\mathcal{D})}$ are those of $\mathcal{C}$, while the *morphisms* of $Sh_{*(\mathcal{C},\mathcal{D})}$ are the above defined equivalence classes $F_* \equiv \lfloor F_*^{\Lambda} \rfloor = \lfloor \langle \boldsymbol{f}_* \rangle \rfloor$, i.e. they are the equivalence classes $\lfloor \langle \boldsymbol{f}_* \rangle \rfloor$ of the equivalence classes $\langle \boldsymbol{f}_* \rangle$ of morphisms $\boldsymbol{f}_*$ of $pro_*^{\sim}$-$\mathcal{D}$. (Recall that each $\boldsymbol{f}_*$ is also an equivalence – "homotopy" – class $[(f_{\boldsymbol{\mu}})]$ of a hyperladder $(f_{\boldsymbol{\mu}})$.) Clearly, by Theorem 4 and Corollary 2,

$$Sh_{*(\mathcal{C},\mathcal{D})}(X, Y) \approx Sh_{*(\mathcal{C},\mathcal{D})}^{\Lambda}(X, Y) \approx pro_*^{\Lambda}\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$$

whenever X and Y admit $\mathcal{D}$-expansions over the same index set Λ. Thus, every *weak shape morphism* $F_* : X \to Y$ admits a (formal) representing diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xleftarrow{\boldsymbol{p}} & X \\ \boldsymbol{f}_* \downarrow & & \downarrow F_*^{\Lambda} \\ \boldsymbol{Y} & \xleftarrow[\boldsymbol{q}]{} & Y \end{array},$$

via an $Sh_{*(\mathcal{C},\mathcal{D})}^{\Lambda}$, i.e. via a pro_*^{Λ}-$\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$. The *identity* weak shape morphism 1_{X*} on an object X is the equivalence class $\lfloor 1_{X*}^{\Lambda} \rfloor$ of the identity $1_{X*}^{\Lambda} \in Sh_{*(\mathcal{C},\mathcal{D})}^{\Lambda}(X, X)$ – provided X admits a $\mathcal{D}$-expansion over Λ. In order to define the *composition* of two weak shape morphisms $F_* \equiv \lfloor F_*^{\Lambda} \rfloor : X \to Y$ and $G_* \equiv \lfloor G_*^{M} \rfloor : Y \to Z$, we need Corollary 2 and the following simple fact:

Lemma 5. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse systems in a category $\mathcal{A}$, both indexed by the same index set, and let $\boldsymbol{Y}'$ and $\boldsymbol{Z}$ be inverse systems in $\mathcal{A}$, both indexed by the same index set. If $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in pro-$\mathcal{A}$ (pro$^{\sim}$-$\mathcal{A}$), then there exist inverse* *systems* $\boldsymbol{X}'$, $\boldsymbol{Y}''$ *and* $\boldsymbol{Z}'$ *in* $\mathcal{A}$, *all indexed by the same index set, such that* $\boldsymbol{X}' \cong \boldsymbol{X}$, $\boldsymbol{Y}'' \cong \boldsymbol{Y}$ $(\cong \boldsymbol{Y}')$ *and* $\boldsymbol{Z}' \cong \boldsymbol{Z}$ *in pro-*$\mathcal{A}$ *(pro*$^{\sim}$*-*$\mathcal{A}$*).*

Proof. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be indexed by a Λ, and let $\boldsymbol{Y}'$ and $\boldsymbol{Z}$ be indexed by an M. Put $N = \Lambda \times M$ and order it coordinatewise, i.e. $\nu = (\lambda, \mu) \leq (\lambda', \mu') = \nu'$ if and only if $\lambda \leq \lambda'$ in Λ and $\mu \leq \mu'$ in M. Observe that, if Λ and M are infinite, cofinite and ordered having no maximal elements, then so is N. Thus, the index set N may serve in the case of *pro*$^{\sim}$-$\mathcal{A}$ too. Now, for every $\nu = (\lambda, \mu) \in N$, put

$$X'_\nu = X_\lambda,\ Y''_\nu = Y_\lambda \text{ (or } Y'_\mu),\ Z'_\nu = Z_\mu,$$

and for every related pair $\nu = (\lambda, \mu) \leq (\lambda', \mu') = \nu'$ in N, put

$$p'_{\nu\nu'} = p_{\lambda\lambda'},\ q''_{\nu\nu'} = q_{\lambda\lambda'} \text{ (or } q'_{\mu\mu'}),\ r'_{\nu\nu'} = r_{\mu\mu'}.$$

It is readily seen that $\boldsymbol{X}' \cong \boldsymbol{X}$, $\boldsymbol{Y}'' \cong \boldsymbol{Y}$ $(\cong \boldsymbol{Y}')$ and $\boldsymbol{Z}' \cong \boldsymbol{Z}$ in *pro*-$\mathcal{A}$ (*pro*$^{\sim}$-$\mathcal{A}$) hold. □

By Lemma 5, X, Y and Z admit $\mathcal{D}$-expansions over the same index set N. Thus, we can define the composition $G_*F_* : X \to Z$ of F_* and G_* to be the equivalence class $\lfloor G_*^N F_*^N \rfloor$ of the composition $G_*^N F_*^N$ of the representatives $F_*^N : X \to Y$ of F_* and $G_*^N : Y \to Z$ of G_* in $Sh^N_{*(\mathcal{C},\mathcal{D})}$, i.e.

$$G_*F_* = \lfloor G_*^M \rfloor \lfloor F_*^\Lambda \rfloor = \lfloor G_*^N \rfloor \lfloor F_*^N \rfloor = \lfloor G_*^N F_*^N \rfloor .$$

Then Corollary 2 guaranties that the composition of weak shape morphisms is well defined. Further, by Theorem 4, $F_* 1_{X*} = F_* = 1_{Y*} F_*$ holds. It remains to verify that this composition is associative. First, by Theorem 4 and its proof, if $H_*^\Lambda = G_*^\Lambda F_*^\Lambda$ and $H_*^{\Lambda'} = \widetilde{h}(H_*^\Lambda)$ (hereby Λ' may be an enlargement of Λ according to Lemma 5, then $H_*^{\Lambda'} = G_*^{\Lambda'} F_*^{\Lambda'}$. Now, let $F_* \equiv \lfloor F_*^\Lambda \rfloor : X \to Y$, $G_* \equiv \lfloor G_*^M \rfloor : Y \to Z$ and $H_* = \lfloor H_*^N \rfloor : Z \to W$ be weak shape morphisms. Then,

$$\begin{aligned} H_*(G_*F_*) &= \lfloor H_*^N \rfloor (\lfloor G_*^M \rfloor \lfloor F_*^\Lambda \rfloor) = \lfloor H_*^N \rfloor \left(\left\lfloor G_*^{M'} F_*^{M'} \right\rfloor\right) \\ &\equiv \lfloor H_*^N \rfloor \left\lfloor U_*^{M'} \right\rfloor = \left\lfloor H_*^{N'} U_*^{N'} \right\rfloor = \left\lfloor H_*^{N'} (G_*^{N'} F_*^{N'}) \right\rfloor = \left\lfloor (H_*^{N'} G_*^{N'}) F_*^{N'} \right\rfloor \\ &= \left\lfloor H_*^{N'} G_*^{N'} \right\rfloor \left\lfloor F_*^{N'} \right\rfloor = \left(\left\lfloor H_*^{N'} \right\rfloor \left\lfloor G_*^{N'} \right\rfloor\right) \left\lfloor F_*^{N'} \right\rfloor = (H_*G_*)F_*. \end{aligned}$$

This completes the construction of the (abstract) weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$. Observe that Theorem 4 and the above construction allows to define the (abstract) *weak shape functor*

$$S_* : \mathcal{C} \to Sh_{*(\mathcal{C},\mathcal{D})}$$

by putting

$$S_*(X) = X$$

and, for an $f : X \to Y$,

$$S_*(f) = F_*,$$

such that

$$F_* = \lfloor S_*^\Lambda(f) \rfloor \equiv \lfloor F_*^\Lambda \rfloor = \lfloor \langle \boldsymbol{f}_* \rangle \rfloor,$$

where $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$ in pro_*^Λ-$\mathcal{D}$ is associated with an $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in pro^Λ-$\mathcal{D}$, that is induced by $f : X \to Y$. Similarly to a "Λ-case", the functor S_* factorizes through the shape functor $S : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$, i.e. there exists a faithful functor

$$T : Sh_{(\mathcal{C},\mathcal{D})} \to Sh_{*(\mathcal{C},\mathcal{D})}$$

such that $S_* = TS$,

$$\begin{array}{ccc} & \mathcal{C} & \\ S \swarrow & & \searrow S_* \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{T}{\longrightarrow} & Sh_{*(\mathcal{C},\mathcal{D})} \end{array}.$$

Thus, one may consider the shape category $Sh_{(\mathcal{C},\mathcal{D})}$ to be a subcategory of the weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$, $Sh_{(\mathcal{C},\mathcal{D})} \subseteq Sh_{*(\mathcal{C},\mathcal{D})}$, on the same object class $Ob\mathcal{C}$, such that the shape morphism sets $Sh_{(\mathcal{C},\mathcal{D})}(X,Y)$ are essentially enriched. Indeed, in general, even in the case of $P \equiv X, Y \equiv Q \in Ob\mathcal{D}$,

$$Sh_{*(\mathcal{C},\mathcal{D})}(P,Q) \supsetneqq Sh_{(\mathcal{C},\mathcal{D})}(P,Q) \approx \mathcal{C}(P,Q)$$

(see Example 3 of the next section and [21], description in Section 6, and apply to $n = 1$). The main thing is that there can exist an isomorphism in $Sh_{*(\mathcal{C},\mathcal{D})}(X,Y)$, while there is no isomorphism in $Sh_{(\mathcal{C},\mathcal{D})}(X,Y)$ (see Corollary 6 of Section 7 below). Of course, if $P, Q \in Ob\mathcal{D}$, then the construction of the weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$ insures that $P \cong Q$ in $\mathcal{D}$ (or $\mathcal{C}$), $Sh(P) = Sh(Q)$ and $Sh_*(P) = Sh_*(Q)$ are mutually equivalent conditions.

7. Weak Shape Versus Coarse Shape

In their recent paper [10], N. Koceić Bilan and the first named author introduced the notion of (abstract) coarse shape for a category pair $(\mathcal{C},\mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{C}$ is dense. The construction of the (abstract) coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ is quite similar to that of the shape category $Sh_{(\mathcal{C},\mathcal{D})}$, so that its realizing category

pro^*-$\mathcal{D} \supseteq pro$-$\mathcal{D}$ ("$*$-pro-category") has morphism sets pro^*-$\mathcal{D}(\boldsymbol{X},\boldsymbol{Y}) \supseteq pro$-$\mathcal{D}(\boldsymbol{X},\boldsymbol{Y})$ which are essentially enriched. There is a faithful functor ("inclusion") $J : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^*_{(\mathcal{C},\mathcal{D})}$, and there are $\mathcal{C}$-objects (even metric continua) such that $Sh^*(X) = Sh^*(Y)$ and $Sh(X) \neq Sh(Y)$. In this section we shall prove that the weak shape is, in general, coarser than the coarse shape, while they coincide on a class of $\mathcal{C}$-objects admitting countable $\mathcal{D}$-expansions (for instance, on metrizable compacta). However, even in such a case, a set of weak shape morphisms is essentially richer than the corresponding set of coarse shape morphisms. First, recall the definition of pro^*-$\mathcal{A}$ (see [10], Section 3.2).

Let $\mathcal{A}$ be a category, and let $(inv$-$\mathcal{A})^*$ be the corresponding $*$-inv-category, i.e. $Ob(inv$-$\mathcal{A})^* = Ob(inv$-$\mathcal{A})$ and $(inv$-$\mathcal{A})^*(\boldsymbol{X},\boldsymbol{Y})$ consists of all $*$ *-morphisms* (originally, S^*-morphisms) (f, f^n_μ) of $\boldsymbol{X}$ to $\boldsymbol{Y}$, which are defined as follows:

$f : M \to \Lambda$ is a function (the index function),

$f^n_\mu : X_{f(\mu)} \to Y_\mu$ is an $\mathcal{A}$-morphism, $\mu \in M$, $n \in \mathbb{N}$, and

$(\forall \mu \leq \mu' \text{ in } M)(\exists \lambda \geq f(\mu), f(\mu') \text{ in } \Lambda)(\exists n \in \mathbb{N})(\forall n' \geq n)$

$f^{n'}_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f^{n'}_{\mu'} p_{f(\mu')\lambda}$.

If f is increasing and $\lambda = f(\mu')$, then (f, f^n_μ) is said to be *special*. A $*$-morphism $(f, f^n_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *equivalent* to a $*$-morphism $(f', f'^n_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f^n_\mu) \sim (f', f'^n_\mu)$, provided

$(\forall \mu \in M)(\exists \lambda \geq f(\mu), f'(\mu) \text{ in } \Lambda)(\exists n \in \mathbb{N})(\forall n' \geq n)$

$f^{n'}_\mu p_{f(\mu)\lambda} = f'^{n'}_\mu p_{f'(\mu)\lambda}$.

The relation $\sim$ is a natural equivalence relation on $(inv$-$\mathcal{A})^*$, and the corresponding quotient category $(inv$-$\mathcal{A})^*/\sim$ is denoted by pro^*-$\mathcal{A}$.

We also need the notion of a "reduced $*$-pro-category" (compare the "$*$-reduced pro-category" in Section 3). Let $(inv$-$\mathcal{A})^*_\sim \subseteq (inv$-$\mathcal{A})^*$ be the subcategory defined as follows:

$$Ob(inv\text{-}\mathcal{A})^*_\sim \subseteq Ob(inv\text{-}\mathcal{A})^* = Ob(inv\text{-}\mathcal{A})$$

consists of all inverse systems in $\mathcal{A}$ over infinite cofinite directed ordered index sets having no maximal element; if $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ belong to $Ob(inv$-$\mathcal{A})^*_\sim$, let

$$(inv\text{-}\mathcal{A})^*_\sim(\boldsymbol{X},\boldsymbol{Y}) = \begin{cases} \emptyset, & \Lambda \neq M\,, \\ (inv\text{-}\mathcal{A})^*(\boldsymbol{X},\boldsymbol{Y}), & \Lambda = M\,. \end{cases}$$

Let

$$pro^*_\sim\text{-}\mathcal{A} = (inv\text{-}\mathcal{A})^*_\sim/(\simeq)$$

be the corresponding quotient category. Clearly, $pro^*_\sim$-$\mathcal{A} \subseteq pro^*$-$\mathcal{A}$, and it "is full" whenever $\boldsymbol{X}$ and $\boldsymbol{Y}$ share the common index set. Notice that, for every prescribed index set Λ, there also exists the (sub)category $(inv\text{-}\mathcal{A})^*_\Lambda \subseteq (inv\text{-}\mathcal{A})^*_\sim$ as well as $pro^*_\Lambda\text{-}\mathcal{A} \subseteq pro^*_\sim\text{-}\mathcal{A}$.

Theorem 5. *For every category $\mathcal{A}$, there exists a functor*

$$\underline{W} : pro^*_\sim\text{-}\mathcal{A} \to pro^\sim_*\text{-}\mathcal{A},$$

which keeps the objects fixed. The same holds in the case of a fixed index set Λ. In the sequential case ($\Lambda = \mathbb{N}$), $\underline{W} : tow^*\text{-}\mathcal{A} \to tow_*\text{-}\mathcal{A}$ is a faithful functor.

In order to prove this theorem, we first need to prove a few technical facts.

Lemma 6. *Every hyperladder $(f_{\boldsymbol{\mu}}) \in inv^\Lambda_*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) = \underline{L}(\boldsymbol{X}, \boldsymbol{Y})$ yields an equivalent hyperladder $(f'_{\boldsymbol{\mu}}) \in inv^\Lambda_*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ such that all of its (nonempty) ladders, $f'_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, have the index functions which are restrictions of the same increasing function $\phi : \Lambda \to \Lambda$, $\phi \geq 1_\Lambda$. Moreover, if $(f_{\boldsymbol{\mu}}) \simeq (g_{\boldsymbol{\mu}})$ then $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$, such that there exists a unique increasing function $\chi : \Lambda \to \Lambda$, $\chi \geq 1_\Lambda$, realizing $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$, i.e. (for the appropriate indices $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ and $\mu \in J^* \subseteq \boldsymbol{\mu}$)*

$$f'_\mu p_{\phi(\mu)\chi(\mu)} = g'_\mu p_{\psi(\mu)\chi(\mu)}.$$

Proof. The proof consists of two steps. In the first one we provide a family (ϕ_μ), $\mu \in \Lambda$, of index functions, such that every ϕ_μ is the index function for all the (nonempty) ladders $f_{\boldsymbol{\mu}}$ with $\mu_1 = \mu$. Then, in the second step, we provide a common index function ϕ for all the (nonempty) ladders.

Step 1. For every $\mu \in \Lambda$, let $\Lambda_\mu = \{\lambda \in \Lambda \mid \lambda \geq \mu\} \subseteq \Lambda$. Given a $\mu_1 \in \Lambda$, we are to define an increasing function

$$\phi_{\mu_1} : \Lambda_{\mu_1} \to \Lambda_{\mu_1},\ \phi_{\mu_1}(\mu) \geq \mu \text{ for every } \mu \in \Lambda_{\mu_1}.$$

Since $(f_{\boldsymbol{\mu}})$ is a hyperladder, for μ_1 and $\mu'_1 = \mu_1$, there exists a $\lambda^1 \geq \mu'_1$ such that, for every $\mu_2 \geq \lambda^1$, the index function of the corresponding ladder $f_{\boldsymbol{\mu}}$ satisfies $f(\mu_1) \leq \lambda^1$. Put $\phi_{\mu_1}(\mu_1) = \lambda^1$. Notice that, by the property of a hyperladder, for every $\mu'_1 \geq \mu_1$, there exists an upper bound $\lambda^1(\mu_1, \mu'_1) \geq \mu'_1$ for the index functions of all the ladders $f_{\boldsymbol{\mu}}$ with $\mu_2 \geq \lambda^1$. Assume that $\phi_{\mu_1}(\mu)$ is defined for all $\mu \in \Lambda_{\mu_1}$, $|\mu| < n \in \mathbb{N}$ in Λ_{μ_1}, so that

$$\phi_{\mu_1}(\mu) \geq \mu, \lambda^1(\mu_1, \mu) \quad \text{and}$$

$$\phi_{\mu_1}(\mu') \leq \phi_{\mu_1}(\mu'') \text{ whenever } \mu_1 \leq \mu' \leq \mu'' \leq \mu.$$

Let $\mu \in \Lambda_{\mu_1}$, $|\mu| = n$ in Λ_{μ_1}. Let $\mu^1 = \mu_1, \ldots, \mu^n$ be all the predecessors of μ in Λ_{μ_1}. Then $|\mu^i| < n$, and thus, $\phi_{\mu_1}(\mu^i)$, $i = 1, \ldots, n$, are already defined. Since Λ_{μ_1} is cofinite, there exists a

$$\lambda \geq \mu, \lambda^1(\mu_1, \mu), \phi_{\mu_1}(\mu^1), \ldots, \phi_{\mu_1}(\mu^n)$$

in Λ_{μ_1}. Put $\phi_{\mu_1}(\mu) = \lambda$. In this way, we have inductively constructed, for every $\mu \in \Lambda$, a desired increasing function $\phi_\mu : \Lambda_\mu \to \Lambda_\mu$, $\phi_\mu \geq 1_{\Lambda_\mu}$. Now, given $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ associated with a pair (μ_1, μ_2), let the ladder $f'_{\boldsymbol{\mu}} = \emptyset$ whenever $\phi_{\mu_1}(\mu_1) \not\leq \mu_2$. If $\phi_{\mu_1}(\mu_1) \leq \mu_2$, let $f'_{\boldsymbol{\mu}}$ consist of all admissible morphisms

$$f_\mu p_{f(\mu)\phi_{\mu_1}(\mu)} \equiv f'_\mu,$$

where f_μ belongs to the ladder $f_{\boldsymbol{\mu}}$. Then, clearly, $f'_{\boldsymbol{\mu}} \neq \emptyset$, i.e. at least $f'_{\mu_1} \in f'_{\boldsymbol{\mu}}$, because of $\mu_1 \leq \phi_{\mu_1}(\mu_1) \leq \mu_2$. One readily sees that $\mu_1 \leq \mu \leq \mu' \leq \mu_2$ (in the domain of a ladder $f'_{\boldsymbol{\mu}}$) implies

$$f'_\mu p_{\phi_{m_1}(\mu)\phi_{\mu_1}(\mu')} = q_{\mu\mu'} f'_{\mu'}.$$

This confirms that the ladders $f'_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, are well defined, and that, for every $\mu \in \Lambda$, the function ϕ_μ provides the index functions for all the ladders $f'_{\boldsymbol{\mu}}$, where $\boldsymbol{\mu}$ is associated with a pair $(\mu_1 = \mu, \mu_2)$. Let us verify that the family $(f'_{\boldsymbol{\mu}})$ is a hyperladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$. For every $\mu_1 \in \Lambda$ and every $\mu'_1 \geq \mu_1$, put $\lambda^1 = \phi_{\mu_1}(\mu'_1)$, and let $\mu_2 \geq \lambda^1$. Then, by construction, the corresponding ladder $f'_{\boldsymbol{\mu}}$ is defined at least for all μ, $\mu_1 \leq \mu \leq \mu'_1$, and $\phi_{\mu_1}(\mu) \leq \phi_{\mu_1}(\mu'_1) \leq \lambda^1$. Further, notice that $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$ because every ladder $f'_{\boldsymbol{\mu}}$ is a restriction of the corresponding ladder $f_{\boldsymbol{\mu}}$ composed with the appropriate bonding morphisms of $\boldsymbol{X}$. Therefore, if $(f_{\boldsymbol{\mu}}) \simeq (g_{\boldsymbol{\mu}})$ and $(g'_{\boldsymbol{\mu}})$ is determined in the same way by $(g_{\boldsymbol{\mu}})$, then $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$. It remains to prove that this relation can be realized by means of a family (χ_μ) of (index) functions. For every $\mu \in \Lambda$, a desired increasing function

$$\chi_\mu : \Lambda_\mu \to \Lambda_\mu, \ \chi_\mu \geq 1_{\Lambda_\mu},$$

can be defined similarly to ϕ_μ. Let $\mu_1 \in \Lambda$. By $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$, for μ_1 and $\mu'_1 = \mu_1$, there exists a $\lambda^1_* \geq \mu_1$ (and $\lambda^1_* \geq \phi_{\mu_1}(\mu_1), \psi_{\mu_1}(\mu_1) \geq \mu_1$). Put $\chi_{\mu_1}(\mu_1) = \lambda^1_*$. Observe that, by the homotopy relation of hyperladders, for every $\mu'_1 \geq \mu_1$, there exists an upper bound $\lambda^1_*(\mu_1, \mu'_1) \geq \mu'_1$ for all indices λ occurring in relation $f'_{\boldsymbol{\mu}} \simeq g'_{\boldsymbol{\mu}}$, whenever $\mu_2 \geq \lambda^1_*$. Assume that $\chi_{\mu_1}(\mu)$ is defined for all $\mu \in \Lambda_{\mu_1}$, $|\mu| < n \in \mathbb{N}$ in Λ_{μ_1}, such that

$$\chi_{\mu_1}(\mu) \geq \mu, \lambda^1_*(\mu_1, \mu) \text{ and}$$

$$\chi_{\mu_1}(\mu') \leq \chi_{\mu_1}(\mu'') \text{ whenever } \mu_1 \leq \mu' \leq \mu'' \leq \mu.$$

Let $\mu \in \Lambda_{\mu_1}$, $|\mu| = n$ in Λ_{μ_1}. Let $\mu^1 = \mu_1, \ldots, \mu^n$ be all the predecessors of μ in Λ_{μ_1}. Then $\chi_{\mu_1}(\mu^i)$, $i = 1, \ldots, n$, are already defined. There exists a

$$\lambda \geq \mu, \lambda^1_*(\mu_1, \mu), \chi_{\mu_1}(\mu^1), \ldots, \chi_{\mu_1}(\mu^n),$$

in Λ_{μ_1}. Put $\chi_{\mu_1}(\mu) = \lambda$. In this way, we have inductively constructed, for every $\mu \in \Lambda$, the desired increasing function $\chi_\mu : \Lambda_\mu \to \Lambda_\mu$, $\chi_\mu \geq 1_{\Lambda_\mu}$. Now, given a $\mu_1 \in \Lambda$ and a $\mu_1' \geq \mu_1$, one can put $\lambda_*^1 = \chi_{\mu_1}(\mu_1')$ and thus, for every $\mu_2 \geq \lambda_*^1$, achieve that $f'_{\boldsymbol{\mu}} \simeq g'_{\boldsymbol{\mu}}$, so that

$$f'_\mu p_{\phi_{m_1}(\mu)\chi_{\mu_1}(\mu)} = g'_\mu p_{\psi_{m_1}(\mu)\chi_{\mu_1}(\mu)},$$

for every $\mu \in J^*$, where J^* contains the segment $[\mu_1, \mu_1']$. Thus, the equivalence relation $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$ is realized via the constructed family (χ_μ), $\mu \in \Lambda$.

Step 2. Let $(f_{\boldsymbol{\mu}}) \in inv_*^\Lambda\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ be a hyperladder having a family of index functions (ϕ_μ) described in Step 1. We are to prove that $(f_{\boldsymbol{\mu}})$ yields a hyperladder $(f'_{\boldsymbol{\mu}}) \in inv_*^\Lambda\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ providing a unique increasing index function $\phi : \Lambda \to \Lambda$, $\phi \geq 1_\Lambda$, for all of its ladders $f'_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$. Moreover, if a $(g_{\boldsymbol{\mu}})$ with a (ψ_μ) yields $(g'_{\boldsymbol{\mu}})$ with a ψ, and if $(f_{\boldsymbol{\mu}}) \simeq (g_{\boldsymbol{\mu}})$ is realized via a family (χ_μ), we shall prove that $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$ can be realized via a unique increasing function $\chi : \Lambda \to \Lambda$, $\chi \geq 1_\Lambda$.

We define a desired index function $\phi : \Lambda \to \Lambda$ by induction on $|\mu| \in \mathbb{N} \cup \{0\}$ (Λ is cofinite) as follows. If $\mu \in \Lambda$ and $|\mu| = 0$, put $\phi(\mu) = \phi_\mu(\mu)$. Let $n \in \mathbb{N}$, and let us assume that $\phi(\mu)$ is defined for all $\mu \in \Lambda$ with $|\mu| < n$, such that $\phi(\mu) \geq \phi_\mu(\mu)$ and $\phi(\mu') \leq \phi(\mu'')$ whenever $\mu' < \mu'' \leq \mu$. Let $\mu \in \Lambda$ and $|\mu| = n \in \mathbb{N}$, and let $\mu_1, \ldots, \mu_n < \mu$ be all the predecessors of μ. Then $|\mu_i| < n$ for all $i = 1, \ldots, n$. Thus, all the $\phi(\mu_i)$ are already defined. Since Λ is directed, there exists a $\lambda \in \Lambda$ such that $\lambda \geq \phi(\mu_1), \ldots, \phi(\mu_k), \phi_\mu(\mu)$. Put $\phi(\mu) = \lambda$. Then $\phi(\mu) \geq \phi_\mu(\mu) \geq \mu$ and, for every pair $\mu' < \mu''$ ($\leq \mu$), $\phi(\mu') \leq \phi(\mu'')$ holds. Namely, if $\mu'' < \mu$, then the latter holds by the inductive assumption, while $\phi(\mu') \leq \phi(\mu)$ holds by $\lambda \geq \phi(\mu_i)$ for all $i = 1, \ldots, n$ (μ' is some μ_i). This completes the inductive definition of the index function $\phi : \Lambda \to \Lambda$, which is increasing and $\phi \geq 1_\Lambda$.

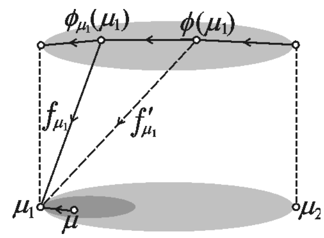
Let us now define the desired ladders $f'_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$. Given the $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ associated with a (μ_1, μ_2), let $f'_{\boldsymbol{\mu}}$ be the maximal admissible restriction of the ladder $f_{\boldsymbol{\mu}}$ composed with the corresponding bonding morphisms

$$p_{\phi_{\mu_1}(\mu)\phi(\mu)} : X_{\phi(\mu)} \to X_{\phi_{\mu_1}(\mu)}, \ \mu \in J' \subseteq J \subseteq \boldsymbol{\mu}.$$

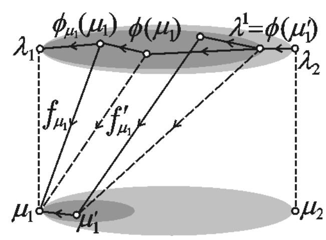
Hence, if a morphism f'_μ of $f'_{\boldsymbol{\mu}}$ exists, then it is of the form

$$f_\mu p_{\phi_{\mu_1}(\mu)\phi(\mu)} : X_{\phi(\mu)} \to Y_\mu,$$

where f_μ belongs to $f_{\boldsymbol{\mu}}$. Obviously, $f'_{\boldsymbol{\mu}}$ is not empty if and only if $f_{\boldsymbol{\mu}}$ is not empty and $\phi(\mu_1) \leq \mu_2$.



Let us verify that the family $(f'_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, is a hyperladder of inv_*^{Λ}-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. For every $\mu_1 \in \Lambda$ and every $\mu_1' \geq \mu_1$, put $\lambda^1 = \phi(\mu_1') \geq \mu_1'$, and let $\mu_2 \geq \lambda^1$. Then $\mu_2 \geq \phi_{\mu_1'}(\mu_1') \geq \mu_1'$. Since $(f_{\boldsymbol{\mu}})$ is a hyperladder with the family (ϕ_μ), the domain set J of the corresponding ladder $f_{\boldsymbol{\mu}}$ must contain all the indices μ, $\mu_1 \leq \mu \leq \mu_1'$. Thus, by construction, the composite morphism $f_\mu p_{\phi_{\mu_1}(\mu)\phi(\mu)}$ belongs to $f'_{\boldsymbol{\mu}}$, for every μ, $\mu_1 \leq \mu \leq \mu_1'$. Consequently, the domain $J' \subseteq J$ of $f'_{\boldsymbol{\mu}}$ must contain all μ, $\mu_1 \leq \mu \leq \mu_1'$, too.



That suffices to conclude that $(f'_{\boldsymbol{\mu}})$ is a hyperladder of inv_*^{Λ}-$\mathcal{A}$. In addition, let $(g_{\boldsymbol{\mu}})$ with a (ψ_μ) yield $(g'_{\boldsymbol{\mu}})$ with a ψ, and let $(f_{\boldsymbol{\mu}}) \simeq (g_{\boldsymbol{\mu}})$ be realized via a (χ_μ). Since $(f'_{\boldsymbol{\mu}})$ and $(g'_{\boldsymbol{\mu}})$ are obtained by composing $(f_{\boldsymbol{\mu}})$ and $(g_{\boldsymbol{\mu}})$ with the bonding morphisms of $\boldsymbol{X}$ respectively, it is obvious that $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$ must hold. It remains to find a desired homotopy realizing function $\chi : \Lambda \to \Lambda$. The definition is by induction – quite similar to that of a ϕ for $(f'_{\boldsymbol{\mu}})$. If $\mu \in \Lambda$ and $|\mu| = 0$, put

$$\chi(\mu) = \lambda \geq \phi(\mu), \psi(\mu), \chi_\mu(\mu).$$

Assume that χ is defined for all $\mu \in \Lambda$ with $|\mu| < n \in \mathbb{N}$, such that $\chi(\mu) \geq \phi(\mu), \psi(\mu), \chi_\mu(\mu)$, and $\chi(\mu') \leq \chi(\mu'')$ whenever $\mu' < \mu'' \leq \mu$. Let $\mu \in \Lambda$ and $|\mu| = n$, and let $\mu_1, \ldots, \mu_n$ be all the predecessors of μ. Then put

$$\chi(\mu) = \lambda \geq \phi(\mu), \psi(\mu), \chi(\mu_1), \ldots, \chi(\mu_n).$$

It is readily seen that $\chi : \Lambda \to \Lambda$ is increasing and $\chi \geq 1_\Lambda$, as well as that $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$ is also realized via χ. Indeed, for every $\mu_1 \in \Lambda$ and every $\mu_1' \geq \mu_1$, one may put $\lambda_*^1 = \chi_\mu(\mu_1')$ and, for every $\mu_2 \geq \lambda_*^1$, achieve that $f'_{\boldsymbol{\mu}} \simeq g'_{\boldsymbol{\mu}}$ and

$$f'_\mu p_{\phi(\mu)\chi(\mu)} = g'_\mu p_{\psi(\mu)\chi(\mu)},$$

for every $\mu \in J^*$, where J^* contains the segment $[\mu_1, \mu_1']$. □

Lemma 7. *For each Λ, every special $*$-morphism $(f, f_\mu^n) \in inv_\Lambda^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ with $f \geq 1_\Lambda$ induces a hyperladder $(f'_{\boldsymbol{\mu}}) \in inv_*^\Lambda\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ having f to be the index function for all the (nonempty) ladders. Moreover, $(f, f_\mu^n) \sim (g, g_\mu^n)$ implies $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$ which realizes via a unique increasing function $\chi : \Lambda \to \Lambda$, $\chi \geq 1_\Lambda$. If $\Lambda = \mathbb{N}$, then $(f, f_\mu^n) \sim (g, g_\mu^n)$ is equivalent to $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$.*

Proof. Let a special $*$-morphism $(f, f_\mu^n) \in inv_\Lambda^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ with $f \geq 1_\Lambda$ be given. For every $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, associated with a pair (μ_1, μ_2), put $f'_{\boldsymbol{\mu}} = \emptyset$ whenever $f(\mu_1) \not\leq \mu_2$. If $f(\mu_1) \leq \mu_2$, let $f'_{\boldsymbol{\mu}}$ be defined by the maximal restriction of f to the initial subset $J \subseteq \boldsymbol{\mu}$ determined by $f(\mu) \leq \mu_2$, $\mu \in J$, and by the commutativity condition of (f, f_μ^n) for $n = |\mu_2|$. Thus, each $f'_\mu \in f'_{\boldsymbol{\mu}}$ is the corresponding $f_\mu^{|\mu_2|}$. Clearly, such a ladder $f'_{\boldsymbol{\mu}} \neq \emptyset$ because at least $f'_{\mu_1} = f_{\mu_1}^{|\mu_2|} \in f'_{\boldsymbol{\mu}}$. Notice that, if $|\mu_2|$ is not sufficiently large, then f_{μ_1} is the only member of $f_{\boldsymbol{\mu}}$. Let us show that the family $(f'_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, is a hyperladder, i.e. a morphism of $inv_*^\Lambda\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. For every $\mu_1 \in \Lambda$ and every $\mu_1' \geq \mu_1$, choose a $\lambda^1 \geq f(\mu_1')$ such that $\left|\lambda^1\right| \geq n_{(\mu,\mu')}$ for every related pair $\mu \leq \mu'$, $\mu_1 \leq \mu \leq \mu' \leq \mu_1'$, where $n_{(\mu,\mu')}$ comes from the commutativity condition of (f, f_μ^n). Such a λ^1 exists, because Λ is directed and cofinite. Let $\mu_2 \geq \lambda^1$. Then $|\mu_2| \geq \left|\lambda^1\right|$, and thus, the commutativity condition of (f, f_μ^n) implies the commutativity of the corresponding ladder $f'_{\boldsymbol{\mu}}$ as well as the condition for a hyperladder. Let $(f, f_\mu^n) \sim (g, g_\mu^n)$ and let $(g'_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, be the hyperladder induced in the same way by (g, g_μ^n). We have to prove that $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$ via a unique increasing function $\chi : \Lambda \to \Lambda$, $\chi \geq 1_\Lambda$. According to Lemma 6 and its proof, it suffices to prove that $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$. Let $\mu_1 \in \Lambda$ and let $\mu_1' \geq \mu_1$. By the equivalence $(f, f_\mu^n) \sim (g, g_\mu^n)$, for every $\mu \in \Lambda$, there exist a $\lambda(\mu) \geq f(\mu), g(\mu)$ and an $n(\mu) \in \mathbb{N}$ such that, for every $n' \geq n$, the appropriate commutativity condition is fulfilled. Choose a $\lambda_*^1 \geq \lambda(\mu)$ for all μ, $\mu_1 \leq \mu \leq \mu_1'$, such that $\left|\lambda_*^1\right| \geq n(\mu)$ for all of those μ. Then, for every $\mu_2 \geq \lambda_*^1$, $|\mu_2| \geq n(\mu)$ holds. Hence, the corresponding ladders $f'_{\boldsymbol{\mu}}$ and $g'_{\boldsymbol{\mu}}$ are equivalent, $f'_{\boldsymbol{\mu}} \simeq g'_{\boldsymbol{\mu}}$, and the needed condition for the equivalence of the hyperladders, $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$, is also fulfilled (the index functions are fixed). Finally, let $\Lambda = \mathbb{N}$ and let the induced hyperladders be equivalent, $(f'_{\boldsymbol{\mu}}) \simeq (g'_{\boldsymbol{\mu}})$. Then, by construction, $(f, f_\mu^n) \sim (g, g_\mu^n)$ must hold too, because every $n' \in \mathbb{N}$, $n' \geq n$, admits a new $\mu_2 \in \Lambda = \mathbb{N}$ such that $|\mu_2| = \mu_2 = n'$. □

Lemma 8. *For every special $*$-morphism $(f, f_\mu^n) \in inv_\Lambda^*$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, there exists a special $*$-morphism $(f', f_\mu'^n) \in inv_\Lambda^*$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ such that $f' \geq 1_\Lambda$ and $(f, f_\mu^n) \sim (f', f_\mu'^n)$.*

Proof. The construction of $f' : \Lambda \to \Lambda$ is by induction. Let $\mu \in \Lambda$ be an initial element, $|\mu| = 0$. Put $f'(\mu) = \lambda$, where $\lambda \geq \mu, f(\mu)$. Given an $m \in \mathbb{N}$, assume that $f'(\mu)$ is defined for all $\mu \in \Lambda$, $|\mu| < m$, such that $f'(\mu) \geq \mu, f(\mu)$ and $f'(\mu') \leq f'(\mu'')$ whenever $\mu' \leq \mu'' \leq \mu$. Let $\mu \in \Lambda$, $|\mu| = m$. Let $\mu_1, \ldots, \mu_m$ be all the predecessors of μ. Then, $|\mu_i| < m$ and $f'(\mu_i)$ are already defined for every $i = 1, \ldots, m$. Since Λ is cofinite, there exists a $\lambda \geq \mu, f(\mu), f'(\mu_1), \ldots, f'(\mu_m)$. Put $f'(\mu) = \lambda$. In this way a desired increasing index function $f' : \Lambda \to \Lambda$, $f' \geq 1_\Lambda$, is defined. Notice that, for every $\mu \in \Lambda$, $f'(\mu) \geq f(\mu)$ holds. Thus, the morphisms

$$f_\mu'^n : X_{f'(\mu)} \to Y_\mu, \;\; f_\mu'^n = f_\mu^n p_{f(\mu)f'(\mu)},$$

are well defined for all $\mu \in \Lambda$ and all $n \in \mathbb{N}$. The verification that $(f', f_\mu'^n)$ is a special $*$-morphism and that $(f, f_\mu^n) \sim (f', f_\mu'^n)$ is trivial. □

Proof of Theorem 5. Recall that $Ob(pro_\sim^*\text{-}\mathcal{A}) = Ob(pro_*^\sim\text{-}\mathcal{A}) = Ob(pro^\sim\text{-}\mathcal{A})$. Thus, one may put $\underline{W}(\boldsymbol{X}) = \boldsymbol{X}$ for every object $\boldsymbol{X}$. Let $\boldsymbol{f}^* \in pro_\sim^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, $M = \Lambda$. By Lemma 10 of [10] and Lemma 8, there exists a special representative (f, f_μ^n) of $\boldsymbol{f}^*$ such that $f \geq 1_\Lambda$. By Lemmata 7, $\boldsymbol{f}^*$ induces a morphism $\boldsymbol{f}_* = [(f'_{\boldsymbol{\mu}})] \in pro_*^\sim\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, where $(f'_{\boldsymbol{\mu}})$ is a hyperladder given by (f, f_μ^n). Put $\underline{W}(\boldsymbol{f}^*) = \boldsymbol{f}_*$. We have to prove that $\underline{W}(\boldsymbol{1}_{\boldsymbol{X}}^*) = \boldsymbol{1}_{\boldsymbol{X}*}$ and $\underline{W}(\boldsymbol{g}^*\boldsymbol{f}^*) = \underline{W}(\boldsymbol{g}^*)\underline{W}(\boldsymbol{f}^*)$. The first equality obviously holds because, by construction and the proof of Lemma 7, every ladder of the constructed hyperladder consists of morphisms of a given $*$-morphism composed by the bonding morphisms of $\boldsymbol{X}$. Let $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ be a special representative of $\boldsymbol{f}^*$ such that $f \geq 1_\Lambda$, and let $(g, g_\nu^n) : \boldsymbol{Y} \to \boldsymbol{Z}$, $N = \Lambda$, be a special representative of $\boldsymbol{g}^*$ such that $g \geq 1_\Lambda$. Let $(h, h_\nu^n) : \boldsymbol{X} \to \boldsymbol{Z}$, be the composition of (f, f_μ^n) and (g, g_ν^n), i.e. $h = fg \geq 1_\Lambda$ and $h_\nu^n = g_\nu^n f_{g(\nu)}^n$ for all $\mu \in \Lambda$ and all $n \in \mathbb{N}$. Let the hyperladders $(f'_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$, $(g'_{\boldsymbol{\nu}}) : \boldsymbol{Y} \to \boldsymbol{Z}$ and $(h'_{\boldsymbol{\nu}}) : \boldsymbol{X} \to \boldsymbol{Z}$ be given by (f, f_μ^n), (g, g_ν^n) and (h, h_ν^n) respectively, according to Lemma 7. It suffices to prove that $(g'_{\boldsymbol{\nu}})(f'_{\boldsymbol{\mu}}) \simeq (h'_{\boldsymbol{\nu}})$. Recall that each ladder of $(g'_{\boldsymbol{\nu}})(f'_{\boldsymbol{\mu}}) = (g'_{\boldsymbol{\nu}} f'_{\boldsymbol{\nu}})$ is of the form $g'_{\boldsymbol{\nu}} f'_{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \boldsymbol{\Lambda}$. Therefore, a morphism belonging to a $g'_{\boldsymbol{\nu}} f'_{\boldsymbol{\nu}}$ is of the form $g_\nu^{|\nu_2|} f_{g(\nu)}^{|\nu_2|}$. On the other hand, a morphism of each ladder $h'_{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \boldsymbol{\Lambda}$, is of the form $h_\nu^{|\nu_2|}$. Since $h_\nu^n = g_\nu^n f_{g(\nu)}^n$, for every $\nu \in \Lambda$ and every $n \in \mathbb{N}$, even the conclusion $(g'_{\boldsymbol{\nu}})(f'_{\boldsymbol{\mu}}) = (h'_{\boldsymbol{\nu}})$ holds. In the case $\Lambda = \mathbb{N}$, i.e. for the functor $\underline{W} : tow_\sim^*\text{-}\mathcal{A} \to tow_*^\sim\text{-}\mathcal{A}$, the last statement of Lemma 7 implies that it is faithful. □

Observe that Lemma 6 is (implicitly) used only for this final statement. However, it will be indispensable for the proof of Theorem 8 below.

Theorem 6. *For every category pair* $(\mathcal{C}, \mathcal{D})$*, where* $\mathcal{D} \subseteq \mathcal{C}$ *is dense and full, there exists a functor*

$$W : Sh^*_{(\mathcal{C},\mathcal{D})} \to Sh_{*(\mathcal{C},\mathcal{D})},$$

which keeps the objects fixed.

Proof. Since $Ob(Sh^*_{(\mathcal{C},\mathcal{D})}) = Ob(Sh_{*(\mathcal{C},\mathcal{D})})$ $(= Ob\mathcal{C})$, one may put $W(X) = X$ for each $\mathcal{C}$-object X. Let $F^* : X \to Y$ be a coarse shape morphism. By Theorem 3 of [10], there exists a realizing morphism $\boldsymbol{f}^* \in pro^*_{\sim}\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$, $M = \Lambda$, of F^* (having even a level representative $(1_\Lambda, f^n_\lambda)$). Let $\boldsymbol{f}_* = \underline{W}(\boldsymbol{f}^*) \in pro^\Lambda_*\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$, by Theorem 5, and let $F^\Lambda_* = \langle \boldsymbol{f}_* \rangle \in Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}(\boldsymbol{X}, \boldsymbol{Y})$ be the equivalence class of $\boldsymbol{f}_*$. Further, let

$$F_* = \lfloor F^\Lambda_* \rfloor = \lfloor \langle \boldsymbol{f}_* \rangle \rfloor \in Sh_{*(\mathcal{C},\mathcal{D})}(\boldsymbol{X}, \boldsymbol{Y})$$

be the equivalence class of F^Λ_*. Now, put $W(F^*) = F_*$. Since $\underline{W} : pro^*_{\sim}\text{-}\mathcal{D} \to pro^{\sim}_*\text{-}\mathcal{D}$ is a functor (Theorem 5), one readily verifies that W is also a functor. □

According to the related facts from [10], our Section 6 and Theorem 6, the following diagram of functors commutes:

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \swarrow S & \downarrow S^* & \searrow S_* & \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{J}{\longrightarrow} & Sh^*_{(\mathcal{C},\mathcal{D})} & \underset{W}{\longrightarrow} & Sh_{*(\mathcal{C},\mathcal{D})} \end{array}.$$

Hereby, $JW = T$ (see Sections 5 and 6), and J as well as T is faithful.

Corollary 3. *Let* $(\mathcal{C}, \mathcal{D})$ *be a category pair, where* $\mathcal{D} \subseteq \mathcal{C}$ *is dense and full. Then, for every pair of* $\mathcal{C}$*-objects* X, Y*, the following implications hold:*

$$(Sh(X) = Sh(Y)) \Rightarrow (Sh^*(X) = Sh^*(Y)) \Rightarrow (Sh_*(X) = Sh_*(Y)).$$

Theorem 7. *Let* $(\mathcal{C}, \mathcal{D})$ *be a category pair, where* $\mathcal{D} \subseteq \mathcal{C}$ *is dense and full, such that every* $\mathcal{C}$*-object admits a* countable $\mathcal{D}$*-expansion. Then the functor* $W : Sh^*_{(\mathcal{C},\mathcal{D})} \to Sh_{*(\mathcal{C},\mathcal{D})}$ *is faithful.*

For the proof, we need the next lemma.

Lemma 9. *Every countable inverse system* $\boldsymbol{X}$ *in a category* $\mathcal{A}$ *is isomorphic in pro-*$\mathcal{A}$ *to an inverse sequence* $\boldsymbol{X}'$ *in* $\mathcal{A}$.

Proof. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a countable inverse system in a category $\mathcal{A}$. By [4], Exercises, A.1, p. 229, there exists an inverse sequence $\boldsymbol{X}'$ which is cofinal in $\boldsymbol{X}$, and therefore, $\boldsymbol{X}' \cong \boldsymbol{X}$ in *pro*-$\mathcal{A}$. However, in [4] there is no proof of this fact. Therefore, we provide the following one. Since Λ is countable, one may consider (by forgetting the ordering) that $\Lambda = \{\lambda_{1,}\lambda_2, \ldots \lambda_i, \ldots\}$, $i \in \mathbb{N}$. Let us denote $\lambda'_1 = \lambda_1$, and put

$$\Lambda^{(1)} = \{\lambda \in \Lambda \mid \lambda \leq \lambda'_1\} \subseteq \Lambda.$$

If $\Lambda^{(1)} = \Lambda$, then λ'_1 is the maximal element of Λ. Thus, to obtain a desired sequence one may put

$$\boldsymbol{X}' = (X'_i = X_{\lambda'_1}, p'_{ii'} = 1, \mathbb{N}).$$

Now, assuming that Λ has no maximal element, let us choose a $\lambda'_2 \in \Lambda \setminus \Lambda^{(1)}$ such that $\lambda'_2 > \lambda'_1$ and $\lambda'_2 \geq \lambda_2$, and put

$$\Lambda^{(2)} = \{\lambda \in \Lambda \setminus \Lambda^{(1)} \mid \lambda \leq \lambda'_2\} \subseteq \Lambda \setminus \Lambda^{(1)}.$$

By induction on $i \in \mathbb{N}$, choose a $\lambda'_{i+1} \in \Lambda \setminus (\Lambda^{(1)} \cup \cdots \cup \Lambda^{(i)})$ such that $\lambda'_{i+1} > \lambda'_i$ and $\lambda'_{i+1} \geq \lambda_{i+1}$, and put

$$\Lambda^{(i+1)} = \{\lambda \in \Lambda \setminus (\Lambda^{(1)} \cup \cdots \cup \Lambda^{(i)} \mid \lambda \leq \lambda'_{i+1}\} \subseteq \Lambda \setminus (\Lambda^{(1)} \cup \cdots \cup \Lambda^{(i)}).$$

Notice that $\Lambda = \bigsqcup_{i \in \mathbb{N}} \Lambda^{(i)}$ (disjoint union) and the subset $\Lambda' \equiv \{\lambda'_i \mid i \in \mathbb{N}\} \subseteq \Lambda$ is cofinal in Λ (with respect to the given ordering). Clearly, $\lambda'_i < \lambda'_{i'}$ if and only if $i < i'$. Further, $\lambda \leq \lambda'$ in Λ if and only if either there exists an $i \in \mathbb{N}$ such that $\lambda \leq \lambda'$ in $\Lambda^{(i)}$, or there exists a pair $i, i' \in \mathbb{N}$ such that $\lambda \in \Lambda^{(i)}$, $\lambda' \in \Lambda^{(i')}$, $\lambda \leq \lambda'$ and $i < i'$. For every $i \in \mathbb{N}$, put $X'_i = X_{\lambda'_i}$ and, for every pair $i \leq i'$, put $p'_{ii'} = p_{\lambda'_{i'}\lambda'_i} : X'_{i'} \to X'_i$. Then the inverse sequence $\boldsymbol{X}' = (X'_i, p'_{ii'}, \mathbb{N})$ is isomorphic to $\boldsymbol{X}$ in *pro*-$\mathcal{A}$. For instance, the morphism $\boldsymbol{u} : \boldsymbol{X}' \to \boldsymbol{X}$ represented by $(u, u_\lambda) : \boldsymbol{X}' \to \boldsymbol{X}$, where $u : \Lambda \to \Lambda'$, $u[\Lambda^{(i)}] = \{i\}$, $i \in \mathbb{N}$, and $u_\lambda : X'_{u(\lambda)} \to X_\lambda$, $\lambda \in \Lambda$, are the appropriate bonding morphisms, is an isomorphism having the inverse $\boldsymbol{u}^{-1} : \boldsymbol{X} \to \boldsymbol{X}'$ represented by $(u', u'_i) : \boldsymbol{X} \to \boldsymbol{X}'$, where $u' : \mathbb{N} \to \Lambda$, $u'(i) = \lambda'_i$, $i \in \mathbb{N}$, and $u'_i : X_{v(i)} \to X'_i$, $i \in \mathbb{N}$, are the identities. □

Proof of Theorem 7. Let $X, Y \in Ob\mathcal{C}$. Let $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be countable $\mathcal{D}$-expansions of X, Y respectively. By Lemma 9, there exist inverse sequences $\boldsymbol{X}'$, $\boldsymbol{Y}'$ which are isomorphic in *pro*-$\mathcal{D}$ to $\boldsymbol{X}$, $\boldsymbol{Y}$ respectively.

Then there exist sequential $\mathcal{D}$-expansions $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ of X, Y respectively. By construction of the weak shape category (Sections 5 and 6),

$$Sh_{*(\mathcal{C},\mathcal{D})}(X,Y) \approx pro_*^{\mathbb{N}}\text{-}\mathcal{D}(\boldsymbol{X}',\boldsymbol{Y}') = tow\text{-}\mathcal{D}(\boldsymbol{X}',\boldsymbol{Y}').$$

Consequently, tow_*-$\mathcal{D}$ is a realizing category for $Sh_{*(\mathcal{C},\mathcal{D})}$. By the last statement of Theorem 5, the functor $\underline{W} : tow^*\text{-}\mathcal{D} \to tow_*\text{-}\mathcal{D}$ is faithful. Finally, Theorem 5 and its proof imply that the corresponding functor $W : Sh^*_{(\mathcal{C},\mathcal{D})} \to Sh_{*(\mathcal{C},\mathcal{D})}$ is faithful. □

Corollary 4. *In the standard case $\mathcal{C} = HcM$ and $\mathcal{D} = HcANR$ (or $HcPOL$), the functor $W : Sh^*(HcM) \to Sh_*(HcM)$ is faithful.*

In the sequential case ($\Lambda = \mathbb{N}$), we shall prove even more: The classifications of objects by the weak shape type and by the coarse shape type coincide. Theorem 5 has established a functor of $pro^*_\sim$-$\mathcal{A}$ to $pro^\sim_*$-$\mathcal{A}$. In the sequential case there also exists a converse functor. However, in general, it is *not* faithful.

Theorem 8. *For every category $\mathcal{A}$, there exists a functor*

$$\underline{U} : tow_*\text{-}\mathcal{A} \to tow^*\text{-}\mathcal{A},$$

which keeps the objects fixed. In general, $\underline{U}$ is not a faithful functor.

Proof. Clearly, we put $\underline{U}(\boldsymbol{X}) = \boldsymbol{X}$ for every object $\boldsymbol{X} \in Ob(tow_*\text{-}\mathcal{A}) = Ob(tow^*\text{-}\mathcal{A})$. Let $\boldsymbol{f}_* \in tow_*\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$. By Lemma 6, there exists a representing hyperladder $(f_{\boldsymbol{\mu}}) \in inv_*^{\mathbb{N}}\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ of $\boldsymbol{f}_*$, having a unique increasing index function $\phi : \mathbb{N} \to \mathbb{N}$, $\phi \geq 1_{\mathbb{N}}$, for all the ladders $f_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda} = \boldsymbol{N}$. First, we shall construct a special $*$-morphism $(f, f_\mu^n) \in inv_{\mathbb{N}}^*\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ by means of the hyperladder $(f_{\boldsymbol{\mu}})$. Put

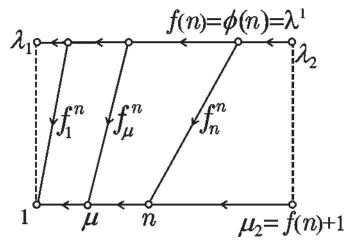
$$f = \phi : \mathbb{N} \to \mathbb{N}.$$

Let $\mu \in \mathbb{N}$ and $n \in \mathbb{N}$ be given. If $n < \mu$, let the morphism

$$f_\mu^n : X_{f(\mu)} \to Y_\mu$$

be chosen arbitrarily; if $n \geq \mu$, let f_μ^n be the corresponding morphism of the ladder $f_{\boldsymbol{\mu}}$, where $\boldsymbol{\mu}$ is associated with $(1, \mu_2)$ and $\mu_2 \geq n$ is the minimal index such that $f_{\boldsymbol{\mu}}$ fulfills the condition of the hyperladder $(f_{\boldsymbol{\mu}})$. (More precisely: $\mu_1 = 1$, $\mu_1' = n$, $\lambda^1 = \phi(n) = f(n)$, $\mu_2 = \lambda^1$ – minimal.)

Notice that, for every $\mu \in \mathbb{N}$ and every $n \in \mathbb{N}$, $n \geq \mu$ implies that all the morphisms $f_1^n, \ldots, f_\mu^n, \ldots, f_n^n$ belongs to the same ladder $f_{\boldsymbol{\mu}}$. Let us verify that the family (f, f_μ^n) is a special $*$-morphism of $\boldsymbol{X}$ to $\boldsymbol{Y}$. Indeed, given any



related pair $\mu \leq \mu'$ in $\mathbb{N}$, choose $\lambda = f(\mu') \geq \mu'$ and $n = \mu_2 = f(\mu') \geq \mu'$. Then f_μ^n and $f_{\mu'}^n$ belong to the ladder $f_{\boldsymbol{\mu}}$, where $\boldsymbol{\mu}$ is associated with $(1, f(\mu'))$, and thus, they commute. Further, if $n' \geq n$, then the new $\mu_2 = n'$ insures the needed commutativity condition. Let $\boldsymbol{f}^* = [(f, f_\mu^n)] \in tow^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, and put $\underline{U}(\boldsymbol{f}_*) = \boldsymbol{f}^*$. To show that $\underline{U}$ is well defined, consider a pair of equivalent hyperladders, $(f_{\boldsymbol{\mu}}) \simeq (g_{\boldsymbol{\mu}})$, via an increasing $\chi : \mathbb{N} \to \mathbb{N}$, $\chi \geq 1_{\mathbb{N}}$. Let (f, f_μ^n) and (g, g_μ^n) by special $*$-morphisms given by $(f_{\boldsymbol{\mu}})$ and $(g_{\boldsymbol{\mu}})$ respectively, according to the preceding construction. Then $(f, f_\mu^n) \sim (g, g_\mu^n)$. Namely, given a $\mu \in \mathbb{N}$, the needed $\lambda \geq f(\mu), g(\mu)$ and $n \in \mathbb{N}$ may be the same $\lambda_*^1 = \chi(\mu)$, which exists by $(f_{\boldsymbol{\mu}}) \simeq (g_{\boldsymbol{\mu}})$ via χ. Therefore, the correspondence $\boldsymbol{f}_* \mapsto \boldsymbol{f}^* = \underline{U}(\boldsymbol{f}_*)$ is well defined. Further, it is obvious by construction that $\underline{U}(\mathbf{1}_{\boldsymbol{X}*}) = \mathbf{1}_{\boldsymbol{X}}^*$. Finally, $\underline{U}(\boldsymbol{g}_* \boldsymbol{f}_*) = \underline{U}(\boldsymbol{g}_*)\underline{U}(\boldsymbol{f}_*)$ also holds. Namely, if an $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ yields (f, f_μ^n) and a $(g_{\boldsymbol{\nu}}) : \boldsymbol{Y} \to \boldsymbol{Z}$ yields (g, g_ν^n), then $(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}}) = (g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}}) : \boldsymbol{X} \to \boldsymbol{Z}$ yields $(h, h_{\boldsymbol{\nu}}^n)$ such that

$$(h, h_{\boldsymbol{\nu}}^n) \sim (fg, g_\nu^n f_{g(\nu)}^n) = (g, g_\nu^n)(f, f_\mu^n).$$

Indeed, one can easily verify that, for every $\nu \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ (any $n \geq fg(\nu)$ suits) such that $h_{\boldsymbol{\nu}}^{n'} = g_\nu^{n'} f_{g(\nu)}^{n'}$, for every $n' \geq n$. To see that the functor $\underline{U}$ is not faithful, let us observe that $\underline{U}(\boldsymbol{f}_*)$ is defined via the subfamily of a special representative $(f_{\boldsymbol{\mu}}) \in \boldsymbol{f}_*$, where each $\boldsymbol{\mu}$ *is* associated with a pair $(1, \mu_2)$. Therefore, if $\boldsymbol{f}_*, \boldsymbol{g}_* : \boldsymbol{X} \to \boldsymbol{Y}$ admit a pair of special representatives $(f_{\boldsymbol{\mu}})$, $(g_{\boldsymbol{\mu}})$ such that the homotopy condition is fulfilled for $\mu_1 = 1$, and there exists a $\mu_1 > 1$ such that the corresponding homotopy condition is not fulfilled, then $\underline{U}(\boldsymbol{f}_*) = \underline{U}(\boldsymbol{g}_*)$, while $\boldsymbol{f}_* \neq \boldsymbol{g}_*$. □

Theorem 9. *Let $(\mathcal{C}, \mathcal{D})$ be a category pair, where $\mathcal{D} \subseteq \mathcal{C}$ is dense and full, such that every $\mathcal{C}$-object admits a countable $\mathcal{D}$-expansion. Then there exists a functor*

$$U : Sh_{*(\mathcal{C},\mathcal{D})} \to Sh^*_{(\mathcal{C},\mathcal{D})},$$

which keeps the objects fixed. In general, U is not a faithful functor.

Proof. Since $Ob(Sh_{*(\mathcal{C},\mathcal{D})}) = Ob(Sh^*_{(\mathcal{C},\mathcal{D})})$ $(= Ob\mathcal{C})$, one may put $U(X) = X$ for each $\mathcal{C}$-object X. Let $F_* : X \to Y$ be a weak shape morphism. By construction of the the weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$ (Sections 5 and 6) and by Lemma 9, there exists a realizing morphism $\boldsymbol{f}_* \in tow_*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y})$ of F_*. Let $\boldsymbol{f}^* = \underline{U}(\boldsymbol{f}_*) \in tow^*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y})$, by Theorem 8. Now, put $U(F_*) = F^* : X \to Y$ in $Sh^*_{(\mathcal{C},\mathcal{D})}$, where F^* is the equivalence class $\langle \boldsymbol{f}^* \rangle$ of $\boldsymbol{f}^*$. Since $\underline{U} : tow_*\text{-}\mathcal{D} \to tow^*\text{-}\mathcal{D}$ is a functor (Theorem 8), one trivially verifies that U is also a functor. Finally, U is not faithful because $\underline{U}$ is not faithful. □

The next fact is an immediate consequence of Corollary 3 and Theorem 9.

Corollary 5. *If $X \in Ob\mathcal{C}$ admits a* countable *$\mathcal{D}$-expansion, then its weak and coarse shape types coincide. In other words, if $X, Y \in Ob\mathcal{C}$ admit* countable *$\mathcal{D}$-expansions, then*

$$(Sh_*(X) = Sh_*(Y)) \Leftrightarrow (Sh^*(X) = Sh^*(Y)).$$

Corollary 6. *Let X and Y be compact metrizable spaces. Then the following statements are equivalent:*

(i) $Sh_*(X) = Sh_*(Y)$;

(ii) $Sh^*(X) = Sh^*(Y)$;

(iii) $S^*(X) = S^*(Y)$.

There exists a metrizable continuum X such that its shape type is strictly finer than the above three (coinciding) types of X.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows by Corollary 5, while the equivalence (ii) $\Leftrightarrow$ (iii) follows by Theorem 7 of [10]. The last statement follows by the facts proved in Section 4 of [16] □

Remark 2. An easy analysis shows that the categories $\mathcal{S}(n)$, $n \in \mathbb{N}$, constructed in [21], are *not* mutually equivalent, whenever $n \neq n'$. However, by combining Theorem 4.7 of [21] and Corollary 6, we infer that the isomorphisms classifications in all of them coincide, that is, they coincide with the classification by the S^*-equivalence. It is the reason why we hereby have generalized and considered only the case $n = 1$ of [21]. Clearly, the category $\mathcal{S}(1)$ equals to the weak shape category $Sh_{*(HcM,HcANR)} \equiv Sh_*(HcM)$.

At the end of this section, we want to show explicitly that a set $Sh_{*(\mathcal{C},\mathcal{D})}$ (X,Y) of weak shape morphisms is essentially richer than the corresponding set $Sh_{(\mathcal{C},\mathcal{D})}(X,Y)$ $(Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y))$ of (coarse) shape morphisms – even in the case of a $Y \in Ob\mathcal{D}$.

Lemma 10. *Every hyperladder $(f_{\boldsymbol{\mu}}) \in inv_*^{\Lambda}\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ yields an increasing function $\phi : \Lambda \to \Lambda$, $\phi \geq 1_\Lambda$, and, for every $\mu \in \Lambda$, a family F_μ of $\mathcal{A}$-morphisms*

$$f_\mu^{\boldsymbol{\mu}} : X_{\phi(\mu)} \to Y_\mu, \ \boldsymbol{\mu} \in \boldsymbol{\Lambda},$$

such that the following condition is fulfilled:

$$(*) \ (\forall \mu_1 \in \Lambda)(\forall \mu_1' \geq \mu_1)(\forall \mu_2 \geq \phi(\mu_1'))f_\mu^{\boldsymbol{\mu}} p_{\phi(\mu)\phi(\mu')} = q_{\mu\mu'} f_{\mu'}^{\boldsymbol{\mu}},$$

whenever $\mu_1 \leq \mu \leq \mu' \leq \mu_1'$ and $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ is associated with the chosen pair (μ_1,μ_2). Conversely, every increasing function $\phi : \Lambda \to \Lambda$, $\phi \geq 1_\Lambda$, and every family (F_μ), $\mu \in \Lambda$, of families

$$F_\mu = \{f_\mu^{\boldsymbol{\mu}} \in \mathcal{A}(X_{\phi(\mu)}, Y_\mu) \mid \boldsymbol{\mu} \in \boldsymbol{\Lambda}\},$$

satisfying condition $(*)$, yield a hyperladder $(f_{\boldsymbol{\mu}}) \in inv_*^{\Lambda}\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$.

Proof. Let $(f_{\boldsymbol{\mu}})$ be a hyperladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$. By Lemma 6, $(f_{\boldsymbol{\mu}})$ yields a hyperladder $(f'_{\boldsymbol{\mu}})$ of $\boldsymbol{X}$ to $\boldsymbol{Y}$ having a unique increasing index function $\phi : \Lambda \to \Lambda$, $\phi \geq 1_\Lambda$, for all of its nonempty ladders. (The morphisms of an $f'_{\boldsymbol{\mu}}$ are those of $f_{\boldsymbol{\mu}}$ composed with the appropriate bonding morphisms of $\boldsymbol{X}$.) Given a $\mu \in \Lambda$, put

$$F_\mu = \{f_\mu^{\boldsymbol{\mu}} \mid f_\mu^{\boldsymbol{\mu}} = f'_\mu \in f'_{\boldsymbol{\mu}}, \boldsymbol{\mu} \in \boldsymbol{\Lambda}\}.$$

Then, the special property of the hyperladder $(f'_{\boldsymbol{\mu}})$ (one can put λ^1, for $\mu_1' \geq \mu_1$, to be $\phi(\mu_1')$) obviously implies condition $(*)$. Conversely, let an increasing $\phi : \Lambda \to \Lambda$, $\phi \geq 1_\Lambda$, and a family $(F_\mu)_{\mu \in M}$,

$$F_\mu = \{f_\mu^{\boldsymbol{\mu}} \in \mathcal{A}(X_{\phi(\mu)}, Y_\mu) \mid \boldsymbol{\mu} \in \boldsymbol{\Lambda}\},$$

be given, such that condition $(*)$ holds. Let $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ be associated with a related pair (μ_1,μ_2). If $\phi(\mu_1) \not\leq \mu_2$, put $f_{\boldsymbol{\mu}}$ to be the empty ladder. If $\phi(\mu_1) \leq \mu_2$, then, in every full chain (linearly ordered) C from μ_1 to μ_2, there exists a maximal μ_C such that $\phi(\mu_C) \leq \mu_2$. For every such C and every $\mu \in C$, $\mu \leq \mu_C$, put $f_\mu = f_\mu^{\boldsymbol{\mu}} : X_{\phi(\mu)} \to Y_\mu$. Then the commutativity in condition $(*)$ insures that the morphisms f_μ define a nonempty ladder $f_{\boldsymbol{\mu}}$ of $\boldsymbol{X}$ to $\boldsymbol{Y}$. Observe that the family $(f_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$, obtained in this way has ϕ as the unique index function for all the (nonempty) ladders. Finally, condition $(*)$ immediately implies that $(f_{\boldsymbol{\mu}})$ is a hyperladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$. □

According to Lemma 10, an alternative convenient description of a hyperladder $(f_{\boldsymbol{\mu}})$ may be given by an ordered pair $(f,(F_\mu)_{\mu\in\Lambda})$, where the family $(F_\mu)_{\mu\in\Lambda}$ of ($|\Lambda|$ *cardinality*) families

$$F_\mu = \{f_\mu^{\boldsymbol{\mu}} \in \mathcal{A}(X_{f(\mu)}, Y_\mu) \mid \boldsymbol{\mu} \in \boldsymbol{\Lambda}\}$$

fulfills condition $(*)$. On the other hand, every $*$-morphism (f, f_μ^n) is also an ordered pair $(f,(F_\mu)_{\mu\in\Lambda})$, where the family $(F_\mu)_{\mu\in\Lambda}$ of (*countable*) families

$$F_\mu = \{f_\mu^n \in \mathcal{A}(X_{f(\mu)}, Y_\mu) \mid n \in \mathbb{N}\}$$

is subjected to the appropriate commutativity condition. The conclusion follows by considering the corresponding equivalence relations (see also the proof of Theorem 9). (Recall that cardinality $|\boldsymbol{\Lambda}|$ of $\boldsymbol{\Lambda}$ equals cardinality $|\Lambda|$ of Λ.) To be even more specific, we give the following example.

Example 3. Consider the (standard) weak shape category Sh_*, i.e. the case $\mathcal{C} = HTop$ and $\mathcal{D} = HPol$. Let X be a topological space which does *not* admit any *countable* $HPol$-expansion and let Q be a polyhedron consisting of two points. Then,

$$card(Sh(X,Q)) = card(HTop(X,Q)) \equiv card([X,Q]) = 2.$$

Further, by [10], Claims 1 and 2 of Section 4,

$$card(Sh^*(X,Q)) = (card([X,Q]))^{\aleph_0} = 2^{\aleph_0}.$$

Finally, by Lemma 10.

$$card(Sh_*(X,Q)) = (card([X,Q]))^{|\Lambda|} > 2^{\aleph_0}.$$

Corollary 5 and Example 3 show that, in order to provide a pair of spaces belonging to different coarse shape types and to the same weak shape type, one has to consider a class of spaces which do not admit countable polyhedral (or ANR) expansions.

8. Weak Shape Isomorphisms

In this section we shall characterize (realizing) weak shape isomorphisms by establishing an (abstract) analogue of the well known Morita Lemma, see [17]. The corresponding criterion is somewhat stronger because it does not involve any level representative. Further, we shall give a criterion for the weak shape isomorphyness of a (realizing) pair of *sequential* expansions ($\Lambda = \mathbb{N}$), which does not explicitly involve any (realizing) weak shape morphism. It suffices to formulate and prove the appropriate theorems for pro_*^Λ-$\mathcal{A}$ and tow_*-$\mathcal{A}$ respectively, where $\mathcal{A}$ is an arbitrary category.

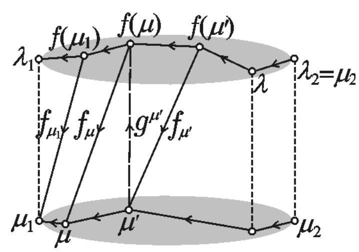
Theorem 10. *Let $\boldsymbol{f}_* = [(f_{\boldsymbol{\mu}})] \in pro_*^\Lambda$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. If $\boldsymbol{f}_*$ is an isomorphism, then every representative $(f_{\boldsymbol{\mu}})$ of $\boldsymbol{f}_*$ fulfills the following condition:*

$$\text{(WI) } (\forall \mu_1 \in \Lambda)(\forall \mu \geq \mu_1)(\exists \mu' \geq \mu)(\exists \lambda \geq \mu')(\forall \mu_2 \geq \lambda)$$

there exists a $g^{\mu'} : Y_{\mu'} \to X_{f(\mu)}$ commuting with the ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$, where $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ is associated with the chosen pair (μ_1, μ_2). More precisely,

$$f_\mu g^{\mu'} = q_{\mu\mu'} \text{ and } g^{\mu'} f_{\mu'} p_{f(\mu')\lambda} = p_{f(\mu)\lambda}.$$

Conversely, if there exists a representative $(f_{\boldsymbol{\mu}})$ of $\boldsymbol{f}_*$, having a unique cofinal index function, such that condition (WI) holds, then $\boldsymbol{f}_*$ is an isomorphism.



Proof. Let $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$ be an isomorphism of pro_*^Λ-$\mathcal{A}$, and let $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a representative of $\boldsymbol{f}_*$ in inv_*^Λ-$\mathcal{A}$. Then, for every representative $(g_{\boldsymbol{\lambda}}) : \boldsymbol{Y} \to \boldsymbol{X}$ of $\boldsymbol{f}_*^{-1}$,

$$(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) = (g_{\boldsymbol{\lambda}} f_{\boldsymbol{\lambda}}) \simeq (1_{\boldsymbol{X\lambda}}) \text{ and } (f_{\boldsymbol{\mu}})(g_{\boldsymbol{\lambda}}) = (f_{\boldsymbol{\mu}} g_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{Y\mu}})$$

hold in inv_*^Λ-$\mathcal{A}$. That means

$$(\forall \lambda_1 \in \Lambda)(\forall \lambda_1' \geq \lambda_1)(\exists \lambda_*^1 \geq \lambda_1')(\forall \lambda_2 \geq \lambda_*^1)\ g_{\boldsymbol{\lambda}} f_{\boldsymbol{\lambda}} \simeq 1_{\boldsymbol{X\lambda}}$$

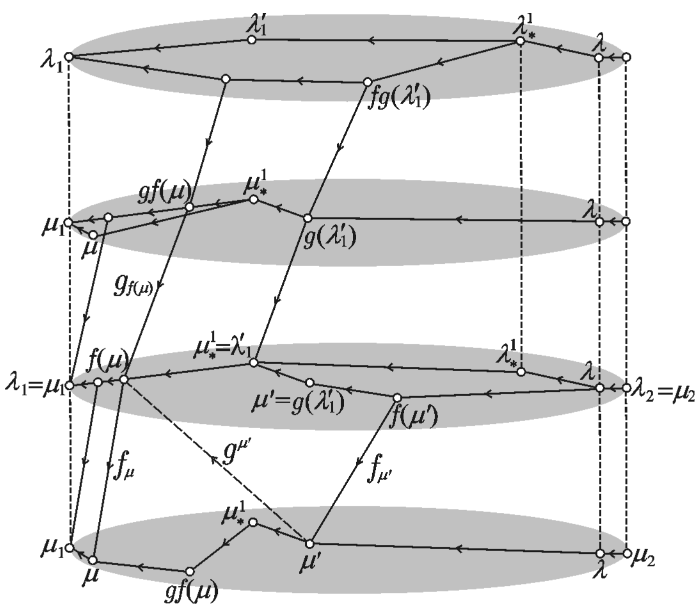
as desired, and

$$(\forall \mu_1 \in \Lambda)(\forall \mu_1' \geq \mu_1)(\exists \mu_*^1 \geq \mu_1')(\forall \mu_2 \geq \mu_*^1)\ f_{\boldsymbol{\mu}} g_{\boldsymbol{\mu}} \simeq 1_{\boldsymbol{Y\mu}}$$

as desired (see Definition 3). By Lemma 6, there exists a representative $(g_{\boldsymbol{\lambda}})$ having a unique index function $g \geq 1_\Lambda$. Let any $\mu_1 \in \Lambda$ and any $\mu \geq \mu_1$ be given. By $(f_{\boldsymbol{\mu}})(g_{\boldsymbol{\lambda}}) \simeq (1_{\boldsymbol{Y}\boldsymbol{\mu}})$, for μ_1 and $\mu_1' = \mu$, there exists a $\mu_*^1 \geq \mu_1'$ having the mentioned property. Put $\mu' = g(\mu_*^1)$. Then $\mu' \geq \mu_*^1 \geq \mu_1' = \mu$ because of $g \geq 1_\Lambda$. Further, by $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X}\boldsymbol{\lambda}})$, for $\lambda_1 = \mu_1$ and $\lambda_1' = \mu_*^1 \geq \lambda_1$, there exists a $\lambda_*^1 \geq \lambda_1'$ having the mentioned property. Since Λ is directed, there exists a $\lambda \geq \lambda_*^1, g(\lambda_1')$. Then $\lambda \geq \mu'$. Let $\mu_2 = \lambda_2 \geq \lambda$ be chosen arbitrarily, and consider the corresponding ladders $f_{\boldsymbol{\mu}}$ and $(g_{\boldsymbol{\lambda}})$, where $\boldsymbol{\lambda} = \boldsymbol{\mu}$ is associated with the pair $(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$. Notice that

$$f(\mu) \leq gf(\mu) \leq \mu_*^1 = \lambda_1' \leq g(\lambda_1').$$

An appropriate picture is given below.



Put

$$g^{\mu'} = g_{f(\mu)} q_{gf(\mu)\mu'} : Y_{\mu'} \to X_{f(\mu)},$$

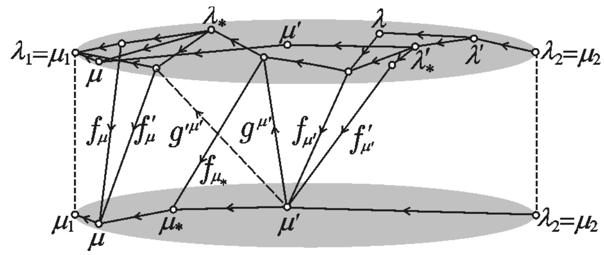
and the needed commutativity in condition (WI) for $(f_{\boldsymbol{\mu}})$ is straightforward. Conversely, let a morphism $\boldsymbol{f}_* \in pro_*^\Lambda\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ admit a representative $(f_{\boldsymbol{\mu}}) \in inv_*^\Lambda\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, having a unique cofinal (increasing) index function $f : \Lambda \to \Lambda$ for all the ladders $f_{\boldsymbol{\mu}}$, such that condition (WI) for $(f_{\boldsymbol{\mu}})$ holds. (f - cofinal means: every λ admits a μ such that $f(\mu) \geq \lambda$.) We have to construct a hyperladder $(g_{\boldsymbol{\lambda}}) : \boldsymbol{Y} \to \boldsymbol{X}$, such that

$$(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X}\boldsymbol{\lambda}}) \text{ and } (f_{\boldsymbol{\mu}})(g_{\boldsymbol{\lambda}}) \simeq (1_{\boldsymbol{Y}\boldsymbol{\mu}}).$$

Let us first prove the following simple fact: *If a hyperladder* $(f_{\boldsymbol{\mu}}) \in inv_*^{\Lambda}$-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ *, having a unique cofinal index function, fulfills condition* (WI) *, then every hyperladder* $(f'_{\boldsymbol{\mu}})$ *equivalent to* $(f_{\boldsymbol{\mu}})$ *,* $(f'_{\boldsymbol{\mu}}) \simeq (f_{\boldsymbol{\mu}})$ *, also fulfills* (WI) *.*

Let $\mu_1 \in \Lambda$ and let $\mu \geq \mu_1$. By $(f'_{\boldsymbol{\mu}}) \simeq (f_{\boldsymbol{\mu}})$, there exists a $\lambda_* \geq \mu$ such that, for every $\mu_2 \geq \lambda_*$, the appropriate homotopy condition is fulfilled. Since the index function f of $(f_{\boldsymbol{\mu}})$ is cofinal (and increasing), there exists a $\mu_* \geq \mu$ such that $f(\mu_*) \geq \lambda_*$. By condition (WI) of $(f_{\boldsymbol{\mu}})$, for μ_1 and $\mu_* \geq \mu_1$, there exist a $\mu' \geq \mu_*$ and a $\lambda \geq \mu'$ such that, for every $\mu_2 \geq \lambda$, there exists an $\mathcal{A}$-morphism $g^{\mu'} : Y_{\mu'} \to X_{f(\mu_*)}$ satisfying the appropriate commutativity conditions. Again, by $(f'_{\boldsymbol{\mu}}) \simeq (f_{\boldsymbol{\mu}})$, now for μ_1 and $\mu' \geq \mu_1$, there exists a $\lambda'_* \geq \mu'$ such that, for every $\mu_2 \geq \lambda'_*$, the appropriate homotopy condition is fulfilled. Observe that $\lambda'_* \geq f(\mu_*) \geq \lambda_*$. Finally, choose a $\lambda' \geq \lambda, \lambda'_*$, and let $\mu_2 \geq \lambda'$. Now, put

$$g'^{\mu'} = p_{f'(\mu)f(\mu_*)}g^{\mu'} : Y_{\mu'} \to X_{f('\mu)}.$$



Then it is straightforward to verify that

$$f'_{\mu}g'^{\mu'} = q_{\mu\mu'} \text{ and } g'^{\mu'}f'_{\mu'}p_{f'(\mu')\lambda'} = p_{f'(\mu)\lambda'}.$$

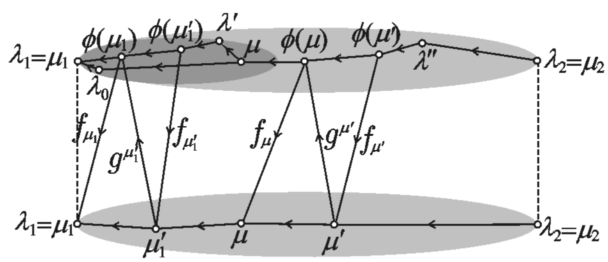
Therefore, $(f'_{\boldsymbol{\mu}})$ fulfills condition (WI). According to this fact and Lemma 6, it suffices to consider a hyperladder $(f_{\boldsymbol{\mu}}) \in \boldsymbol{f}_*$ which has a unique index function $\phi : \Lambda \to \Lambda$, $\phi \geq 1_{\Lambda}$, for all the nonempty ladders $f_{\boldsymbol{\mu}}$, such that $(f_{\boldsymbol{\mu}})$ fulfills condition (WI). Let $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ be associated with a related pair (λ_1, λ_2), and let us consider the ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$, where $\boldsymbol{\mu} = \boldsymbol{\lambda}$. Our intention is to construct a ladder $g_{\boldsymbol{\lambda}} : \boldsymbol{Y} \to \boldsymbol{X}$ by fitting together the morphisms $g^{\mu'}$ existing by condition (WI) for $(f_{\boldsymbol{\mu}})$ over the given segment $\boldsymbol{\mu}$. First, if $J = \emptyset$ or, for $\mu = \mu_1 \in J \neq \emptyset$, any appropriate $\mu' \equiv \mu'_1 \notin J$ or any appropriate $\lambda \equiv \lambda' \not\leq \lambda_2 = \mu_2$, put $g'_{\boldsymbol{\lambda}}$ to be the empty ladder, $g_{\boldsymbol{\lambda}} = \emptyset$. If $J \neq \emptyset$ and, for $\mu = \mu_1$, an appropriate ("minimal") $\mu' \equiv \mu'_1 \in J$ and an appropriate ("minimal") $\lambda \equiv \lambda' \leq \lambda_2$, put

$$g_{\lambda} = p_{\lambda\phi(\mu_1)}g^{\mu'} : Y_{\mu'_1} \to X_{\lambda},\ \lambda_1 \leq \lambda \leq \phi(\mu_1),$$

where $g^{\mu'} : Y_{\mu'_1} \to X_{\phi(\mu_1)}$ exists by condition (WI). In order to define the needed morphisms g_λ for some other λ, $\lambda_1 < \lambda < \lambda_2$ and $\lambda \not\leq \phi(\mu_1)$, whenever it is possible, let us first consider a full chain C in $\boldsymbol{\lambda}$ starting at λ_1, and the minimal $\lambda_0 \in C$ such that $\lambda_0 \not\leq \phi(\mu_1)$ (if it exists). Then choose a ("minimal") $\mu \geq \lambda', \lambda_0$. Notice that $\mu \geq \lambda' \geq \phi(\mu'_1) \geq \mu'_1$. Let $\mu' \geq \mu$ and $\lambda'' \geq \mu'$ be chosen (" minimal") by condition (WI) for μ_1 and μ. If $\mu \notin J$ or $\mu' \notin J$ or $\lambda'' \not\leq \lambda_2$, let there be no new morphisms into an X_λ. If $\mu \in J$ and $\mu' \in J$ and $\lambda'' \leq \lambda_2$, put

$$g_\lambda = p_{\lambda\phi(\mu)} g^{\mu'} : Y_{\mu'} \to X_\lambda,$$
$$\lambda_0 \leq \lambda \leq \phi(\mu) \text{ or } \phi(\mu_1) < \lambda \leq \phi(\mu),$$

where $g^{\mu'} : Y_{\mu'} \to X_{\phi(\mu)}$ exists by condition (WI).



By chasing the appropriate diagram in the above picture, one readily sees ($\phi(\mu) \geq \mu$) that
$$g_{\phi(\mu_1)} q_{\mu'_1\mu'} = p_{\phi(\mu_1)\phi(\mu)} g_{\phi(\mu)}.$$
Consequently, by construction, for every defined pair g_λ, $g_{\lambda'}$, if $\lambda \leq \lambda'$ then

$$g_\lambda q_{\mu_*\mu_{**}} = p_{\lambda\lambda'} g_{\lambda'},$$

where $\mu_* \leq \mu_{**}$ belong to $\{\mu'_1, \mu'\}$. Further, let us consider the minimal $\lambda'_0 \in C$ such that $\lambda'_0 \not\leq \phi(\mu)$ (if it exists). Then repeat the above construction. If some new g_λ's occur, they must commute mutually as well as with the other g_λ according to the related indices. By continuing this procedure, the construction must stop in finitely many steps, because $\boldsymbol{\lambda}$ is a finite set.

To complete the construction, we need a slight correction as follows: After defining g_λ, $\lambda_1 \leq \lambda \leq \phi(\mu_1)$, consider the minimal elements $\lambda_0^1, \ldots, \lambda_0^k$ in all full chains $C_1, \ldots, C_k$ in $\boldsymbol{\lambda}$ (starting at λ_1) respectively, such that $\lambda_0^i \not\leq \phi(\mu_1)$ for all $i = 1, \ldots, k$ (if they exist). Then choose a ("minimal") $\mu \geq \lambda, \lambda_0^1, \ldots, \lambda_0^k$. Now proceed as before, i.e. apply condition (WI) for μ_1 and μ. In the next step, one also has to consider all the "missing" $\lambda_0'^1, \ldots, \lambda_0'^k$ (the existing ones only) and choose a $\mu'' \geq \lambda', \lambda_0'^1, \ldots, \lambda_0'^k$, and then proceed as before. The construction stops in finitely many steps. Since the obtained morphisms g_λ commute according to the related indices, they define a ladder $g_{\boldsymbol{\lambda}}$ of $\boldsymbol{Y}$ to $\boldsymbol{X}$, having an increasing index function $g : I \to \boldsymbol{\mu} = \boldsymbol{\lambda}$, where $g(\lambda)$ is the corresponding "μ'" which comes from condition (WI).

We are to prove that the obtained family $(g_{\boldsymbol{\lambda}})$, $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, is a hyperladder of $\boldsymbol{Y}$ to $\boldsymbol{X}$. Indeed, since $(f_{\boldsymbol{\mu}})$ is a hyperladder, for every $\lambda_1 \in \Lambda$ and every $\lambda_1' \geq \lambda_1$, the above construction without a given $\lambda_2 \geq \lambda_1'$, i.e. with no given $\boldsymbol{\lambda}$ in advance, reaches the given λ_1'. Namely, for $\mu_1 = \lambda_1$ there exists a μ, $\mu_1 \leq \mu \leq \lambda_1'$, such that $\phi(\mu) \geq \lambda_1'$. Now, since $(f_{\boldsymbol{\mu}})$ is a hyperladder, one can reach, by construction, a $\mu^1 \geq \mu, \lambda_1'$ large enough, such that, for every $\mu_2 = \lambda_2 \geq \mu^1$, the corresponding ladder $g_{\boldsymbol{\mu}}$ has the index function $g : I \to \boldsymbol{\mu} = \boldsymbol{\lambda}$, such that $\lambda_1' \in I \subseteq \boldsymbol{\lambda}$ and $g(\lambda_1') \leq \mu^1$.

Finally, the equivalence relations $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X}\boldsymbol{\lambda}})$ and $(f_{\boldsymbol{\mu}})(g_{\boldsymbol{\lambda}}) \simeq (1_{\boldsymbol{Y}\boldsymbol{\mu}})$ follow immediately by condition (WI) used in the construction. □

Corollary 7. *A morphism $\boldsymbol{f}_* = [(f_{\boldsymbol{\mu}})] \in tow_*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ is an isomorphism of $tow_*\text{-}\mathcal{A}$ if and only if there exists a (equivalently, every) representative $(f_{\boldsymbol{\mu}}) \in inv_*^{\mathbb{N}}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ of $\boldsymbol{f}_*$ satisfying (satisfies) condition (WI).*

Proof. It suffices to observe that, in the case of inverse sequences, the auxiliary fact (proved in the proof of Theorem 10.) holds unconditionally, i.e. if a hyperladder $(f_{\boldsymbol{\mu}}) \in inv_*^{\mathbb{N}}\text{-}(\boldsymbol{X}, \boldsymbol{Y})$ fulfills condition (WI), then every hyperladder $(f'_{\boldsymbol{\mu}})$ equivalent to $(f_{\boldsymbol{\mu}})$ also fulfills (WI). (In other words, in the case of inverse sequences, (WI) is a property of a morphism $\boldsymbol{f}_*$ of $tow_*\text{-}\mathcal{A}$.) □

Corollary 8. *Let X be a topological space. Then the following assertions are equivalent:*

(i) The shape of X is trivial, $Sh(X) = 0$;

(ii) The coarse shape of X is trivial, $Sh^*(X) = 0$;

(iii) The weak shape of X is trivial, $Sh_*(X) = 0$.

Proof. It suffices to prove that (iii) implies (i). Let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ be an appropriate $HPol$-expansion of X, and let $\boldsymbol{q} : Y \to \boldsymbol{Y} = (Y_\lambda, [q_{\lambda\lambda'}], \Lambda)$ be the trivial expansion of a singleton $Y \equiv \{*\}$ ($\equiv Y_\lambda$, $\lambda \in \Lambda$, $[q_{\lambda\lambda'}] = q_{\lambda\lambda'} : \{*\} \to \{*\}$, $\lambda \leq \lambda'$). Since $Sh_*(X) = 0 = Sh_*(\{*\})$, there exists an isomorphism $\boldsymbol{g}_* : \boldsymbol{Y} \to \boldsymbol{X}$ of $pro_*^{\Lambda}\text{-}HPol(\boldsymbol{Y}, \boldsymbol{X})$. By Theorem 10, condition (WI) for $\boldsymbol{g}_*$, for every $\lambda \in \Lambda$ there exists a $\lambda' \geq \lambda$ such that the bonding morphism $[p_{\lambda\lambda'}]$ is trivial, i.e. every representing mapping $p_{\lambda\lambda'} \in [p_{\lambda\lambda'}]$ is homotopically trivial. The conclusion follows by the proof of Theorem II.2.3.7 of [15]. □

For another criterion, in the case of inverse sequences ($\Lambda = \mathbb{N}$) in a category $\mathcal{A}$, we need a few auxiliary notions. For every ordered pair (ϕ, ψ) of increasing functions $\phi, \psi : \mathbb{N} \to \mathbb{N}$, such that $\phi > 1_{\mathbb{N}}$ and $\psi > 1_{\mathbb{N}}$, and for every $\lambda \in \mathbb{N}$, the sequence (in $\mathbb{N}$)

$$\lambda, \phi(\lambda), \psi\phi(\lambda), \phi\psi\phi(\lambda), \dots,$$
$$(\psi\phi)^{n-1}(\lambda), \phi(\psi\phi)^{n-1}(\lambda), \dots, ((\psi\phi)^0 \equiv 1_{\mathbb{N}}),$$

denoted shortly by $(\phi, \psi)_\lambda$, is said to be the λ- *trace* of (ϕ, ψ). Given a pair of inverse sequences $\boldsymbol{X}, \boldsymbol{Y}$ in a category $\mathcal{A}$ and a μ-trace $(\phi, \psi)_\mu$, $\mu \in \mathbb{N}$, a *serpent on* $(\phi, \psi)_\mu$ is a set consisting of $2n$ $\mathcal{A}$-morphisms

$$f_i^n : X_{\phi(\mu')} \to Y_{\mu'} \text{ and } g_i^n : Y_{\psi\phi(\mu')} \to X_{\phi(\mu')},$$

where $i = 1, \dots, n$, $n \in \mathbb{N}$, $\mu' \in \{\mu_*, \psi\phi(\mu_*), \dots, (\psi\phi)^n(\mu_*)\}$, and μ_* is an odd member of $(\phi, \psi)_\mu$, such that the following diagram in $\mathcal{A}$ commutes:

$$\begin{array}{ccccccccc} X_{\phi(\mu_*)} & \leftarrow & X_{\phi\psi\phi(\mu_*)} & \leftarrow \cdots \leftarrow & X_{\phi(\psi\phi)^{n-1}(\mu_*)} & & \\ f_1^n \downarrow & \nwarrow g_1^n & \downarrow f_2^n & \nwarrow g_2^n \quad \cdots & \downarrow f_n^n & \nwarrow g_n^n & . \\ Y_{\mu_*} & \leftarrow & Y_{\psi\phi(\mu_*)} & \leftarrow \cdots \leftarrow & Y_{(\psi\phi)^{n-1}(\mu_*)} & \leftarrow Y_{(\psi\phi)^n(\mu_*)} & \end{array}$$

More precisely, we should speak about an $(\boldsymbol{X}, \boldsymbol{Y})$-$n$-serpent on a trace $(\phi, \psi)_\mu$ ending at (an odd member) μ_*.

Let $\boldsymbol{\mu} = \boldsymbol{\lambda} \in \boldsymbol{N}$ be associated with a related pair $(\mu_1, \mu_2) = (\lambda_1 \lambda_2)$, and let $\mu_1' \in \mathbb{N}$, $\mu_1 \le \mu_1' \le \mu_2$. Let $\boldsymbol{\mu}$ contain $2n$ consequtive members of a μ_1-trace $(\phi, \psi)_{\mu_1}$, for some $n \in \mathbb{N}$, such that $(\psi\phi)^{n-1}(\mu_1) \ge \mu_1'$. Then every corresponding $(\boldsymbol{X}, \boldsymbol{Y})$-$n$-serpent on $(\phi, \psi)_{\mu_1}$ ending at μ_1 is said to be *a serpent within* $\boldsymbol{\mu}$ *reaching* μ_1'.

Theorem 11. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be inverse sequences in a category* $\mathcal{A}$. *Then* $\boldsymbol{X} \cong \boldsymbol{Y}$ *in tow*$_*$-$\mathcal{A}$, *if and only if there exists a pair of increasing functions* $\phi, \psi : \mathbb{N} \to \mathbb{N}$, $\phi, \psi > 1_{\mathbb{N}}$, *such that the following condition is fulfilled:*

$$\textit{(WI)}_* \ (\forall \mu_1 \in \Lambda)(\forall \mu_1' \ge \mu_1)(\forall \mu_2 \ge \phi(\psi\phi)^2(\mu_1'))$$

there exists a serpent on the μ_1-*trace* $(\phi, \psi)_{\mu_1}$ *within* $\boldsymbol{\mu}$ *reaching* μ_1', *where* $\boldsymbol{\mu}$ *is associated with the chosen pair* (μ_1, μ_2).

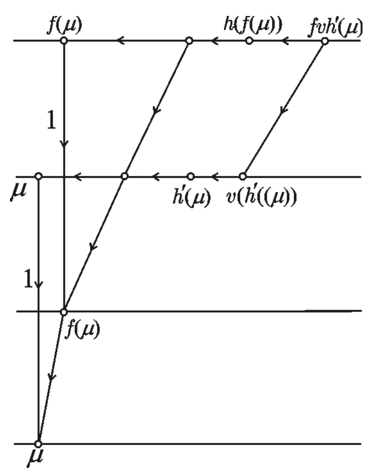
Proof. Let $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow_*-$\mathcal{A}$. Then there exists a pair of isomorphisms $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g}_* = \boldsymbol{f}_*^{-1} : \boldsymbol{Y} \to \boldsymbol{X}$ such that $\boldsymbol{g}_*\boldsymbol{f}_* = \boldsymbol{1}_{\boldsymbol{X}^*}$ and $\boldsymbol{f}_*\boldsymbol{g}_* = \boldsymbol{1}_{\boldsymbol{Y}^*}$ in tow_*-$\mathcal{A}$. Hence, for any choice of the representatives $(f_{\boldsymbol{\mu}})$ of $\boldsymbol{f}_*$ and $(g_{\boldsymbol{\lambda}})$ of $\boldsymbol{g}_*$,

$$(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X\lambda}}) \text{ and } (f_{\boldsymbol{\mu}})(g_{\boldsymbol{\lambda}}) \simeq (1_{\boldsymbol{Y\mu}})$$

hold. By Lemma 6, we can choose hyperladders $(f_{\boldsymbol{\mu}})$, $(g_{\boldsymbol{\lambda}})$ having the unique (increasing) index functions $f, g : \mathbb{N} \to \mathbb{N}$ respectively, such that $f, g \geq 1_{\mathbb{N}}$, and that, in addition, the above homotopy relations are realized via a pair of increasing functions $h, h' : \mathbb{N} \to \mathbb{N}$ respectively, such that $h, h' \geq 1_{\mathbb{N}}$. Further, without loss of generality, we may also assume that all these index functions are strictly greater than $1_{\mathbb{N}}$, because the appropriate shifts are allowed (inductively). Put $\phi = f$ and

$$\psi(\lambda) = \begin{cases} vh'(\lambda), & \lambda \in f[\mathbb{N}]\,, \\ g(\lambda), & \lambda \in \mathbb{N} \setminus f[\mathbb{N}]\,, \end{cases}$$

where $v : \mathbb{N} \to \mathbb{N}$ is a shift function satisfying $fvh' \geq hf$. Then, ψ is increasing and $\psi \geq g > 1_{\mathbb{N}}$.



Let $\mu_1 \in \mathbb{N}$ and let $\mu_1' \geq \mu_1$. Then there exists a unique (minimal) $n \in \mathbb{N}$ such that $(\psi\phi)^{n-1}(\mu_1) \geq \mu_1'$. If $n > 1$, then $\mu_1 < (\psi\phi)^{n-1}(\mu_1) \leq \mu_1'$. Notice that $n = 1$ if and only if $\mu_1' = \mu_1$. By $(f_{\boldsymbol{\mu}})(g_{\boldsymbol{\lambda}}) \simeq (1_{\boldsymbol{Y\mu}})$, for μ_1 and $(\psi\phi)^{n-1}(\mu_1) \geq \mu_1$, there exists a $\mu_*^1 \geq (\psi\phi)^{n-1}(\mu_1)$ (for instance, $\mu_*^1 = h'((\psi\phi)^{n-1}(\mu_1)))$ such that, for every $\mu_2 \geq \mu_*^1$, the corresponding ladders $f_{\boldsymbol{\mu}}g_{\boldsymbol{\mu}}$ and $1_{\boldsymbol{Y\mu}}$ are homotopic in the desired way. In this particular case, by the previous adaptation, it assures that the choice

$$\mu^1 = \psi\phi((\psi\phi)^{n-1}(\mu_1)) = (\psi\phi)^n(\mu_1) \geq h'((\psi\phi)^{n-1}(\mu_1)) = \mu_*^1$$

implies

$$f_\mu g'_{\phi(\mu)} = q_{\mu\psi\phi(\mu)},$$

where $\mu \in \{(\psi\phi)^{i-1}(\mu_1) \mid i = 1, \ldots, n\}$ and

$$g'_{\phi(\mu)} \equiv g_{f(\mu)} q_{gf(\mu)vh'(\mu)} = g_{\phi(\mu)} q_{g\phi(\mu)\psi(\mu)}.$$

Further, by $(g_{\boldsymbol\lambda})(f_{\boldsymbol\mu}) \simeq (1_{\boldsymbol{X\lambda}})$, for $\lambda_1 = \mu_1$ and $\phi(\psi\phi)^{n-1}(\lambda_1) \geq \lambda_1$, there exists a $\lambda_*^1 \geq \phi(\psi\phi)^{n-1}(\lambda_1)$ (for instance, $\lambda_*^1 = h(\phi(\psi\phi)^{n-1}(\lambda_1)))$ such that, for every $\lambda_2 \geq \lambda_*^1$, the corresponding ladders $g_{\boldsymbol\lambda} f_{\boldsymbol\lambda}$ and $1_{\boldsymbol{X\lambda}}$ are homotopic as desired. According to the previous adaptation, it assures that the choice

$$\lambda^1 = \phi\psi(\phi(\psi\phi)^{n-1}(\lambda_1)) = \phi(\psi\phi)^n(\lambda_1) \geq h(\phi(\psi\phi)^{n-1}(\lambda_1)) = \lambda_*^1$$

implies

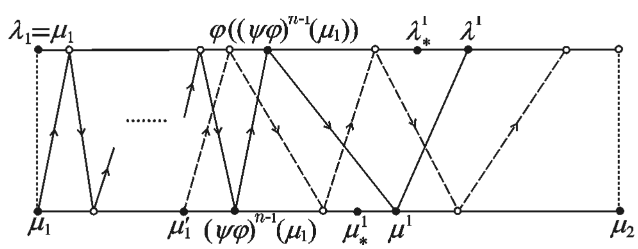
$$g'_{\phi(\mu)} f_{\psi\phi(\mu)} = p_{\phi(\mu)\phi\psi\phi(\mu)},$$

where $\mu \in \{(\psi\phi)^{i-1}(\mu_1) \mid i = 1, \ldots, n\}$. By choosing a $\mu_2 = \lambda_2 \geq \lambda^1 \geq \max\{\lambda_*^1, \mu^1\} \geq \mu_*^1$, all the above relations hold for the ladders $f_{\boldsymbol\mu}$ and $g_{\boldsymbol\lambda}$, $\boldsymbol\lambda = \boldsymbol\mu$. Now, let $\mu_2 \geq \phi(\psi\phi)^2(\mu_1')$. Observe that it implies $\mu_2 \geq \lambda^1$. Indeed, if $n > 1$, then

$$\lambda^1 = \phi(\psi\phi)^n(\mu_1) = \phi(\psi\phi)^2(\psi\phi)^{n-2}(\mu_1) < \phi(\psi\phi)^2(\mu_1') \leq \mu_2,$$

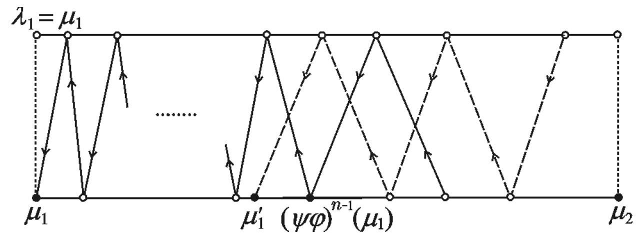
while, if $n = 1$, then $\mu_1' = \mu_1$ and $\lambda^1 = \phi(\psi\phi)(\mu_1') < \phi(\psi\phi)^2(\mu_1') \leq \mu_2$. For every $i = 1, \ldots, n$, put

$$f_i^n = f_{(\psi\phi)^{i-1}(\mu_1)} : X_{\phi(\psi\phi)^{i-1}(\mu_1)} \to Y_{(\psi\phi)^{i-1}(\mu_1)},$$
$$g_i^n = g'_{\phi(\psi\phi)^{i-1}(\mu_1)} : Y_{(\psi\phi)^1(\mu_1)} \to X_{\phi(\psi\phi)^{i-1}(\mu_1)}.$$



These morphisms make a desired serpent on the trace $(\phi, \psi)_{\mu_1}$ within $\boldsymbol\mu$ reaching μ_1', that verifies condition $(\mathrm{WI})_*$.

Conversely, let for inverse sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$ in a category $\mathcal{A}$, there exist a pair of increasing functions $\phi, \psi : \mathbb{N} \to \mathbb{N}$, $\phi, \psi > 1_{\mathbb{N}}$, such that condition $(\mathrm{WI})_*$ is fulfilled. We are to construct a pair of hyperladders $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$, $(g_{\boldsymbol{\lambda}}) : \boldsymbol{Y} \to \boldsymbol{X}$ such that $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X}\boldsymbol{\lambda}})$ and $(f_{\boldsymbol{\mu}})(g_{\boldsymbol{\lambda}}) \simeq (1_{\boldsymbol{Y}\boldsymbol{\mu}})$. Let $\boldsymbol{\mu} \in \boldsymbol{N}$ be associated with a related pair $\mu_1 < \mu_2$ in $\mathbb{N}$. Choose the maximal (if it exists) $\mu_1' \in \mathbb{N}$ such that $\mu_1' \geq \mu_1$ and $\phi(\psi\phi)^2(\mu_1') \leq \mu_2$. If such a μ_1' does not exist, let the corresponding ladders $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$ and $g_{\boldsymbol{\lambda}} : \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{\lambda} = \boldsymbol{\mu}$, be empty. If such μ_1' exists, it is unique. By condition $(\mathrm{WI})_*$, there exists an $(\boldsymbol{X}, \boldsymbol{Y})$-$n$-serpent on the trace $(\phi, \psi)_{\mu_1}$ within $\boldsymbol{\mu}$ reaching μ_1' (and ending at μ_1).



It provides the morphisms

$$f_i^n : X_{\phi(\psi\phi)^{i-1}(\mu_1)} \to Y_{(\psi\phi)^{i-1}(\mu_1)},\ g_i^n : Y_{(\psi\phi)^{i}(\mu_1)} \to X_{\phi(\psi\phi)^{i-1}(\mu_1)},$$

$i = 1, \ldots, n$. Put $f_{\mu_1} = f_1^n$ and, for every $i = 1, \ldots, n-1$ and every μ, $(\psi\phi)^{i-1}(\mu_1) < \mu \leq (\psi\phi)^i(\mu_1)$, put $f_\mu = q_{\mu(\psi\phi)^i(\mu_1)} f_{i+1}^n$. Further, for every λ, $\lambda_1 \leq \lambda \leq \phi(\mu_1)$, put $g_\lambda = p_{\lambda\phi(\mu_1)} g_1^1$, and for every $i = 1, \ldots, n-1$ and every λ, $\phi(\psi\phi)^{i-1}(\mu_1) < \lambda \leq \phi(\psi\phi)^i(\mu_1)$, put $g_\lambda = p_{\lambda\phi(\psi\phi)^i(\mu_1)} g_{i+1}^n$. In this way, the ladders $f_{\boldsymbol{\mu}}$ and $g_{\boldsymbol{\lambda}}$ are well defined for all $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \boldsymbol{N}$. The construction guarantees that the families $(f_{\boldsymbol{\mu}})$ and $(g_{\boldsymbol{\lambda}})$ are hyperladders of $\boldsymbol{X}$ to $\boldsymbol{Y}$ and of $\boldsymbol{Y}$ to $\boldsymbol{X}$ respectively. Namely, given a $\mu_1 \in \mathbb{N}$ and a $\mu_1' \geq \mu_1$, put $\lambda^1 = \phi(\psi\phi)^2(\mu_1')$. Then, for every $\mu_2 \geq \lambda^1$, the corresponding ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$ fulfills the desired condition. Further, given a $\lambda_1 \in \mathbb{N}$ and a $\lambda_1' \geq \lambda_1$, put $\mu_1 = \lambda_1$ and $\mu_1' = \lambda_1'$, and take $\mu^1 = \lambda^1 = \phi(\psi\phi)^2(\mu_1')$ as before. Then, for every $\lambda_2 \geq \mu^1$, the ladder $g_{\boldsymbol{\lambda}} \in (g_{\boldsymbol{\lambda}})$ has the desired property. Finally, the equivalence relations $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X}\boldsymbol{\lambda}})$ and $(f_{\boldsymbol{\mu}})(g_{\boldsymbol{\lambda}}) \simeq (1_{\boldsymbol{Y}\boldsymbol{\mu}})$ follow immediately by construction. □

Observe that Theorem 11 directly relates the (abstract) weak shape to the S^*-equivalence naturally generalized to any *tow*-category (compare Corollary 6 from above and [10], Theorem 7.)

Corollary 9. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in a category $\mathcal{A}$. Then the following are equivalent:*

(i) $\boldsymbol{X} \cong \boldsymbol{Y}$ *in* tow_*-$\mathcal{A}$*;*

(ii) $\boldsymbol{X} \cong \boldsymbol{Y}$ *in* tow^*-$\mathcal{A}$*;*

(iii) $S^*(\boldsymbol{X}) \cong S^*(\boldsymbol{Y})$.

9. Some Weak Shape Invariants

In [12] (see also [16], Remark) it is proved that several important shape invariant properties of metric compacta are actually S (S^*)-invariant properties. We are going to prove that those, and a few others, are weak shape invariants in the most general setting, i.e. whenever they make sense.

Lemma 11. *Let X and Y be topological spaces such that X is weak shape dominated by Y, $Sh_*(X) \leq Sh_*(Y)$. Then the following statements hold:*

(i) If Y is connected, then so is X.

(ii) If $Sh(Y) = 0$, then also $Sh(X) = 0$.

(iii) If the shape dimension $sd(Y) \leq n$, then also $sd(X) \leq n$.

(iv) If Y is n-shape connected, then so is X.

Proof. Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be $HPol$-expansions of X and Y respectively, having the same index set Λ (admissible for the reduced pro-category $pro^{\sim}_*$-$HPol$; see Proposition 1 of Section 3). Since $Sh_*(X) \leq Sh_*(Y)$, there exists a pair of morphisms $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g}_* : \boldsymbol{Y} \to \boldsymbol{X}$ of $pro^{\sim}_*$-$HPol$ such that $\boldsymbol{g}_*\boldsymbol{f}_* = \boldsymbol{1}_{\boldsymbol{X}^*}$. By Lemma 6, there exists a pair of representatives $(f_{\boldsymbol{\mu}}) \in \boldsymbol{f}_*$, $(g_{\boldsymbol{\lambda}}) \in \boldsymbol{g}_*$ having unique increasing index functions $f, g \geq 1_\Lambda$ respectively for all the ladders. Moreover, there exists an increasing function $\chi : \Lambda \to \Lambda$, $\chi \geq fg \geq 1_\Lambda$, realizing the homotopy relation $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X\lambda}})$. Therefore,

$$(\forall \lambda_1 \in \Lambda)(\forall \lambda_1' \geq \lambda_1)(\exists \lambda_*^1 = \chi(\lambda_1') \geq \lambda_1', fg(\lambda_1'))(\forall \lambda_2 \geq \lambda_*^1)$$
$$g_{\boldsymbol{\lambda}} f_{\boldsymbol{\lambda}} \simeq 1_{\boldsymbol{X\lambda}}$$

as desired (see Definition 3), where $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ is associated with (λ_1, λ_2). In particular, this means that

$$g_{\lambda_1} f_{g(\lambda_1)} p_{fg(\lambda_1)\lambda_*^1} \simeq p_{\lambda_1 \lambda_*^1}.$$

$$\begin{array}{ccccc} & Y_{g(\lambda_1)} & & & \\ g_{\lambda_1} \swarrow & & \nwarrow f_{g(\lambda_1)} \cdots & & . \\ X_{\lambda_1} & \leftarrow & X_{fg(\lambda_1)} & \leftarrow & X_{\lambda^1_*} \end{array}$$

Let Y be connected. Then one may assume that each Y_μ, $\mu \in \Lambda$, is connected. Thus, the above homotopy relation implies that $p_{\lambda_1 \lambda^1_*}(X_{\lambda^1_*})$ is contained in a single component of X_{λ_1}. Consequently, X is connected, and (i) is proved.

Let Y have the trivial shape, $Sh(Y) = 0$. Hence, one may assume that each Y_μ, $\mu \in \Lambda$, is a singleton. Now, a consideration quite similar to the above one shows that the bonding mapping $p_{\lambda_1 \lambda^1_*}$ is homotopically trivial, $p_{\lambda_1 \lambda^1_*} \simeq 0$. This implies that the shape of X is trivial, $Sh(X) = 0$. Thus, statement (ii) is proved. (Recall that, by Corollary 8, the trivial shape and weak trivial shape are equivalent properties).

Let the shape dimension $sd(Y) \leq n$. According to Theorem II.1.2. of [15], one may assume that each polyhedron Y_μ, $\mu \in \Lambda$, has dimension $\dim Y_\mu \leq n$. Then, the above homotopy relation proves that, for every $\lambda \equiv \lambda_1$, there exists a $\lambda' \equiv \lambda^1_* \geq \lambda$ such that the bonding mapping $p_{\lambda\lambda'}$ factors up to homotopy through $Y_{g(\lambda)}$. By the mentioned theorem again, $sd(X) \leq n$ holds, and (iii) is proved.

Finally, let Y be n-shape connected, i.e. Y is connected and $pro\text{-}\pi_k(Y, y_0) = 0$, $k = 1, \ldots, n$, for some point $y_0 \in Y$. This means that, for every $\mu \in \Lambda$, there exists a $\mu' \in \Lambda$ such that the induced homomorphisms $(q_{\mu\mu'})_\# : \pi_k(Y_{\mu'}, y_{\mu'}) \to \pi_k(Y_\mu, y_\mu)$, $k = 1, \ldots, n$, are trivial, $(q_{\mu\mu'})_\# = 0$; hereby, $y_\mu \equiv q_\mu(y_0)$, $\mu \in \Lambda$. Choose any $x_0 \in X$ and denote $x_\lambda \equiv p_\lambda(x_0)$, $\lambda \in \Lambda$. Now, for every $\lambda \equiv \lambda_1 \in \Lambda$, choose a $\mu' \geq \mu \equiv g(\lambda_1)$ such that $(q_{\mu\mu'})_\# = 0$. Notice that there exists a $\lambda'_1 \geq \lambda_1$ such that $g(\lambda'_1) \geq \mu'$ (for instance, $\lambda'_1 = \mu'$). By the hyperladders homotopy relation $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X_\lambda}})$, for λ_1 and $\lambda'_1 \geq \lambda_1$, there exists a $\lambda^1_* \geq \lambda'_1$ such that, for every $\lambda_2 \geq \lambda^1_*$, the desired homotopy relation holds. In particular, it implies that

$$g_{\lambda_1} q_{g(\lambda_1) g(\lambda'_1)} f_{g(\lambda'_1)} p_{fg(\lambda'_1)\lambda^1_*} \simeq p_{\lambda_1 \lambda^1_*}.$$

Put $\lambda' \equiv \lambda^1_*$, and consider a path $\omega_{\mu'}$ in $Y_{\mu'}$ joining

$$y_{\mu'} \text{ and } q_{\mu' g(\lambda'_1)} f_{g(\lambda'_1)} p_{fg(\lambda'_1)\lambda'}(x_{\lambda'}).$$

By the homotopy extension property, there exists a mapping

$$f : (X_{\lambda'}, x_{\lambda'}) \to (Y_{\mu'}, y_{\mu'})$$

which is (freely) homotopic to

$$q_{\mu' g(\lambda'_1)} f_{g(\lambda'_1)} p_{fg(\lambda'_1)\lambda'} : (X_{\lambda'}, x_{\lambda'}) \to (Y_{\mu'}, y'_{\mu'}),$$

where $y'_{\mu'} \equiv q_{\mu' g(\lambda'_1)} f_{g(\lambda'_1)} p_{fg(\lambda'_1)\lambda'}(x_{\lambda'})$, such that the corresponding homotopy restricted to $\{x_{\lambda'}\} \times I$ reduces to $\omega_{\mu'}$. According to the above homotopy relation, the mappings $g_\lambda q_{\mu\mu'} f : (X_{\lambda'}, x_{\lambda'}) \to (X_\lambda, g_\lambda(y_\mu))$ and $p_{\lambda\lambda'} : (X_{\lambda'}, x_{\lambda'}) \to (X_\lambda, x_\lambda)$ are (freely) homotopic. By [19], Theorems 1.8.7 and 7.3.14, the homomorphism

$$(p_{\lambda\lambda'})_\# : \pi_k(X_{\lambda'}, x_{\lambda'}) \to \pi_k(X_\lambda, x_\lambda)$$

is the composition of

$$(g_\lambda q_{\mu\mu'} f)_\# = (g_\lambda)_\# (q_{\mu\mu'})_\# f_\# : \pi_k(X_{\lambda'}, x_{\lambda'}) \to \pi_k(X_\lambda, g(y_\mu)),$$

with an isomorphism $\pi_k(X_\lambda, g(y_\mu)) \to \pi_k(X_\lambda, x_\lambda)$, $k = 1, \ldots, n$. Since $(q_{\mu\mu'})_\# = 0$, it follows that $(p_{\lambda\lambda'})_\# = 0$. Thus, X is n-shape connected, and the lemma is proved. □

Lemma 12. *Let $(\mathcal{C}, \mathcal{D})$ be a category pair such that $\mathcal{D} \subseteq \mathcal{C}$ is dense and full. Let $X, Y \in Ob\mathcal{C}$ and let X be (abstract) weak shape dominated by Y, $Sh_{*(\mathcal{C},\mathcal{D})}(X) \leq Sh_{*(\mathcal{C},\mathcal{D})}(Y)$. Then the following statements hold:*

(i) If Y is movable, then so is X.

(ii) For every full subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$, if Y is $\mathcal{D}_0$-movable, then so is X.

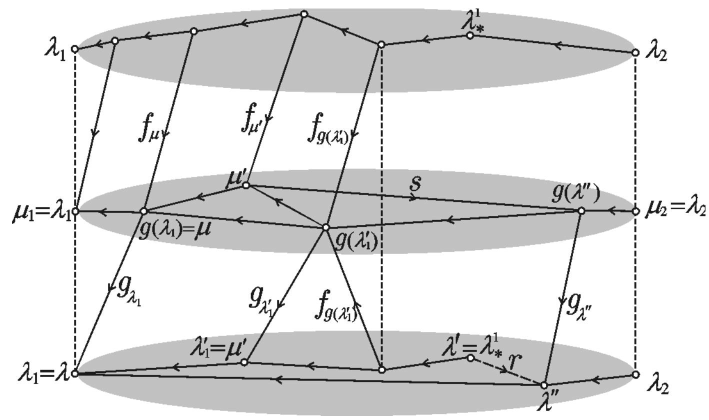
(iii) If Y is semi-stable, then so is X.

(iv) If Y is strongly movable, then so is X.

Proof. By replacing $HPol$ with $\mathcal{D}$, the first (common) part of the proof of Lemma 11 holds in this setting as well. To prove (i), let us recall that Y is movable provided, for every $\mu \in \Lambda$, there exists a $\mu' \geq \mu$ such that, for every $\mu'' \geq \mu$, there exists a $\mathcal{D}$-morphism $s : Y_{\mu'} \to Y_{\mu''}$ satisfying $q_{\mu\mu''} s = q_{\mu\mu'}$. Let $\lambda \in \Lambda$. Denote $\lambda_1 \equiv \lambda$, and let $\mu_1 = \lambda_1$. Then $g(\lambda_1) \geq \mu_1$. Let μ' be the movability index for $\mu \equiv g(\lambda_1)$. Notice that $\mu' \geq g(\lambda_1) \geq \lambda_1$.

Let $\lambda'_1 = \mu'$. Then $g(\lambda'_1) \geq \mu'$. By the equivalence relation $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X\lambda}})$, for λ_1 and $\lambda'_1 \geq \lambda_1$, there exists a $\lambda^1_* \geq \lambda'_1, fg(\lambda_1)$ (for instance $\lambda^1_* = \chi(\lambda'_1)$) such that, for every $\lambda_2 \geq \lambda^1_*$, the appropriate relation holds. Put $\lambda' = \lambda^1_*$, and let $\lambda'' \geq \lambda = \lambda_1$. Then $g(\lambda'') \geq g(\lambda_1)$, and there exists an $s : Y_{\mu'} \to Y_{g(\lambda'')}$ such that $q_{g(\lambda_1) g(\lambda'')} s = q_{g(\lambda_1)\mu'}$. Choose any $\lambda_2 \geq \lambda^1_*, g(\lambda'')$ $(\geq \lambda'_1)$, and let $g_{\boldsymbol{\lambda}}$, $f_{\boldsymbol{\mu}}$ be the corresponding ladders, where $\boldsymbol{\mu} = \boldsymbol{\lambda}$ is associated with $(\mu_1, \mu_2) = (\lambda_1, \lambda_2)$. Put

$$r = g_{\lambda''} s q_{\mu' g(\lambda'_1)} f_{g(\lambda'_1)} p_{fg(\lambda'_1)\lambda'} : X_{\lambda'} \to X_{\lambda''}.$$



Then,

$$\begin{aligned} p_{\lambda\lambda''} r &= p_{\lambda_1\lambda''} g_{\lambda''} s q_{\mu' g(\lambda_1')} f_{g(\lambda_1')} p_{fg(\lambda_1')\lambda'} \\ &= g_{\lambda_1} q_{g(\lambda_1)g(\lambda'')} s q_{\mu' g(\lambda_1')} f_{g(\lambda_1')} p_{fg(\lambda_1')\lambda'} \\ &= g_{\lambda_1} q_{g(\lambda_1)\mu'} q_{\mu' g(\lambda_1')} f_{g(\lambda_1')} p_{fg(\lambda_1')\lambda'} \\ &= g_{\lambda_1} q_{g(\lambda_1)g(\lambda_1')} f_{g(\lambda_1')} p_{fg(\lambda_1')\lambda'} = p_{\lambda_1\lambda_1'} g_{\lambda_1'} f_{g(\lambda_1')} p_{fg(\lambda_1')\lambda'} \\ &\equiv p_{\lambda_1\lambda_1'} g_{\lambda_1'} f_{g(\lambda_1')} p_{fg(\lambda_1')\lambda^1_*} \equiv p_{\lambda_1\lambda_1'} p_{\lambda_1'\lambda^1_*} = p_{\lambda_1\lambda^1_*} \equiv p_{\lambda\lambda'}. \end{aligned}$$

This implies that X is movable.

Let Y be $\mathcal{D}_0$-movable, i.e. for every $\mu \in \Lambda$, there exists a $\mu' \geq \mu$ such that, for every $\mu'' \geq \mu$, every $P \in Ob\mathcal{D}_0$ and every $\mathcal{D}$-morphism $h : P \to Y_{\mu'}$, there exists a $\mathcal{D}$-morphism $s : P \to Y_{\mu''}$ such that $q_{\mu\mu''} s = q_{\mu\mu'} h$. Let $\lambda \in \Lambda$. Denote $\lambda_1 \equiv \lambda$ and $\mu \equiv g(\lambda_1) \geq \mu_1 = \lambda_1$. Let $\mu' \geq \mu$ be the $\mathcal{D}_0$-movability index for μ. Notice $g(\mu') \geq \mu' \geq \lambda_1$ because of $g \geq 1_\Lambda$. Let $\lambda^1_* \geq \mu', fg(\mu')$ be an index existing by the equivalence relation $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X\lambda}})$, for λ_1 and $\lambda_1' \equiv \mu' \geq \lambda_1$ (for instance, $\lambda^1_* = \chi(\mu')$). Put $\lambda' = \lambda^1_*$. Let a $\lambda'' \geq \lambda$, a $\mathcal{D}_0$-object P and a $\mathcal{D}$-morphism $h : P \to X_{\lambda'}$ be given. Notice that $g(\lambda'') \geq \mu$. Let $\lambda_2 = \mu_2 \geq \lambda^1_*, g(\lambda'')$. Since Y is $\mathcal{D}_0$-movable, for the morphism

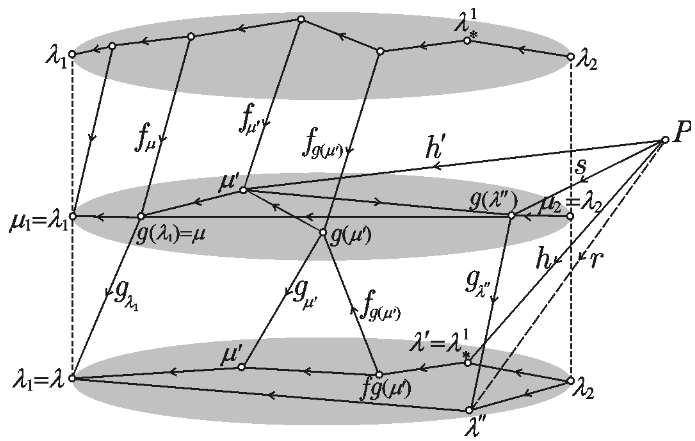
$$h' = q_{\mu' g(\mu')} f_{g(\mu')} p_{fg(\mu')\lambda'} h : P \to Y_{\mu'},$$

there exists a morphism $s : P \to Y_{g(\lambda'')}$ such that $q_{\mu g(\lambda'')} s = q_{\mu\mu'} h'$. Put

$$r = g_{\lambda''} s : P \to X_{\lambda''}.$$

Then,

$$p_{\lambda\lambda''} r = p_{\lambda_1\lambda''} g_{\lambda''} s = g_{\lambda_1} q_{g(\lambda_1)g(\lambda'')} s \equiv g_{\lambda_1} q_{\mu g(\lambda'')} s$$



$$= g_{\lambda_1} q_{\mu\mu'} h' \equiv g_{\lambda_1} q_{g(\lambda_1)\mu'} q_{\mu' g(\lambda'')} f_{g(\mu')} p_{fg(\mu')\lambda'} h$$
$$= g_{\lambda_1} q_{g(\lambda_1)g(\mu')} f_{g(\mu')} p_{fg(\mu')\lambda_*^1} h = p_{\lambda_1\mu'} g_{\mu'} f_{g(\mu')} p_{fg(\mu')\lambda_*^1} h$$
$$= p_{\lambda_1\mu'} p_{\mu'\lambda_*^1} h = p_{\lambda_1\lambda_*^1} h \equiv p_{\lambda\lambda'} h.$$

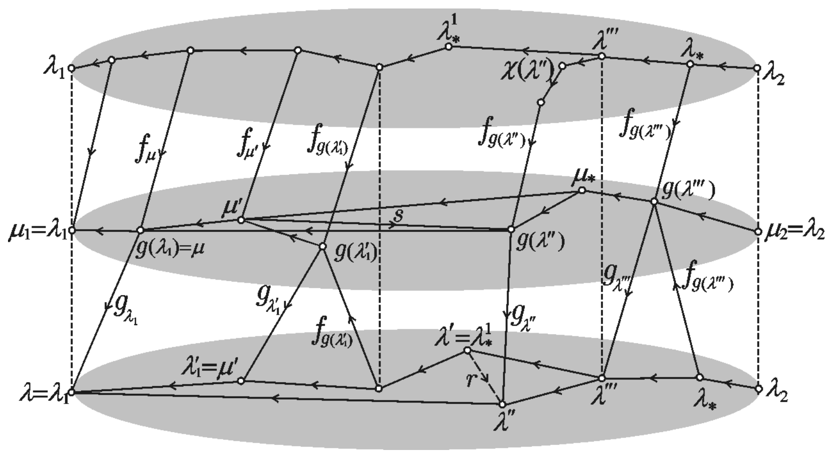
This implies that X is $\mathcal{D}_0$-movable, and statement (ii) is proved.

Let Y be semi-stable (the complementary part of the strong movability: [20], Section 3, and [22], Remark 4), i.e. for every $\mu \in \Lambda$, there exists a $\mu' \geq \mu$ such that, for every $\mu'' \geq \mu$, there exist a $\mu_* \geq \mu', \mu''$ and a $\mathcal{D}$-morphism $s : Y_{\mu'} \to Y_{\mu''}$ satisfying $s q_{\mu'\mu_*} = q_{\mu''\mu_*}$. In order to prove that X is semi-stable, let us repeat the construction in the proof of statement (i), and then, let $\mu_* \geq \mu', g(\lambda'')$ be an existing index for $\boldsymbol{Y}$. Then $\mu_* \geq \lambda''$. Let $\chi(\lambda'')$ be the index coming from $(g_{\boldsymbol{\lambda}})(f_{\boldsymbol{\mu}}) \simeq (1_{\boldsymbol{X\lambda}})$ for λ_1 and $\lambda'' \geq \lambda_1$. Since Λ is cofinite and $g \geq 1_\Lambda$ increasing, there exists a $\lambda''' \geq \lambda', \chi(\lambda'')$ such that $g(\lambda''') \geq \mu_*$. Then $\lambda''' \geq \lambda''$. Put $\lambda_* = fg(\lambda''') \geq \lambda'''$. Then $\lambda_* \geq \lambda', \lambda''$. Choose any $\lambda_2 \geq \lambda_*$, and consider the corresponding ladders $g_{\boldsymbol{\lambda}}$ and $f_{\boldsymbol{\mu}}$, where $\boldsymbol{\mu} = \boldsymbol{\lambda}$ is associated with $(\mu_1, \mu_2) = (\lambda_1, \lambda_2)$.

Then,

$$r p_{\lambda'\lambda_*} = g_{\lambda''} s q_{\mu' g(\lambda_1')} f_{g(\lambda_1')} p_{fg(\lambda_1')\lambda'} p_{\lambda' fg(\lambda''')}$$
$$= g_{\lambda''} s q_{\mu' g(\lambda_1')} f_{g(\lambda_1')} p_{fg(\lambda_1') fg(\lambda''')}$$
$$= g_{\lambda''} s q_{\mu' g(\lambda_1')} q_{g(\lambda_1') g(\lambda''')} f_{g(\lambda''')} = g_{\lambda''} s q_{\mu' g(\lambda''')} f_{g(\lambda''')}$$
$$= g_{\lambda''} s q_{\mu'\mu_*} q_{\mu_* g(\lambda''')} f_{g(\lambda''')} = g_{\lambda''} q_{g(\lambda'')\mu_*} q_{\mu_* g(\lambda''')} f_{g(\lambda''')} = g_{\lambda''} q_{g(\lambda'') g(\lambda''')} f_{g(\lambda''')}$$
$$= g_{\lambda''} f_{g(\lambda'')} p_{fg(\lambda'') fg(\lambda''')} = p_{\lambda'' fg(\lambda''')} \equiv p_{\lambda''\lambda_*}.$$

Therefore, X is semi-stable, and (iii) is proved. Finally, to prove assertion (iv),



it suffices to fit together the constructions in the proofs of (i) and (iii). This completes the proof of the lemma. □

Theorem 12. *Each of the following properties of a topological space:*
connectedness;
triviality of shape;
shape dimension $sd \leq n$;
n-shape connectedness;
movability;
n-movability;
semi-stability;
strong movability,
is a weak shape invariant.

Proof. Apply Lemmata 11 and 12. □

Theorem 13. *(i) Let $(\boldsymbol{X}, \boldsymbol{o})$ and $(\boldsymbol{Y}, \boldsymbol{o})$ be pro-(pointed sets) such that $\boldsymbol{X} \leq \boldsymbol{Y}$ in $pro^{\sim}_{*}$-Set_0. If $(\boldsymbol{Y}, \boldsymbol{o})$ is ML (having the Mittag-Leffler property), then so is $(\boldsymbol{X}, \boldsymbol{o})$.*

(ii) Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be pro-groups such that $\boldsymbol{G} \leq \boldsymbol{H}$ in $pro^{\sim}_{*}$-Grp. If $\boldsymbol{H}$ is ML, then so is $\boldsymbol{G}$.

Proof. By [15], Theorem II.6.6, a pointed pro-set has the Mittag-Leffler property if and only if it is movable. Now, the proof of Lemma 12 (i) implies the conclusion that our assertion (i) is true. Further, by [15], Corollary II.6.5, a pro-group has the Mittag-Leffler property if and only if it is movable with respect to free groups. Now, the proof of Lemma 12 (ii) implies the conclusion that our assertion (ii) is true. □

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