International Journal of Pure and Applied Mathematics

Volume 39 No. 3 2007, 425-440

ON ITERATIVE SOLUTIONS OF NONLINEAR EQUATIONS OF THE ϕ-STRONGLY ACCRETIVE TYPE IN UNIFORMLY SMOOTH BANACH SPACES

Shin Min Kang[1 §], Zeqing Liu[2], Chahn Yong Jung[3]

[1]Department of Mathematics
Research Institute of Natural Science
Gyeongsang National University
Jinju, 660-701, KOREA
e-mail: smkang@nongae.gsnu.ac.kr

[2]Department of Mathematics
Liaoning Normal University
P.O. Box 200, Dalian, Liaoning, 116029, P.R. CHINA
e-mail: zeqingliu@sina.com.cn

[3]Department of Business Administration
Gyeongsang National University
Jinju, 660-701, KOREA
e-mail: bb5734@nongae.gsnu.ac.kr

Abstract: Let X be a real uniformly smooth Banach space and $T : X \to X$ be a demicontinuous ϕ-strongly accretive operator. It is proved that the Ishikawa iteration method with errors converges strongly to the solutions of the equations $f = Tx$ and $f = x + Tx$, respectively. A related result deals with the iterative approximation of fixed points of ϕ-hemicontractive operators. Our results extend some known results in the literature.

AMS Subject Classification: 47H17, 47H15, 47H05

Key Words: Ishikawa iteration sequence with errors, demicontinuous operator, ϕ-strongly accretive operator, ϕ-hemicontractive operator, uniformly smooth Banach space

Received: July 19, 2007

§Correspondence author

1. Introduction

For a real Banach space X we denote by J the normalized duality mapping from X into 2^{X^*}given by

$$Jx = \left\{f^* \in X^* : \langle x, f^* \rangle = \|f^*\|^2 = \|x\|^2\right\}, \quad \forall x \in X,$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, I denotes the identity operator, the symbols $D(T), R(T)$ and $F(T)$ stand for the domain, the range and the set of all fixed points of an operator T, respectively.

Definition 1.1. Let $T : D(T) \subset X \to X$ be an operator.

(1) T is said to be *accretive* if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq 0;$$

(2) T is said to be *strongly accretive* if there exists a constant $k \in (0,1)$ such that for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \geq k \|x-y\|^2;$$

(3) T is called *ϕ-strongly accretive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\|x-y\|) \|x-y\|;$$

(4) T is said to be *pseudocontractive* if for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2;$$

(5) T is said to be *strongly pseudocontractive* (respectively, *ϕ-strongly pseudocontractive*) if $(I-T)$ is strongly accretive (respectively, ϕ-strongly accretive);

(6) T is called *ϕ-hemicontractive* if $F(T) \neq \emptyset$ and there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x \in D(T)$, $q \in F(T)$ there exists $j(x-q) \in J(x-q)$ satisfying

$$\langle Tx - q, j(x-q) \rangle \leq \|x-q\|^2 - \phi(\|x-q\|) \|x-q\|;$$

(7) T is said to be *m-accretive* if it is accretive and $R(I + \lambda T) = X$ for all $\lambda > 0$.

The classes of operators introduced above have been studied by various researchers (see, for example [1-48]). It is well known that any strongly accretive and strongly pseudocontractive operators are ϕ-strongly accretive and ϕ-strongly pseudocontractive, respectively, but the converses do not hold. It is clear that every ϕ-strongly pseudocontractive operator with a nonempty fixed point set is ϕ-hemicontractive. In [9], Chidume and Osilike constructed an operator which is ϕ-hemicontractive but not ϕ-strongly pseudocontractive. The accretive operators were introduced independently in 1967 by Browder [2] and Kato [18]. Interest in accretive operators stems mainly from their firm connection with the existence theory for nonlinear evolution equations in Banach spaces. It is well known that many physically significant problems can be modeled in terms of an initial value problem of the form

$$\frac{du}{dt} + Tu = 0, \quad u\left(0\right) = u_0, \tag{1.1}$$

where T is accretive in an appropriate Banach space. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem (1.1) is solvable if T is locally Lipschitz and accretive on X. Browder established also that if $T : X \to X$ is locally Lipschitz and accretive, then T is m-accretive. Martin [39] and Morales [40] generalized indeed the results of Browder to both continuous accretive operators and strongly accretive and continuous operators.

If $T : X \to X$ is strongly accretive and the following equation

$$f = Tx \tag{1.2}$$

has a solution, methods for approximating the solution have been investigated by a few researchers. Chidume [5] obtained that if $X = L_p$(or l_p), $p \geq 2$, and $T : X \to X$ is a Lipschitz strongly accretive operator, then the Mann iteration method can be used to approximate the solution of (1.2). Since the publication of Chidume's result, several researchers have extended it in various directions. In [12], Deng generalized the result to the Ishikawa iteration sequence. Tan and Xu [44] extended the results of Chidume [5] and Deng [12] to real p-uniformly smooth Banach spaces, where $1 < p < 2$. In [10], Chidume and Osilike improved the results of Chidume [5], Deng [12] and Tan and Xu [44] to all real p-uniformly smooth Banach spaces, where $p > 1$. Osilike [41] generalized the above results to Lipschitz ϕ-strongly accretive operators. Xu [45] improved the results of Chidume and Osilike [10] by using the Ishikawa and Mann iteration methods with errors.

The aim of this paper is to study the iterative approximation of solutions to the equations (1.2) and $f = x + Tx$ in the case when X is a uniformly smooth Banach space and $T : X \to X$ is a demicontinuous ϕ-strongly accretive operator. A related result deals with the iterative approximation of fixed points of ϕ-hemicontractive operators. Our results are improvements and extension of the results that have appeared recently. In particular, the results of Chang [3], Chang et al [4], Chidume [5-8], Chidume and Osilike [10], Deng [12-14], Deng and Ding [15], Osilike [41], Tan and Xu [44], Xu [45], Zhou [47], Zhou and Jia [48] and others will be special cases of our theorems.

2. Preliminaries

The following lemmas play a crucial role in the proofs of our main results.

Lemma 2.1. (see [25]) *Suppose that X is a real uniformly smooth Banach space and $T : X \to X$ is a demicontinuous ϕ-strongly accretive operator. Then the equation $Tx = f$ has a unique solution for any $f \in X$.*

Lemma 2.2. (see [45]) *Let X be a real uniformly smooth Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in X, \forall j \in J. \tag{2.3}$$

Lemma 2.3. (see [2], [46]) *Let X be a real Banach space. Then the following conditions are equivalent:*

(a) *X is uniformly smooth;*

(b) *X^* is uniformly convex;*

(c) *J is single valued and uniformly continuous on any bounded subset of X.*

Lemma 2.4. (see [25]) *Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative and bounded sequence and $\phi : [0,\infty) \to [0,\infty)$ be strictly increasing and $\phi(0) = 0$. Assume that $A(a_n) = \frac{\phi(a_n)}{1+a_n+\phi(a_n)}$ for all $n \ge 0$. Then the following statements are equivalent:*

(d) $\inf\{A(a_n) : n \ge 0\} = 0$;

(e) $\inf\{\phi(a_n) : n \ge 0\} = 0$;

(f) *There exists a subsequence $\{a_{n_k}\}_{k=0}^{\infty}$ of $\{a_n\}_{n=0}^{\infty}$ such that $a_{n_k} \to 0$ as $k \to \infty$.*

Lemma 2.5. (see [20) *Let* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ *and* $\{\gamma_n\}_{n=0}^{\infty}$ *be three non-negative real sequences satisfying the inequality*

$$\alpha_{n+1} \leq (1-\omega_n)\,\alpha_n + \beta_n\omega_n + \gamma_n$$

for all $n \geq 0$, *where* $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty}\omega_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n < \infty$. *Then* $\lim_{n\to\infty}\alpha_n = 0$.

3. Main Results

Our main results are as follows:

Theorem 3.1. *Let* X *be a real uniformly smooth Banach space and* $T : X \to X$ *be a demicontinuous* ϕ*-strongly accretive operator. Assume that* $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ *are bounded sequences in* X*and* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *are sequences in* $[0,1]$ *satisfying the conditions*

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = 0; \tag{3.1}$$

$$\sum_{n=0}^{\infty} b_n = +\infty, \quad \sum_{n=0}^{\infty} c_n < +\infty; \tag{3.2}$$

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad \forall n \geq 0. \tag{3.3}$$

For any given $f, x_0 \in X$, *define* $S = f + x - Tx$, $\forall x \in X$ *and the Ishikawa iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$ *by*

$$\begin{aligned} y_n &= a'_n x_n + b'_n S x_n + c'_n v_n, \\ x_{n+1} &= a_n x_n + b_n S y_n + c_n u_n, \quad \forall n \geq 0. \end{aligned} \tag{3.4}$$

If at least one of the following conditions

$$\text{the sequences } \{x_n - Tx_n\}_{n=0}^{\infty} \text{ and } \{y_n - Ty_n\}_{n=0}^{\infty} \text{ are bounded}; \tag{3.5}$$

$$\text{the sequences } \{Tx_n\}_{n=0}^{\infty} \text{ and } \{Ty_n\}_{n=0}^{\infty} \text{ are bounded}; \tag{3.6}$$

is fulfilled, then the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the solution of the equation* $Tx = f$.

Proof. It follows from Lemma 2.1 that the equation $Tx = f$ has a unique solution $q \in X$. It is easy to verify that S is demicontinuous and q is a unique fixed point of S. Thus for any $x, y \in X$ there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Sx - Sy, j(x-y)\rangle \leq \|x-y\|^2(1 - A(\|x-y\|)), \tag{3.7}$$

where $A(x,y) = \frac{\phi(\|x-y\|)}{1+\|x-y\|+\phi(\|x-y\|)} \in [0,1)$ for all $x, y \in X$. This implies that

$$\langle (I - S - A(x,y))x - (I - S - A(x,y))y, j(x-y)\rangle \geq 0,$$

and it follows from Lemma 1.1 of Kato [18] that

$$\|x-y\| \leq \|x - y + r[(I - S - A(x,y))x - (I - S - A(x,y))y]\| \tag{3.8}$$

for all $x, y \in X$ and $r > 0$. Since T is ϕ-strongly accretive, so that for any $x, y \in X$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|,$$

which implies that

$$\phi(\|x-y\|) \leq \|Tx - Ty\|$$

for any $x, y \in X$. Observe that

$$\|Sx - Sy\| \leq \|x-y\| + \|Tx - Ty\| \leq \phi^{-1}(\|Tx - Ty\|) + \|Tx - Ty\|$$

and

$$\|Sx - Sy\| \leq \|x - Tx\| + \|y - Ty\|$$

for all $x, y \in X$. Hence either of (3.5) and (3.6) ensures that both $\{Sx_n\}_{n=0}^{\infty}$ and $\{Sy_n\}_{n=0}^{\infty}$ are bounded. Put $d_n = b_n + c_n$, $d'_n = b'_n + c'_n$ and

$$D = \max\Big\{ \sup_{n\geq 0} \|Sx_n - q\|, \sup_{n\geq 0} \|Sy_n - q\|, \sup_{n\geq 0} \|u_n - q\|, \sup_{n\geq 0} \|v_n - q\|, \|x_0 - q\| \Big\}. \tag{3.9}$$

By induction, (3.3), (3.4) and (3.9), we easily infer that for all $n \geq 0$,

$$\max\{\|x_n - q\|, \|y_n - q\|\} \leq D. \tag{3.10}$$

Set $B = 4D^2$, $t_n = \|j(x_{n+1} - q) - j(y_n - q)\|$ and $s_n = D^2(d_n + 4d'_n + 8c'_n) + 2Dt_n$ for all $n \geq 0$. Observe that

$$\|(x_{n+1} - q) - (y_n - q)\| \leq 2D(d_n + d'_n) \to 0$$

as $n \to \infty$. It follows from Lemma 2.3 that

$$\lim_{n\to\infty} t_n = 0. \tag{3.11}$$

In view of (3.1), (3.2) and (3.11), we have

$$\lim_{n\to\infty} s_n = 0. \tag{3.12}$$

Using Lemma 2.2, (3.3), (3.4), (3.9) and (3.10), we conclude that

$$\|y_n - q\|^2 \leq (1 - d'_n)^2 \|x_n - q\|^2 + 2\langle d'_n(Sx_n - q)$$

$$+ c'_n(v_n - Sx_n), j(y_n - q)\rangle \le (1-d'_n)^2\|x_n - q\|^2 + 2D^2(d'_n + 2c'_n) \quad (3.13)$$

for all $n \ge 0$. Using again Lemma 2.2, (3.3), (3.4), (3.9), (3.10) and (3.13), we obtain that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \le (1-d_n)^2\|x_n - q\|^2 + 2\langle d_n(Sy_n - q) \\
&+ c_n(u_n - Sy_n), j(x_{n+1} - q)\rangle \le (1-d_n)^2\|x_n - q\|^2 + 2d_n\langle Sy_n - q, j(y_n - q)\rangle \\
&+ 2d_n\langle Sy_n - q, j(x_{n+1} - q) - j(y_n - q)\rangle + 2c_n\langle u_n - Sy_n, j(x_{n+1} - q)\rangle \\
&\le (1 - \min\{A(y_n, q), A(x_n, q)\}d_n)\|x_n - q\|^2 + d_n s_n + Bc_n \quad (3.14)
\end{aligned}$$

for all $n \ge 0$. Put $\inf_{n\ge 0}\min\{A(y_n, q), A(x_n, q)\} = r$. We assert that $r = 0$. Otherwise $r > 0$. It follows from (3.14) that

$$\|x_{n+1} - q\|^2 \le (1 - rd_n)\|x_n - q\|^2 + d_n s_n + Bc_n \quad (3.15)$$

for all $n \ge 0$. Let $\alpha_n = \|x_n - q\|^2$, $\beta_n = d_n s_n$, $\gamma_n = Bc_n$ and $\omega_n = rd_n$. In virtue of Lemma 2.5, (3.1), (3.2), (3.12) and (3.15), we infer immediately that $\alpha_n \to 0$ as $n \to \infty$. This means that $r = 0$, which is a contradiction.

Now we claim that there exists a positive integers sequence $\{n_i\}_{i=0}^{\infty}$ such that

$$\lim_{i\to\infty}\|x_{n_i} - q\| = \lim_{i\to\infty}\|y_{n_i} - q\| = 0. \quad (3.16)$$

Observe that

$$0 = \inf_{n\ge 0}\min\{A(y_n, q), A(x_n, q)\} = \min\Big\{\inf_{n\ge 0} A(y_n, q), \inf_{n\ge 0} A(x_n, q)\Big\},$$

which implies that either $\inf_{n\ge 0} A(y_n, q) = 0$ or $\inf_{n\ge 0} A(x_n, q) = 0$. Suppose that $\inf_{n\ge 0} A(y_n, q) = 0$. In view of Lemma 2.4, we conclude that there exists a subsequence $\{\|y_{n_i} - q\|\}_{i=0}^{\infty}$ of $\{\|y_n - q\|\}_{n=0}^{\infty}$ satisfying $\lim_{i\to\infty}\|y_{n_i} - q\| = 0$. From (3.1)-(3.4), we infer that

$$\|x_{n_i} - q\| \le \|y_{n_i} - q\| + \|(x_{n_i} - q) - (y_{n_i} - q)\| \le \|y_{n_i} - q\| + 2D(d'_{n_i} + c'_{n_i}) \to 0$$

as $i \to \infty$.

Similarly we can prove that (3.16) holds also in case $\inf_{n\ge 0} A(x_n, q) = 0$.

Using (3.1)-(3.3), (3.12) and (3.16), we conclude easily that given $\varepsilon > 0$ there exists a positive integer k satisfying

$$\begin{aligned}
&\|x_{n_k} - q\| < \frac{\varepsilon}{\sqrt{2}}, \quad s_n < \frac{9\varepsilon^2}{32}\cdot\frac{\phi(\frac{\varepsilon}{2\sqrt{2}})}{1 + D + \phi(D)}, \\
&B\sum_{m=0}^{\infty} c_{n+m} < \frac{\varepsilon^2}{2}, \quad \max\{d_n + c_n, d'_n + c'_n\} < \frac{\varepsilon}{8D\sqrt{2}}
\end{aligned} \quad (3.17)$$

for all $n \geq n_k$. Next we prove by induction that for any $i \geq 1$,

$$\|x_{n_k+i} - q\|^2 < \frac{\varepsilon^2}{2} + B \sum_{m=0}^{i-1} c_{n_k+m}. \tag{3.18}$$

Suppose that $\|x_{n_k+1} - q\|^2 \geq \frac{\varepsilon^2}{2} + Bc_{n_k}$. Obviously, $\|x_{n_k+1} - q\| \geq \frac{\varepsilon}{\sqrt{2}}$. It follows from (3.4) and (3.17) that

$$\begin{aligned}\|x_{n_k} - q\| &\geq \|x_{n_k+1} - q\| - d_{n_k}\|Sy_{n_k} - x_{n_k}\| - c_{n_k}\|u_{n_k} - Sy_{n_k}\| \\ &\geq \frac{\varepsilon}{\sqrt{2}} - 2D(d_{n_k} + c_{n_k}) > \frac{3\varepsilon}{4\sqrt{2}}\end{aligned}$$

and

$$\begin{aligned}\|y_{n_k} - q\| &\geq \|x_{n_k} - q\| - d'_{n_k}\|Sx_{n_k} - x_{n_k}\| - c'_{n_k}\|u_{n_k} - Sx_{n_k}\| \\ &> \frac{3\varepsilon}{4\sqrt{2}} - 2D(d'_{n_k} + c'_{n_k}) > \frac{\varepsilon}{2\sqrt{2}}.\end{aligned}$$

From (3.14) and the above inequalities we obtain that

$$\begin{aligned}\frac{\varepsilon^2}{2} + Bc_{n_k} &\leq \|x_{n_k+1} - q\|^2 \\ &\leq \left(1 - \frac{\phi(\|y_{n_k} - q\|)}{1 + \|y_{n_k} - q\| + \phi(\|y_{n_k} - q\|)} d_{n_k}\right)\|x_{n_k} - q\|^2 + d_{n_k}s_{n_k} + Bc_{n_k} \\ &\leq \|x_{n_k} - q\|^2 + \left[s_{n_k} - \frac{9\varepsilon^2}{32} \cdot \frac{\phi(\frac{\varepsilon}{2\sqrt{2}})}{1 + D + \phi(D)}\right] d_{n_k} + Bc_{n_k} < \frac{\varepsilon^2}{2} + Bc_{n_k},\end{aligned}$$

which is a contradiction. That is, (3.18) holds for $i = 1$. Assume that (3.18) holds for some $i > 1$. Suppose that

$$\|x_{n_k+i+1} - q\|^2 \geq \frac{\varepsilon^2}{2} + B \sum_{m=0}^{i} c_{n_k+m}. \tag{3.19}$$

Then $\|x_{n_k+i+1} - q\| \geq \frac{\varepsilon}{\sqrt{2}}$ and

$$\begin{aligned}\|x_{n_k+i} - q\| &\geq \|x_{n_k+i+1} - q\| - d_{n_k+i}\|Sy_{n_k+i} - x_{n_k+i}\| \\ &\quad - c_{n_k+i}\|u_{n_k+i} - Sy_{n_k+i}\| > \frac{3\varepsilon}{4\sqrt{2}}\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\|y_{n_k+i} - q\| &\geq \|x_{n_k+i} - q\| - d'_{n_k+i}\|Sx_{n_k+i} - x_{n_k+i}\| \\ &\quad - c'_{n_k+i}\|v_{n_k+i} - Sx_{n_k+i}\| > \frac{\varepsilon}{2\sqrt{2}}.\end{aligned} \tag{3.21}$$

Thus (3.14), (3.19)-(3.21) yield that

$$\begin{aligned}
&\frac{\varepsilon^2}{2} + B\sum_{m=0}^{i} c_{n_k+m} \le \|x_{n_k+i+1} - q\|^2 \\
&\le \|x_{n_k+i} - q\|^2 + [s_{n_k+i} - A(\|y_{n_k+i} - q\|)\|x_{n_k+i} - q\|^2]d_{n_k+i} + Bc_{n_k+i} \\
&\le \|x_{n_k+i} - q\|^2 + \left[s_{n_k+i} - \frac{\phi(\frac{\varepsilon}{2\sqrt{2}})}{1 + D + \phi(D)} \cdot \frac{9\varepsilon^2}{32}\right]d_{n_k+i} + Bc_{n_k+i} \\
&< \frac{\varepsilon^2}{2} + B\sum_{m=0}^{i} c_{n_k+m},
\end{aligned}$$

which is a contradiction. That is, (3.18) holds for $i + 1$. Therefore (3.18) holds for any integers $i \ge 1$. Using (3.17) and (3.18), we have

$$\|x_{n_k+i} - q\| < \sqrt{\frac{\varepsilon^2}{2} + B\sum_{m=0}^{i-1} c_{n_k+m}} < \varepsilon$$

for all integers $i \ge 1$. That is, $\|x_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 3.2. *Suppose that* $X, T, \{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ *are as in Theorem 3.1. Then for any given* $f \in X$, *the Ishikawa iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$ *defined for arbitrary* $x_0 \in X$ *by*

$$\begin{aligned}
y_n &= a'_n x_n + b'_n(f - Tx_n) + c'_n v_n, \\
x_{n+1} &= a_n x_n + b_n(f - Ty_n) + c_n u_n, \quad \forall n \ge 0
\end{aligned} \tag{3.22}$$

converges strongly to the solution of the equation $x + Tx = f$ *provided that at least one of* (3.6) *and the following*

$$\text{the sequences } \{x_n + Tx_n\}_{n=0}^{\infty} \text{ and } \{y_n + Ty_n\}_{n=0}^{\infty} \text{ are bounded} \tag{3.23}$$

holds.

Proof. Put $S = I + T$. It is easy to see that S is a demicontinuous ϕ-strongly accretive operator and

$$f - Tx = f - (S - I)x = f + x - Sx$$

for all $x \in X$. It follows from Theorem 3.1 that $\{x_n\}_{n=0}^{\infty}$ converges strongly to the solution of the equation $Sx = f$. This completes the proof. □

Theorem 3.3. *Suppose that* X, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ *are as in Theorem 3.1 and* $T : X \to X$

is a demicontinuous ϕ-strongly pseudocontractive operator. Then the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in X$ by

$$y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \quad n \geq 0 \tag{3.24}$$

converges strongly to the fixed point of T provided that either (3.5) *or* (3.6) *holds.*

Proof. Set $S = I - T$ and $f = 0$. It is easy to verify that S is a demicontinuous ϕ-strongly accretive operator. Lemma 2.1 ensures that the equation $Sx = 0$ has a unique solution $q \in X$. That is, T has a unique fixed point q in X. Observe that $Tx = f + (I - S)x$ for all $x \in X$. Thus Theorem 3.3 follows from Theorem 3.1. This completes the proof. □

Reviewing the proofs of Theorems 3.1-3.3 and Corollaries 3.1-3.3, we can see that the following results hold.

Theorem 3.4. *Suppose that X, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ are as in Theorem 3.1, $T : X \to X$ is ϕ-strongly accretive and the equation $Tx = f$ has a solution for some $f \in X$. Let $S = f + x - Tx$ for all $x \in X$ and the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be as in Theorem 3.1. If either* (3.5) *or* (3.6) *holds, then the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ converges strongly to the solution of the equation $Tx = f$.*

Theorem 3.5. *Suppose that X, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ are as in Theorem 3.1, $T : X \to X$ is ϕ-strongly accretive and the equation $x + Tx = f$ has a solution for some $f \in X$. If the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ defined by* (3.22) *satisfies* (3.6) *or* (3.23), *then it converges strongly to the solution of the equation $x + Tx = f$.*

Theorem 3.6. *Let X be a real uniformly smooth Banach space and K be a nonempty convex subset of X. Suppose that $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ are as in Theorem 3.1 and $T : K \to K$ is ϕ-hemicontractive. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be bounded sequences in K. If the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ defined from an arbitrary $x_0 \in K$ by* (3.24) *satisfies* (3.5) *or* (3.6), *then it converges strongly to the unique fixed point of T.*

Proof. Since T is ϕ-hemicontractive, there exists $q \in K$ with $q = Tq$. Thus for any $x \in K$, there exist $j(x - q) \in J(x - q)$ and a strictly increasing function

$\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - q, j(x - q)\rangle \leq \|x - q\|^2 - \phi(\|x - q\|)\|x - q\|, \tag{3.25}$$

which implies that

$$\langle (I - T - A(x, q))x - (I - T - A(x, q))q, j(x - q)\rangle \geq 0,$$

where $A(x, q) = \frac{\phi(\|x-q\|)}{1+\|x-q\|\phi(\|x-q\|)} \in [0, 1)$ for all $x \in K$. Note that (3.25) means that T has a unique fixed point in K. It follows from the above inequality and Lemma 1.1 of Kato [18] that

$$\|x - q\| \leq \|x - q + r[(I - T - A(x, q))x - (I - T - A(x, q))q]\|$$

for any $x \in K$ and $r > 0$. The remainder of the argument is now identical with the proof of Theorem 3.1 and is therefore omitted. This completes the proof. □

Remark 3.1. Theorems 3.1-3.6 extend, improve and unify Theorems 3.2, 3.3, 4.1 and 5.2 of [3], Theorems 3.2, 3.3, 4.1 and 5.2 of [4], Theorem of [5], Theorems 1 and 2 of [6], Theorems 1 and 2 of [7], Theorems 1, 2, 3 and 4 of [8], Theorems 1, 2, 3 and 4 of [10], Theorems 1 and 2 of [12], Theorems 1 and 2 of [13], Theorems 1 and 2 of [14], Theorems 1 and 2 of [15], Theorem 1 of [41], Theorems 4.1 and 4.2 of [44], Theorems 3.1 and 3.3 of [45], Theorems 1 and 2 of [47] and Theorem 2.1 of [48] in the following ways:

(1) The Mann iteration method in [3-8, 10] and the Ishikawa iteration method in [3, 4, 7, 8, 10, 12-15, 41, 44, 47, 48] are replaced by the more general Ishikawa iteration method with errors.

(2) The Lipschitz continuity in [3, 4, 6, 8, 10, 12-15, 41, 44] either are superfluous or are replaced by the more general demicontinuity. The continuity in [8] is replaced by the more general demicontinuity.

(3) Theorems 3.1-3.6 and Corollaries 3.1-3.6 hold in real uniformly smooth Banach spaces whereas the results of [5, 6,, 9, 11-14, 41, 44] have been proved in the restricted L_p (or l_p) spaces, p-uniformly smooth Banach spaces ($p > 1$), respectively.

(4) The assumptions that both $a_n \leq b_n$ in [3, Theorem 3.2], [7, Theorem 2], [8, Theorem 2] and [10, Theorems 1 and 2], and $\sum_{n=0}^{\infty} a_n b(a_n) < +\infty$ in [7, Theorems 1 and 2] and [8, Theorems 1, 2 and 3], and $b_n \leq a_n^{q-1}$ and the equation $Tx = f$ has a solution in [3, 4, 41] are omitted.

(5) The strongly pseudocontractive operators in [3-8, 10, 12-15, 44, 45, 47, 48] and the strongly accretive operators in [3, 4, 6, 8, 10, 12-15, 44, 47] are replaced by the more general ϕ-hemicontractive operators and ϕ-strongly

accretive operators, respectively.

(6) The boundedness of $R(I-T)$ in [8, 45, 47] are replaced by conditions (3.5) and (3.6), respectively.

(7) The domain K of T may not be closed and bounded.

Remark 3.2. The iteration parameters $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ in Theorems 3.1-3.6 do not depend on any geometric structure of the underlying Banach space X or on any property of the operator T. A prototype for $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ in our theorems is

$$a_n = \frac{n}{n+1},\ b_n = \frac{1}{n+2},\ c_n = \frac{1}{(n+1)(n+2)},\ a'_n = \frac{n}{n+2},\ b'_n = c'_n = \frac{1}{n+2}$$

for all $n \geq 0$.

References

[1] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Nordhoff, Leyden (1976).

[2] F.E. Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, *Bull. Amer. Math. Soc.*, **73** (1967), 875-882.

[3] S.S. Chang, Some problems and results in the study of nonlinear analysis, *Nonlinear Anal. T.M.A.*, **30** (1997), 4197-7208.

[4] S.S. Chang, Y.J. Cho, B.S. Lee, S.M. Kang, Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces, *J. Math. Anal. Appl.*, **224** (1998), 149-165.

[5] C.E. Chidume, Iterative approximation of fixed points of Lipschitz strictly pseudo-contractive mappings, *Proc. Amer. Math. Soc.*, **99** (1987), 283-288.

[6] C.E. Chidume, An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p-spaces, *J. Math. Anal. Appl.*, **151** (1990), 453-461.

[7] C.E. Chidume, Approximation of fixed points of strongly pseudo-contractive mappings, *Proc. Amer. Math. Soc.*, **120** (1994), 545-551.

[8] C.E. Chidume, Iterative solution of nonlinear equations with strongly accretive operators, *J. Math. Anal. Appl.*, **192** (1995), 502-518.

[9] C.E. Chidume, M.O. Osilike, Fixed point iterations for strictly hemi-contractive maps in uniformly smooth Banach spaces, *Numer. Func. Anal. and Optim.*, **15** (1994), 779-790.

[10] C.E. Chidume, M.O. Osilike, Ishikawa iteration process for nonlinear Lipschitz strongly accretive mappings, *J. Math. Anal. Appl.*, **192** (1995), 727-741.

[11] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin/Heidelberg (1985).

[12] L. Deng, On Chidume's open questions, *J. Math. Anal. Appl.*, **174** (1993), 441-449.

[13] L. Deng, An iterative process for nonlinear Lipschitz and strongly accretive mappings in uniformly convex and uniformly smooth Banach spaces, *Acta Appl. Math.*, **32** (1993), 183-196.

[14] L. Deng, Iteration processes for nonlinear Lipschitzian strongly accretive mappings in L_p spaces, *J. Math. Anal. Appl.*, **188** (1994), 128-140.

[15] L. Deng, X.P. Ding, Iterative approximation of Lipschitz strictly pseudo-contractive mappings in uniformly smooth Banach spaces, *Nonlinear Anal. T.M.A.*, **24** (1995), 981-987.

[16] J.A. Gatica, W.A. Kirk, Fixed point theorems for Lipschitz pseudo-contractive mappings, *Proc. Amer. Math. Soc.*, **36** (1972), 111-115.

[17] S. Ishikawa, Fixed point by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147-150.

[18] T. Kato, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan.*, **19** (1967), 508-520.

[19] J.K. Kim, Z. Liu, S.M. Kang, Almost stability of Ishikawa iterative schemes with errors for $\phi-$strongly quasi-accretive and ϕ-hemicontractive operators, *Commun. Korean Math. Soc.*, **19** (2004), 267-281.

[20] L.S. Liu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.*, **194** (1995), 114-125.

[21] Z. Liu, Z. An, Y. Li, S.M. Kang, Iterative approximation of fixed points for ϕ-hemicontractive operators in Banach spaces, *Commun. Korean. Math. Soc.*, **19** (2004), 63-74.

[22] Z. Liu, M. Bouxias, S.M. Kang, Iterative approximation of solutions to nonlinear equations of ϕ-strongly accretive operators in Banach spaces, *Rocky Mountain J. Math.*, **32** (2002), 981-997.

[23] Z. Liu, J.K. Kim, S.M. Kang, Necessary and sufficient conditions for convergence of Ishikawa Iterative schemes with errors to ϕ-hemicontractive mappings, *Commun. Korean Math. Soc.*, **18** (2003), 251-261.

[24] Z. Liu, S.M. Kang, Convergence theorems for ϕ-strongly accretive and ϕ-hemicontractive operators, *J. Math. Anal. Appl.*, **253** (2001), 35-49.

[25] Z. Liu, S.M. Kang, Iterative solutions of nonlinear equations with ϕ-strongly accretive operators in uniformly smooth Banach spaces, *Comput. Math. Appl.*, **45** (2003), 623-634.

[26] Z. Liu, S.M. Kang, Iterative process with errors for nonlinear equations of local $\phi-$strongly accretive operators in arbitrary Banach spaces, *Int. J. Pure Appl. Math.*, **12** (2004), 229-246.

[27] Z. Liu, S.M. Kang, Stability of Ishikawa iteration methods with errors for strong pseudocontractions and nonlinear equations involving accretive operators in arbitrary real Banach spaces, *Math. Comput. Modelling*, **34** (2001), 319-330.

[28] Z. Liu, S.M. Kang, Convergence and stability of the Ishikawa iteration procedures with errors for nonlinear equations of the ϕ-strongly accretive type, *Neural Parallel Sci. Comput.*, **9** (2001), 103-118.

[29] Z. Liu, S.M. Kang, Iterative approximation of fixed points for ϕ-hemicontractive operators in arbitrary Banach spaces, *Acta Sci. Math. (Szeged)*, **67** (2001), 821-831.

[30] Z. Liu, S.M. Kang, Y.J. Cho, Convergence and almost stability of Ishikawa iterative scheme with errors for m-accretive operators, *Comput. Math. Appl.*, **47** (2004), 767-778.

[31] Z. Liu, S.M. Kang, S.H. Shim, Almost stability of the Mann iteration method with errors for strictly hemi-contractive operators in smooth Banach spaces, *J. Korean Math. Soc.*, **40** (2003), 29-40.

[32] Z. Liu, J.S. Ume, Stable and almost stable iteration schemes for strictly hemi-contractive operators in arbitrary Banach spaces, *Numer. Funct. Anal. Optim.*, **23** (2002), 833-848.

[33] Z. Liu, J.S. Ume, S.M. Kang, Strong convergence and pseudo stability for operators of the $\phi-$accretive type in uniformly smooth Banach spaces, *Rostock. Math. Kolloq.*, **59** (2005), 29-40.

[34] Z. Liu, L. Wang, H.G. Kim, S.M. Kang, The equivalence of Mann and Ishikawa iteration schemes with errors for $\phi-$strongly accretive operators in uniformly smooth Banach spaces, *Math. Sci. Res. J.*, **9** (2005), 47-57.

[35] Z. Liu, Y. Xu, Y.J. Cho, Iterative solution of nonlinear equations with ϕ-strongly accretive operators, *Arch. Math.*, **77** (2001), 508-516.

[36] Z. Liu, L. Zhang, S.M. Kang, Convergence theorems and stability results for Lipschitz strongly pseudocontractive operators, *Int. J. Math. and Math. Sci.*, **31** (2002), 611-617.

[37] Z. Liu, L. Zhang, S.M. Kang, Iterative solutions to nonlinear equations of the accretive type in Banach spaces, *East Asian Math. J.*, **17** (2001), 265-273.

[38] W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.

[39] R.H. Martin Jr., A global existence theorem for autonomous differential equations in Banach spaces, *Proc. Amer. Math. Soc.*, **26** (1970), 307-314.

[40] C. Morales, Surjectivity theorems for multi-valued mappings of accretive type, *Comment. Math. Univ. Carolin.*, **26** (1985), 397-413.

[41] M.O. Osilike, Iterative solution of nonlinear equations of the ϕ-strongly accertive type, *J. Math. Anal. Appl.*, **200** (1996), 259-271.

[42] M.O. Osilike, Ishikawa and Mann iteration methods with errors for nonlinear equations of the accretive type, *J. Math. Anal. Appl.*, **213** (1997), 91-105.

[43] S. Reich, An iterative procedure for constructing zeros of accretive set in Banach spaces, *Nonlinear Anal. T.M.A.*, **2** (1978), 85-92.

[44] K.K. Tan, H.K. Xu, Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces, *J. Math. Anal. Appl.*, **178** (1993), 9-21.

[45] Y. Xu, Ishikawa and Mann iterative processes witherrors for nonliara strongly accretive operator equations, *J. Math. Anal. Appl.*, **224** (1998), 91-101.

[46] Z.B.Xu, G.F. Roach, Characteristic inequalities in numformly convex and uniformly smooth Banach spaces, *J. Math. Anal. Appl.*, **157** (1991), 189-210.

[47] H.Y. Zhou, Iterative solution of nonlinear equations involving strongly accretive operators without the Lipschitz assumption, *J. Math. Anal. Appl.*, **213** (1997), 296-307.

[48] H.Y. Zhou, Y.T. Jia, Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption, *Proc. Amer. Math. Soc.*, **125** (1997), 1705-1709.