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HYERS-ULAM-RASSIAS STABILITY OF HOMOMORPHISMS OF VARIOUS FUNCTIONAL EQUATIONS IN QUASI-BANACH ALGEBRAS

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Abstract: In this paper, we investigate the Hyers-Ulam-Rassias stability of homomorphisms of general Cauchy, Jensen's, Cauchy-Jensen's, Apollonius functional equations in quasi-Banach algebras. This idea is also applied to additive mapping of first form and second form. We also investigate isomorphisms between quasi-Banach algebras.

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1. Introduction

The stability problems of functional equations originates from the fundamental question: When is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?

In connection with the above question: In 1940, S.M. Ulam [33] raised a question concerning the stability of homomorphisms:

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Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

The first partial solution to Ulam's question was given by D.H. Hyers [10]. He considered the case of approximately additive functions $f : G_1 \to G_2$ where G_1 and G_2 are Banach spaces and f satisfies Hyers inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in G_1$. It was shown that the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exits for all $x \in G_1$ and that $a : G_1 \to G_2$ is the unique additive mapping satisfying

$$||f(x) - a(x)|| \le \epsilon.$$

Moreover, it was proved that if $f(tx)$ is continuous in t for each fixed $x \in G_1$ then a is linear.

In 1978, Th.M. Rassias [28] provided a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. Th.M. Rassias [28] introduced the following inequality, that we call Cauchy-Rassias inequality.

Assume that there exist constant $\theta \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \theta \left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in G_1$, then Th.M. Rassias showed that there exists a unique $\mathbb{R}$-linear mapping $a : G_1 \to G_2$ such that

$$\|f(x) - a(x)\| \le \frac{2\theta}{2 - 2^p} \|x\|^p$$

for any $x \in G_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in G_1$ then a is linear. The above inequality has produced a lot of influence on the development of what we now call the Hyers-Ulam-Rassias stability of functional equations.

A generalization of this result was proved by J.M. Rassias [23]. The generalized result includes the following theorem.

Theorem 1.1. (see [23]) *Let G_1 be a real normed linear space and G_2 a real complete normed linear space. Assume that $f : G_1 \to G_2$ is an approximately additive mapping for which there exist constants $\theta \ge 0$ and $p, q \in \mathbb{R}$*

such that $r = p + q \neq 1$ *and* f *satisfies the Cauchy-Gavruta-Rassias inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta ||x||^p ||y||^q$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $a : G_1 \to G_2$ *satisfying*

$$\|f(x) - a(x)\| \leq \frac{\theta}{|2^r - 2|} ||x||^r$$

for all $x \in G_1$. *If, in addition,* $f : G_1 \to G_2$ *mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in G_1$, *then* a *is an* $\mathbb{R}$*-linear mapping.*

Other interesting stability results have been achieved by the following authors: Aczel [1], Boreli and Forti [5], Cholewa [6], Czerwik [7], Drljeric [8] and Kannappan [12]. In 1982-1998, J.M. Rassias solved the above Ulam problem for different mappings. In 1983, Skof [32] was the first author to solve the Ulam problem for additive mappings on a restricted domain. However, in 2002, J.M. Rassias and M.J. Rassias [27] considered and investigated quadratic equations involving a product of powers of norms in which an approximate quadratic mapping degenerates to a genuine quadratic mapping. Beside Euler-Lagrange quadratic mappings were introduced by J.M. Rassias [25], the Euler-Lagrange quadratic mappings that were introduced and investigated by J.M. Rassias [25], were named Euler-Lagrange-Rassias mappings and the corresponding Euler-Lagrage quadratic equations were called Euler-Lagrange-Rassias equations. The introdution of Euler-Lagrange mappings and equations in functional equations and inequalities provided an interesting corner stone in analysis and it is of particular interest in probability theory and stochastic analysis. The stability problems of several functional equations have been extensively investigated by a number of authors and many interesting results are available concerning this problem (see [3, 5-11, 13, 27-29]).

In this paper, we prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the general Cauchy functional equation, particular type of Jensen functional equation, Cauchy-Jensen functional equation, Apollonius functional equations in quasi-Banach algebras. We also applied the same idea to the additive mapping of first form and second form. We also investigated isomorphisms between quasi-Banach algebras.

Before we present our main result, we will introduce some basic definitions concerning quasi-Banach spaces and some preliminary results that will be useful in proving our main theorems.

Definition 1.2. (see [2, 31]) Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:

(i) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.

(ii) $||\lambda x|| = |\lambda|||x||$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $||x + y|| \leq K(||x|| + ||y||)$ for all $x, y \in X$.

It follows from condition (iii) that

$$\left\| \sum_{i=1}^{2n} x_i \right\| \leq K^n \sum_{i=1}^{2n} \|x_i\|, \left\| \sum_{i=1}^{2n+1} x_i \right\| \leq K^{n+1} \sum_{i=1}^{2n+1} \|x_i\|$$

for all integers $n \geq 1$ and all $x_1, x_2, \cdots, x_{2n+1} \in X$.

The pair $(X, ||.||)$ is called a quasi-normed space if $||.||$ is a quasi-norm on X. The smallest possible K is called the modulus of concavity of $||.||$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $||.||$ is called a p-norm $(0 < p \leq 1)$ if

$$||x + y||^p \leq ||x||^p + ||y||^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

By the Aoki-Rolewicz Theorem, [31], each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.3. Let $(A, ||.||)$ be a quasi-normed space. The quasi-normed space $(A, ||.||)$ is called a quasi-normed algebra if A is an algebra and there is a constant $C > 0$ such that $||xy|| \leq C||x||||y||$ for all $x, y \in A$.

A quasi-Banach algebra is a complete quasi-normed algebra.

If the quasi-norm $||.||$ is a p-norm then the quasi-Banach algebra is called a p-Banach algebra.

2. Hyers-Ulam-Rassias Stability of Homomorphisms in Quasi-Banach Algebras

Throughout this section, we assume X is a quasi-normed algebra with quasi-norm $||.||_X$ and that Y is a p-Banach algebra with $p-$norm $||.||_Y$. Let K be the modulus of concavity of $||.||_Y$.

(A) *We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebra, associated to the general form of the Cauchy functional equa-*

tion

$$f\left(\sum_{i=1}^{k} x_i\right) = \sum_{i=1}^{k} f(x_i)\,, \tag{E_1}$$

where $k \geq 2 \in N$ and k is finite.

We now define

$$Df_1(x_1, \ldots, x_k) = f\left(\sum_{i=1}^{k} x_i\right) - \sum_{i=1}^{k} f(x_i)\,.$$

Theorem 2.1. *Let $r > \frac{1}{k}, k \in N \geq 2$ and θ be a positive real numbers and let $f : X \to Y$ be a mapping such that*

$$\|Df_1(x_1, \ldots, x_k)\|_Y \leq \theta \prod_{i=1}^{k} ||x_i||_X^r\,, \tag{2.1}$$

$$\|f(x_1x_2) - f(x_1)f(x_2)\|_Y \leq \theta ||x_1||_X^r ||x_2||_X^r\,, \tag{2.2}$$

for all $x_i \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{\theta}{(k^{kpr} - k^p)^{\frac{1}{p}}} ||x||_X^{kr}\,. \tag{2.3}$$

Proof. Setting $x_1 = x_2 = \ldots = x_k = x$ in (2.1), we get

$$\|f(kx) - kf(x)\|_Y \leq \theta ||x||_X^{kr}\,, \quad \text{for all } x \in X. \tag{2.4}$$

Replacing x by x/k, we get

$$\left\|f(x) - kf\left(\frac{x}{k}\right)\right\|_Y \leq \frac{\theta}{k^{kr}} ||x||_X^{kr}\,, \quad \text{for all } x \in X.$$

Replacing x by x/k^l and multipling by k^l, we get

$$\left\|k^l f\left(\frac{x}{k^l}\right) - k^{l+1} f\left(\frac{x}{k^{l+1}}\right)\right\|_Y \leq \frac{\theta k^l}{k^{kr} k^{klr}} ||x||_X^{kr}\,, \quad \text{for all } x \in X.$$

Since Y is a p-Banach algebra,

$$\left\|k^l f\left(\frac{x}{k^l}\right) - k^{l+1} f\left(\frac{x}{k^{l+1}}\right)\right\|_Y^P \leq \frac{\theta^p k^{lp}}{k^{pkr} k^{klrp}} ||x||_X^{kpr}\,, \quad \text{for all } x \in X,$$

$$\left\|k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right)\right\|_Y^P \leq \frac{\theta^p}{k^{pkr}} \sum_{j=l}^{m-1} \frac{k^{pj}}{k^{prj}} ||x||_X^{kpr}\,, \quad \text{for all } x \in X \tag{2.5}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.5) that the sequence $\left\{k^n f(\frac{x}{k^n})\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{k^n f(\frac{x}{k^n})\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) = \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right), \quad \text{for all } x \in X.$$

It follows from (2.1) that

$$\begin{aligned}\|DH(x_1,\ldots,x_k)\|_Y &= \lim_{n\to\infty} k^n \left\| f\left(\frac{1}{k^n}\sum_{i=1}^{k} x_i\right) - \sum_{i=1}^{k} f\left(\frac{x_i}{k^n}\right)\right\|_Y \\ &\le \lim_{n\to\infty} \frac{k^n\theta}{k^{knr}} \prod_{i=1}^{k} ||x_i||_X^r = 0\end{aligned}$$

for all $x_i \in X$. Therefore $H\left(\sum\limits_{i=1}^{k} x_i\right) = \sum\limits_{i=1}^{k} H(x_i)$. Moreover, letting $l = 0$ and allowing the limit $m \to \infty$ in (2.5) we get

$$\begin{aligned}\|f(x) - H(x)\|_Y^p &\le \frac{\theta^p}{k^{kpr}} \sum_{j=0}^{\infty} \frac{k^{pj}}{kprj} ||x||_X^{kpr}, \\ \|f(x) - H(x)\|_Y^p &\le \frac{\theta^p}{k^{kpr} - k^p} ||x||_X^{kpr}, \\ \|f(x) - H(x)\|_Y &\le \frac{\theta}{(k^{kpr} - k^p)^{1/p}} ||x||_X^{kr} \quad \text{which is same as (2.3).}\end{aligned}$$

By the same reasoning as in the proof of Theorem of [28], the mapping $H : X \to Y$ is $\mathbb{R}$ -linear. It follows from (2.2) that

$$\begin{aligned}\|H(x_1x_2) - H(x_1)H(x_2)\|_Y &= \lim_{n\to\infty} k^n \left\| f\left(\frac{x_1x_2}{k^nk^n}\right) - f\left(\frac{x_1}{k^n}\right) f\left(\frac{x_2}{k^n}\right)\right\|_Y \\ &\le \lim_{n\to\infty} \frac{k^n\theta}{k^nr} ||x_1||_X^r ||x_2||_X^r = 0\end{aligned}$$

for all $x, y \in X$. So $H(x_1x_2) = H(x_1)H(x_2)$ for all $x_1, x_2 \in X$.

Now let $T : X \to Y$ be another Cauchy additive mapping satisfying (2.3). Then we have

$$\begin{aligned}\|H(x) - T(x)\|_Y &= k^n \left\| H\left(\frac{x}{k^n}\right) - T\left(\frac{x}{k^n}\right)\right\|_Y \\ &\le k^n K \left[\left\| H\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\|_Y + \left\| T\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\|_Y\right] \\ &\le \frac{2Kk^n\theta}{k^{knr}\left(k^{kpr} - k^p\right)^{\frac{1}{p}}} ||x||_X^{kr}\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. We can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. Thus the mapping $H : X \to Y$ is a unique homomorphism satisfying (2.3). □

Theorem 2.2. *Let $r < \frac{1}{k}, k \in N \geq 2$ and θ be a positive real numbers and let $f : X \to Y$ be a mapping satisfying (2.1) and (2.2). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\|_Y \leq \frac{\theta}{(k^p - k^{kpr})^{1/p}}||x||_X^{kr} \tag{2.6}$$

for all $x \in X$.

Proof. It follows from (2.4) that

$$\left\|f(x) - \frac{1}{k}f(kx)\right\|_Y \leq \frac{\theta}{k}||x||_X^{2r}.$$

Replacing x by $k^l x$ and dividing by k^l, we get

$$\left\|\frac{f(k^l x)}{k^l} - \frac{1}{k^{l+1}}f(k^{l+1}x)\right\|_Y \leq \frac{\theta k^{klr}}{kk^l}||x||_X^{kr}.$$

Since Y is a p-Banach algebra, we obtain

$$\left\|\frac{f(k^l x)}{k^l} - \frac{1}{k^{l+1}}f(k^{l+1}x)\right\|_Y^p \leq \frac{\theta^p k^{klpr}}{k^p k^{lp}}||x||_X^{pkr}.$$

$$\Rightarrow \left\|\frac{f(k^l x)}{k^l} - \frac{1}{k^m}f(k^m x)\right\|_Y^p \leq \frac{\theta^p}{k^p}\sum_{j=l}^{m-1}\frac{k^{kprj}}{k^p j}||x||_X^{kpr}, \tag{2.7}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.7) that the sequence $\left\{\frac{1}{k^n}f(k^n x)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete the sequence $\left\{\frac{1}{k^n}f(k^n x)\right\}$ converges. So we can define the mapping $H : X \to Y$ by

$$H(x) = \lim_{n\to\infty}\frac{f(k^n x)}{k^n}, \quad \text{for all } x \in X.$$

The rest of the proof is similar to the proof of Theorem 2.1 □

(B) *We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Jensen functional equation*

$$3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]. \tag{E_2}$$

Now we define

$$D\ f_2(x,y,z) = 3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z)$$

$$-2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right].$$

Theorem 2.3. *Let $r<\frac{1}{3}$ and θ be a positive real numbers and let $f:X\to Y$ with $f(0)=0$ satisfying (2.2) such that*

$$\|D\ f_2(x,y,z)\|_Y \le \theta||x||_X^r||y||_X^r||z||_X^r \tag{2.8}$$

for all $x,y,z\in X$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in X$, then there exists a unique homomorphism $H:X\to Y$ such that

$$\|f(x)-H(x)\|_Y \le \frac{4^{2r}\theta}{(4^p-4^{3pr})^{1/p}}||x||_X^{3r} \tag{2.9}$$

for all $x\in X$.

Proof. Setting $y=x$ and $z=-2x$ in (2.8), we get

$$\left\|f(-2x)-4f\left(\frac{-x}{2}\right)\right\|_Y \le \theta 2^r||x||_X^{3r}.$$

Replacing x by $-2x$, we get

$$\|f(4x)-4f(x)\|_Y \le \theta 4^r||x||_X^{3r}. \tag{2.10}$$

Replacing x by $4^l x$ and dividing by 4^l, we get

$$\left\|\frac{1}{4^l}f(4^lx)-\frac{1}{4^{l+1}}f(4^{l+1}x)\right\|_Y \le \frac{\theta 4^{2r}}{44^l}4^{3lr}||x||_X^{3r}.$$

Since Y is a p-Banach algebra, we get

$$\left\|\frac{1}{4^l}f(4^lx)-\frac{1}{4^{l+1}}f(4^{l+1}x)\right\|_Y^p \le \frac{\theta^p 4^{2pr}}{4^p4^{pl}}4^{3plr}||x||_X^{3pr}.$$

$$\Rightarrow \left\|\frac{1}{4^l}f(4^lx)-\frac{1}{4^m}f(4^mx)\right\|_Y^p \le \theta^p\frac{4^{2pr}}{4^p}\sum_{j=l}^{m-1}\frac{4^{3prj}}{4^{pj}}||x||_X^{3pr} \tag{2.11}$$

for all non-negative integers m and l with $l<m$ and for all $x\in X$. It follows from (2.11) that the sequence $\left\{\frac{1}{4^n}f(4^nx)\right\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete the sequence $\left\{\frac{1}{4^n}f(4^nx)\right\}$ converges. So we can define the mapping $H:X\to Y$ by

$$H(x)=\lim_{n\to\infty}\frac{f(4^nx)}{4^n}, \quad \text{for all } x\in X.$$

It follows from (2.8) that

$$\begin{aligned}\|D\ H(x,y,z)\|_Y &\le \lim_{n\to\infty}\frac{1}{4^n}\|D\ f_2(4^nx,4^ny,4^nz)\|_Y\\ &\le \lim_{n\to\infty}\frac{\theta}{4^n}4^{3nr}||x||_X^r||y||_X^r||z||_X^r=0\end{aligned}$$

for all $x, y, z \in X$. So

$$3H\left(\frac{x+y+z}{3}\right) + H(x) + H(y) + H(z)$$
$$= 2\left[H\left(\frac{x+y}{2}\right) + H\left(\frac{y+z}{2}\right) + H\left(\frac{z+x}{2}\right)\right],$$

for all $x, y \in X$. Moreover, letting $l = 0$ and allowing the limit $m \to \infty$ in (2.11), we get the inequality (2.9).

By the same reasoning as in the proof of Theorem of [28], the mapping $H : X \to Y$ is $\mathbb{R}$-linear. It follows that from (2.2) that

$$\begin{aligned}\|H(x_1x_2) - H(x_1)H(x_2)\|_Y &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n x_1 4^n x_2) - f(4^n x_1)f(4^n x_2)\|_Y \\ &\le \lim_{n\to\infty} \frac{\theta 4^{2nr}}{4^n} ||x_1||_X^r ||x_2||_X^r = 0\end{aligned}$$

for all $x, y \in X$. So $H(x_1x_2) = H(x_1)H(x_2)$ for all $x_1, x_2 \in X$.

Now, let $T : X \to Y$ be another Jensen additive mapping satisfying (2.9). Then we have

$$\begin{aligned}\|H(x) - T(x)\|_Y &= \frac{1}{4^n} \|H(4^n x) - T(4^n x)\|_Y \\ &\le \frac{K}{4^n} [\|H(4^n x) - f(4^n x)\|_Y + \|T(4^n x) - f(4^n x)\|_Y] \\ &\le \frac{2K4^{3nr}}{4^n} \frac{4^{2r}}{(4^p - 4^{3pr})^{1/p}} ||x||_X^{3r}\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. We can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. □

Theorem 2.4. *Let $r > 1/3$ and θ be a positive real numbers and let $f : X \to Y$ with $f(0) = 0$ satisfying (2.2) and (2.8). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\|_Y \le \frac{4^{2r}\theta}{(4^{3pr} - 4^p)^{1/p}} ||x||_X^{3r} \qquad (2.12)$$

for all $x \in X$.

Proof. It follows from (2.10) that

$$\left\|f(x) - 4f\left(\frac{x}{4}\right)\right\|_Y \le \frac{\theta}{4^r} ||x||_X^{3r}.$$

Replacing x by $\frac{x}{4^l}$ and multipling by 4^l, we get

$$\left\|4^l f\left(\frac{x}{4^l}\right) - 4^{l+1} f\left(\frac{x}{4^{l+1}}\right)\right\|_Y \leq \frac{\theta 4^l}{4^r 4^{3lr}}||x||_X^{3r}.$$

Since Y is a p-Banach algebra, we get

$$\left\|4^l f\left(\frac{x}{4^l}\right) - 4^{l+1} f\left(\frac{x}{4^{l+1}}\right)\right\|_Y^p \leq \frac{\theta^p 4^{pl}}{4^{pr} 4^{3lrp}}||x||_X^{3rp}.$$

$$\Rightarrow \left\|4^l f\left(\frac{x}{4^l}\right) - 4^m f\left(\frac{x}{4^m}\right)\right\|_Y^p \leq \frac{\theta^p}{4^{pr}} \sum_{j=l}^{m-1} \frac{4^{pj}}{4^{3prj}}||x||_X^{3rp} \tag{2.13}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.13) that the sequence $\left\{4^n f(\frac{x}{4^n})\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{4^n f(\frac{x}{4^n})\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) = \lim_{n\to\infty} 4^n f(x/4^n), \quad \text{for all } x \in X.$$

The rest of the proof is similar to the proof of Theorem 2.1. □

(C) *We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebra associated to the Cauchy-Jensen functional equation*

$$2f\left(\frac{x+y}{2} + z\right) = f(x) + f(y) + 2f(z). \tag{E_3}$$

Now we define

$$Df_3(x, y, z) = 2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z).$$

Theorem 2.5. *Let $r > 1/3$ and θ be a positive real numbers and let $f : X \to Y$ be a mapping satisfying (2.2) such that*

$$\|Df_3(x, y, z)\|_Y \leq \theta||x||_X^r||y||_X^r||z||_X^r \tag{2.14}$$

for all $x, y, z \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{\theta}{2(8^{pr} - 2^p)^{1/p}}||x||_X^{3r} \tag{2.15}$$

for all $x \in X$.

Proof. Setting $x = y = z$ in (2.14), we get

$$\|f(2x) - 2f(x)\|_Y \leq \frac{\theta}{2}||x||_X^{3r}. \tag{2.16}$$

Replacing x by $x/2$, we get

$$\|f(x) - 2f(x/2)\|_Y \leq \frac{\theta}{2}\frac{1}{8^r}||x||_X^{3r}.$$

Replacing x by $x/2^l$ and multipling by 2^l, we get

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|_Y \leq \frac{\theta 2^l}{2 \quad 8^r 8^{lr}}||x||_X^{3r}.$$

Since Y is a p-Banach algebra, we get

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|_Y^p \leq \frac{\theta^p 2^{lp}}{2^p 8^{pr} 8^{plr}}||x||_X^{3pr}.$$

$$\Rightarrow \quad \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_Y^p \leq \frac{\theta^p}{2^p 8^{pr}} \sum_{j=l}^{m-1} \frac{2^{pj}}{8^{prj}}||x||_X^{3pr} \tag{2.17}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.17) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) = \lim_{n\to\infty} 2^n f(x/2^n), \quad \text{for all } x \in X.$$

By (2.14), we have

$$\begin{aligned} &\|DH(x,y,z)\|_Y \\ &= \lim_{n\to\infty} 2^n \left\|2f\left(\frac{1}{2^n}\left(\frac{x+y}{2}+z\right)\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - 2f\left(\frac{z}{2^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} \frac{\theta 2^n}{8^{nr}}||x||_X^r||y||_X^r||z||_X^r = 0, \end{aligned}$$

for all $x, y, z \in X$. So

$$2H\left(\frac{x+y}{2}+z\right) = H(x) + H(y) + 2H(z)$$

for all $x, y, z \in X$. Moreover, letting $l = 0$ and allowing the limit $m \to \infty$ in (2.17), we get the inequality (2.15).

By the same reasoning as in the proof of Theorem of [28], the mapping $H : X \to Y$ is $\mathbb{R}$ -linear. It follows from (2.2) that

$$\begin{aligned} \|H(x_1x_2) - H(x_1)H(x_2)\|_Y &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x_1x_2}{2^n2^n}\right) - f\left(\frac{x_1}{2^n}\right) f\left(\frac{x_2}{2^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} \frac{2^n\theta}{2^n r}||x_1||_X^r||x_2||_X^r = 0, \end{aligned}$$

for all $x, y \in X$. So $H(x_1x_2) = H(x_1)H(x_2)$ for all $x_1, x_2 \in X$.

Now let $T : X \to Y$ be another Cauchy-Jensen additive mapping satisfying (2.15). Then we have

$$\|H(x) - T(x)\|_Y = 2^n \left\|H\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_Y$$

$$\leq 2^n K \left(\left\| H\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_Y + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_Y \right)$$
$$\leq \frac{2^n K\theta}{8^{nr}(8^{pr}-2^p)^{1/p}}||x||_X^{3r}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. Thus the mapping $H : X \to Y$ is a unique homomorphism satisfying (2.15). □

Theorem 2.6. *Let $r < \frac{1}{3}$ and θ be a positive real numbers and let $f : X \to Y$ be a mapping satisfying (2.2) and (2.14). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\|_Y \leq \frac{\theta}{2(2^p - 8^{pr})^{1/p}}||x||_X^{3r} \tag{2.18}$$

for all $x \in X$.

Proof. It follows from (2.16) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\|_Y \leq \frac{\theta}{4}||x||_X^{3r}$$

for all $x \in X$. Replacing x by $2^l x$ and dividing by 2^l, we get

$$\left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^{l+1}}f(2^{l+1})x \right\|_Y \leq \frac{\theta 8^{lr}}{4\ 2^l}||x||_X^{3r}.$$

Since Y is a p-Banach algebra, we have

$$\left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^{l+1}}f(2^{l+1}x) \right\|_Y^p \leq \frac{\theta^p 8^{plr}}{4^p 2^{pl}}||x||_X^{3pr}.$$

$$\Rightarrow \quad \left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\|_Y^p \leq \frac{\theta^p}{4^p}\sum_{j=l}^{m-1}\frac{8^{prj}}{2^{pj}}||x||_X^{3pr} \tag{2.19}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.19) that the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) = \lim_{n\to\infty}\frac{1}{2^n}f(2^n x), \quad \text{for all } x \in X.$$

The rest of the proof is similar to the proof of Theorem 2.1. □

(D) *We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebra, associated to the Apollonius type additive mapping*

$$f(z-x) + f(z-y) = -\frac{1}{2}f(x+y) + 2f\left(z - \frac{x+y}{4}\right). \tag{E_4}$$

Now we define

$$D\ f_4(x,y,z) = f(z-x) + f(z-y) + \frac{1}{2}f(x+y) - 2f\left(z - \frac{x+y}{4}\right).$$

Theorem 2.7. *Let $r > \frac{1}{3}$ and θ be a positive real numbers and let $f : X \to Y$ satisfying (2.2) such that*

$$\|D\ f_4(x,y,z)\|_Y \le \theta ||x||_X^r ||y||_X^r ||z||_X^r \tag{2.20}$$

for all $x, y, z \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \le \frac{2^{2r+1}\theta}{(4^{3pr} - 4^p)^{1/p}} ||x||_X^{3r} \tag{2.21}$$

for all $x \in X$.

Proof. Setting $x = y = 2z$ in (2.20), we get

$$||f(4z) - 4f(z)||_Y \le 2\theta 4^r ||z||_X^{3r}\,, \quad \text{for all } z \in X. \tag{2.22}$$

Replacing z by $z/4$ and z by x, we get

$$\left\| f(x) - 4f(\frac{x}{4}) \right\|_Y \le 2\theta \frac{||x||^{3r}}{16^r}\,, \quad \text{for all } x \in X.$$

Replacing x by $\frac{x}{4^l}$ and multipling by 4^l, we get

$$\left\| 4^l f\left(\frac{x}{4^l}\right) - 4^{l+1} f\left(\frac{x}{4^{l+1}}\right) \right\|_Y \le 2\theta \frac{4^l}{16^r 4^{3rl}} ||x||_X^{3r}.$$

Since Y is a p-Banach algebra, we have

$$\left\| 4^l f\left(\frac{x}{4^l}\right) - 4^{l+1} f\left(\frac{x}{4^{l+1}}\right) \right\|_Y^p \le 2^p\theta^p \frac{4^{pl}}{16^{pr} 4^{3prl}} ||x||_X^{3pr}.$$

$$\Rightarrow \quad \left\| 4^l f\left(\frac{x}{4^l}\right) - 4^m f\left(\frac{x}{4^m}\right) \right\|_Y^p \le \frac{2^p\theta^p}{16^{pr}} \sum_{j=1}^{m-1} \frac{4^{pj}}{4^{3prj}} ||x||_X^{3pr} \tag{2.23}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.23) that the sequence $\left\{4^n f(\frac{x}{4^n})\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{4^n f(\frac{x}{4^n})\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) = \lim_{n\to\infty} 4^n f(x/4^n)\,, \quad \text{for all } x \in X.$$

By (2.20), we have

$$\begin{aligned}\|D\ H(x,y,z)\|_Y &\le \lim_{n\to\infty} 4^n \left\| D\ f_4\left(\frac{x}{4^n}, \frac{x}{4^n}, \frac{x}{4^n}\right) \right\|_Y \\ &\le \lim_{n\to\infty} \frac{\theta}{4^n} 4^{3nr} ||x||_X^r ||y||_X^r ||z||_X^r = 0\end{aligned}$$

for all $x, y, z \in X$. So

$$H(z-x) + H(z-y) = -\frac{1}{2}H(x+y) + 2H\left(z - \frac{x+y}{4}\right)$$

for all $x, y \in X$. Moreover, letting $l = 0$ and allowing the limit $m \to \infty$ in (2.23), we get the inequality (2.21).

By the same reasoning as in the proof of Theorem of [28], the mapping $H : X \to Y$ is $\mathbb{R}$-linear. It follows from (2.2) that

$$\begin{aligned}\|H(x_1x_2) - H(x_1)H(x_2)\|_Y &= \lim_{n\to\infty} 16^n \left\| f\left(\frac{x_1x_2}{4^n4^n}\right) - f\left(\frac{x_1}{4^n}\right) f\left(\frac{x_2}{4^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} \frac{16^n\theta}{16^{nr}} ||x_1||_X^r ||x_2||_X^r = 0\,,\end{aligned}$$

for all $x, y \in X$. So $H(x_1x_2) = H(x_1)H(x_2)$ for all $x_1, x_2 \in X$.

Now let $T : X \to Y$ be another Apollonius-type additive mapping satisfying (2.23). Then we have

$$\begin{aligned}\|H(x) - T(x)\|_Y &= 4^n \left\| f\left(\frac{x}{4^n}\right) - T\left(\frac{x}{4^n}\right)\right\|_Y \\ &\leq 4^n K \left[\left\| H\left(\frac{x}{4^n}\right) - f\left(\frac{x}{4^n}\right)\right\|_Y + \left\| T\left(\frac{x}{4^n}\right) - f\left(\frac{x}{4^n}\right)\right\|_Y\right] \\ &\leq \frac{4^{n+r+1}K\theta}{4^{3nr}\left(4^{3pr} - 4^p\right)^{\frac{1}{p}}} ||x||_X^{3r}\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. We can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. Thus the mapping $H : X \to Y$ is a unique homomorphism satisfying (2.21). □

Theorem 2.8. *Let $r < \frac{1}{3}$ and θ be a positive real numbers and let $f : X \to Y$ be a mapping satisfying (2.20) and (2.2). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\|_Y \leq \frac{\theta 2^{2r+1}}{(4^p - 4^{3pr})^{1/p}} ||x||_X^{3r} \tag{2.24}$$

for all $x \in X$.

Proof. It follows from (2.23) that

$$||f(x) - \frac{1}{4}f(4x)||_Y \leq \frac{\theta 4^r}{2} ||x||_X^{3r}$$

for all $x \in X$. Replacing x by $4^l x$ and dividing 4^l, we get

$$\left\|\frac{1}{4^l}f(4^l x) - \frac{1}{4^{l+1}}f(4^{l+1}x)\right\|_Y \leq \frac{\theta}{2}\frac{4^r 4^{3rl}}{4^l} ||x||_X^{3r}.$$

Since Y is a p-Banach algebra, we have

$$\left\|\frac{1}{4^l}f(4^l x)-\frac{1}{4^{l+1}}f(4^{l+1}x)\right\|_Y^p \le \frac{\theta^p}{2^p}\frac{4^{pr}4^{3prl}}{4^{pl}}||x||_X^{3pr}.$$

$$\Rightarrow \quad \left\|\frac{1}{4^l}f(4^l x)-\frac{1}{4^{m}}f(4^{m})\right\|_Y^p \le \frac{4^{pr}\theta^p}{2^p}\sum_{j=1}^{m-1}\frac{4^{3prj}}{4^{p}j}||x||_X^{3pr} \tag{2.25}$$

for all non-negative integers m and l with $l<m$ and for all $x\in X$. It follows from (2.25) that the sequence $\left\{\frac{1}{4^n}f(4^n x)\right\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\left\{\frac{1}{4^n}f(4^n x)\right\}$ converges. So one can define the mapping $H:X\to Y$ by

$$H(x)=\lim_{n\to\infty}\frac{1}{4^n}f(4^n x)\,,\quad \text{for all } x\in X.$$

The rest of the proof is similar to the proof of Theorem 2.1. □

(E) *We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebra, associated to the additive type functional equation*

$$f(2x-y)+f(x-2y)=3f(x)-3f(y)\,. \tag{E_5}$$

Now we define

$$D\ f_5(x,y)=f(2x-y)+f(x-2y)-3f(x)+3f(y).$$

Theorem 2.9. *Let $r>1/2$ and θ be a positive real numbers and let $f:X\to Y$ satisfying (2.2) such that*

$$\|D\ f_5(x,y)\|_Y \le \theta||x||_X^r||y||_X^r \tag{2.26}$$

for all $x,y\in X$. If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in X$, then there exists a unique homomorphism $H:X\to Y$ such that

$$\|f(x)-H(x)\|_Y \le \frac{2K\theta}{(9^{pr}-3^p)^{1/p}}||x||_X^{2r} \tag{2.27}$$

for all $x\in X$.

Proof. Setting $y=x$ in (2.26), we get

$$||f(x)+f(-x)||_Y \le \theta||x||_X^{2r}. \tag{2.28}$$

Letting $y=-x$ in (2.26), we get

$$||3f(x)-2f(3x)-3f(-x)||_Y \le \theta||x||_X^{2r}. \tag{2.29}$$

Thus

$$||f(3x)-3f(x)||_Y \le 2K\theta||x||_X^{2r} \tag{2.30}$$

for all $x\in X$. So

$$\left\|f(x)-3f\left(\frac{x}{3}\right)\right\|_Y \le \frac{2K\theta}{9^r}||x||_X^{2r}$$

for all $x \in X$. Replacing x by $\frac{x}{3^l}$ and multipling by 3^l, we get

$$\left\|3^l f\left(\frac{x}{3^l}\right) - 3^{l+1} f\left(\frac{x}{3^{l+1}}\right)\right\|_Y \leq \frac{2K\theta 3^l}{9^r 9^{rl}}||x||_X^{2r}.$$

Since Y is a p-Banach algebra, we have

$$\left\|3^l f\left(\frac{x}{3^l}\right) - 3^{l+1} f\left(\frac{x}{3^{l+1}}\right)\right\|_Y^p \leq \frac{2^p K^p \theta^p}{9^{pr}} \frac{3^{pl}}{9prl}||x||_X^{2pr}.$$

$$\Rightarrow \left\|3^l f\left(\frac{x}{3^l}\right) - 3^{m} f\left(\frac{x}{3^{m}}\right)\right\|_Y^p \leq \frac{2^p K^p \theta^p}{9^{pr}} \sum_{j=1}^{m-1} \frac{3^{pj}}{9^{prj}}||x||_X^{2pr} \tag{2.31}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.31) that the sequence $\left\{3^n f(\frac{x}{3^n})\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{3^n f(\frac{x}{3^n})\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) = \lim_{n\to\infty} 3^n f(x/3^n), \quad \text{for all } x \in X.$$

It follows from (2.26) that

$$\begin{aligned}\|D\ H(x,y)\|_Y &\leq \lim_{n\to\infty} 3^n \left\|D\ f_5\left(\frac{x}{3^n}, \frac{x}{3^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} \frac{\theta}{3^n} 9^{nr}||x||_X^r||y||_X^r = 0\end{aligned}$$

for all $x, y, z \in X$. So

$$H(2x - y) + H(x - 2y) = 3H(x) - 3H(y)$$

for all $x, y \in X$. Moreover, letting $l = 0$ and allowing the limit $m \to \infty$ in (2.31), we get the inequality (2.27).

By the same reasoning as in the proof of Theorem of [28], the mapping $H : X \to Y$ is $\mathbb{R}$-linear. It follows from (2.2) that

$$\begin{aligned}\|H(x_1x_2) - H(x_1)H(x_2)\|_Y &= \lim_{n\to\infty} 9^n \left\|f\left(\frac{x_1x_2}{3^n3^n}\right) - f\left(\frac{x_1}{3^n}\right) f\left(\frac{x_2}{3^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} \frac{9^n\theta}{9^{nr}}||x_1||_X^r||x_2||_X^r = 0\end{aligned}$$

for all $x, y \in X$. So $H(x_1x_2) = H(x_1)H(x_2)$ for all $x_1, x_2 \in X$.

Now let $T : X \to Y$ be another additive mapping satisfying (2.29). Then we have

$$\begin{aligned}\|H(x) - T(x)\|_Y &= 3^n \left\|f\left(\frac{x}{3^n}\right) - T\left(\frac{x}{3^n}\right)\right\|_Y \\ &\leq 3^n K \left[\left\|H\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\|_Y + \left\|T\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\|_Y\right] \\ &\leq \frac{3^n 4K\theta}{9^{nr}\left(9^{pr} - 3^p\right)^{\frac{1}{p}}}||x||_X^{2r}\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. We can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. Thus the mapping $H : X \to Y$ is a unique homomorphism satisfying (2.27). □

Theorem 2.10. *Let $r < \frac{1}{2}$ and θ be a positive real numbers and let $f : X \to Y$ be a mapping satisfying (2.26) and (2.2). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\|_Y \leq \frac{2K\theta}{(3^p - 9^{pr})^{1/p}} ||x||_X^{2r} \tag{2.32}$$

for all $x \in X$.

Proof. It follows from (2.30) that

$$||f(x) - \frac{1}{3}f(3x)||_Y \leq \frac{2K\theta}{3} ||x||_X^{2r}$$

for all $x \in X$. Replacing x by $3^l x$ and dividing by 3^l, we get

$$\left\| \frac{1}{3^l} f(3^l x) - \frac{1}{3^{l+1}} f(3^{l+1} x) \right\|_Y \leq \frac{2K\theta}{3} \frac{9^{rl}}{3^l} ||x||_X^{2r} .$$

Since Y is a p-Banach algebra, we have

$$\left\| \frac{1}{3^l} f(3^l x) - \frac{1}{3^{l+1}} f(3^{l+1} x) \right\|_Y^p \leq \frac{2^p K^p \theta^p}{3^p} \frac{9^{prl}}{3^{pl}} ||x||_X^{2pr} ,$$

$$\left\| \frac{1}{3^l} f(3^l x) - \frac{1}{3^m} f(3^m x) \right\|_Y^p \leq \frac{2^p K^p \theta^p}{3^p} \sum_{j=1}^{m-1} \frac{9^{prj}}{3^p j} ||x||_X^{2pr} \tag{2.33}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.33) that the sequence $\left\{\frac{1}{3^n} f(3^n x)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{3^n} f(3^n x)\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$H(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) , \quad \text{for all } x \in X.$$

The rest of the proof is similar to the proof of Theorem 2.1. □

(F) *We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebra, associated to the additive mapping of first form*

$$f(x + y) + f(x - y) = 2f(x) . \tag{E_6}$$

Now we define

$$D\ f_6(x, y) = f(x + y) + f(x - y) - 2f(x).$$

Theorem 2.11. *Let $r > \frac{1}{2}$ and θ be a positive real numbers and let*

$f : X \to Y$ satisfying (2.2) such that

$$\|D\ f_6(x,y)\|_Y \le \theta ||x||_X^r ||y||_X^r \tag{2.34}$$

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that

$$\|f(x) - f(0) - H(x)\|_Y \le \frac{\theta}{(4^{pr} - 2^p)^{1/p}} ||x||_X^{2r} \tag{2.35}$$

for all $x \in X$.

Proof. Setting $y = x$ in (2.34), we get

$$||f(2x) + f(0) - 2f(x)||_Y \le \theta ||x||_X^{2r}. \tag{2.36}$$

Replacing x by $x/2$, we get

$$||f(x) + f(0) - 2f(x/2)||_Y \le \frac{\theta}{4^r} ||x||_X^{2r},$$

$$||g(x) - 2g(x/2)||_Y \le \frac{\theta}{4^r} ||x||_X^{2r},$$

where $g(x) = f(x) - f(0)$.

Replacing x by $\frac{x}{2^l}$ and multipling by 2^l, we get

$$\left\|2^l g\left(\frac{x}{2^l}\right) - 2^{l+1} g\left(\frac{x}{2^{l+1}}\right)\right\|_Y \le \frac{\theta 2^l}{4^r 4^{rl}} ||x||_X^{2r}.$$

Since Y is a p-Banach algebra, we have

$$\left\|2^l g\left(\frac{x}{2^l}\right) - 2^{l+1} g\left(\frac{x}{2^{l+1}}\right)\right\|_Y^p \le \frac{\theta^p 2^{lp}}{4^{pr} 4^{prl}} ||x||_X^{2pr}.$$

$$\Rightarrow \quad \left\|2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right)\right\|_Y^p \le \frac{\theta^p}{4^{pr}} \sum_{j=1}^{m-1} \frac{2^{pj}}{4^{prj}} ||x||_X^{2pr} \tag{2.37}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.37) that the sequence $\left\{2^n g(\frac{x}{2^n})\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{2^n g(\frac{x}{2^n})\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$\begin{aligned} H(x) &= \lim_{n\to\infty} 2^n g(x/2^n), \quad \text{for all } x \in X, \\ &= \lim_{n\to\infty} 2^n [f(x/2^n) - f(0)], \quad \text{for all } x \in X, \\ &= \lim_{n\to\infty} 2^n f(x/2^n), \quad \text{for all } x \in X. \end{aligned}$$

It follows from (2.34) that

$$\|D\ H(x,y)\|_Y \le \lim_{n\to\infty} 2^n \left\|D\ f_6\left(\frac{x}{2^n}, \frac{x}{2^n}\right)\right\|_Y$$

$$\leq \lim_{n\to\infty} \frac{\theta 2^n}{4^{nr}} ||x||_X^r ||y||_X^r = 0$$

for all $x, y \in X$. So

$$H(x+y) + H(x-y) = 2H(x).$$

for all $x, y \in X$. Moreover, letting $l = 0$ and allowing the limit $m \to \infty$ in (2.37), we get the inequality (2.35).

By the same reasoning as in the proof of Theorem of [28], the mapping $H : X \to Y$ is $\mathbb{R}$-linear. It follows from (2.2) that

$$\begin{aligned} \|H(x_1x_2) - H(x_1)H(x_2)\|_Y &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{x_1x_2}{2^n 2^n}\right) - f\left(\frac{x_1}{2^n}\right) f\left(\frac{x_2}{2^n}\right) \right\|_Y \\ &\leq \lim_{n\to\infty} \frac{4^n \theta}{4^{nr}} ||x_1||_X^r ||x_2||_X^r = 0 \end{aligned}$$

for all $x, y \in X$. So $H(x_1x_2) = H(x_1)H(x_2)$ for all $x_1, x_2 \in X$.

Now let $T : X \to Y$ be another additive mapping satisfying (2.35). Then we have

$$\begin{aligned} \|H(x) - T(x)\|_Y &= 2^n \left\| f\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|_Y \\ &\leq 2^n K \left[\left\| H\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_Y + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_Y \right] \\ &\leq \frac{2^{n+1} K\theta}{4^{nr} \left(4^{pr} - 2^p\right)^{\frac{1}{p}}} ||x||_X^{2r} \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. We can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. Thus the mapping $H : X \to Y$ is a unique homomorphism satisfying (2.35). □

Theorem 2.12. *Let $r < \frac{1}{2}$ and θ be a positive real numbers and let $f : X \to Y$ be a mapping satisfying (2.34) and (2.2). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\|_Y \leq \frac{\theta}{(2^p - 4^{pr})^{1/p}} ||x||_X^{2r} \tag{2.38}$$

for all $x \in X$.

Proof. It follows from (2.36) that

$$\begin{aligned} ||f(x) - \frac{f(0)}{2} - \frac{1}{2} f(2x)||_Y &\leq \frac{\theta}{2} ||x||_X^{2r}, \\ ||g(x) - \frac{1}{2} g(2x)||_Y &\leq \frac{\theta}{2} ||x||_X^{2r}, \end{aligned}$$

where $g(x) = f(x) - f(0)$ for all $x \in X$.

Replacing x by $2^l x$ and dividing by 2^l, we get

$$\left\| \frac{1}{2^l} g(2^l x) - \frac{1}{2^{l+1}} g(2^{l+1} x) \right\|_Y \leq \frac{\theta 4^{rl}}{2 2^l} ||x||_X^{2r}.$$

Since Y is a p-Banach algebra, we have

$$\left\| \frac{1}{2^l} g(2^l x) - \frac{1}{2^{l+1}} g(2^{l+1} x) \right\|_Y^p \leq \frac{\theta^p 4^{prl}}{2^p 2^{lp}} ||x||_X^{2pr}.$$

$$\Rightarrow \quad \left\| \frac{1}{2^l} g(2^l x) - \frac{1}{2^m} g(2^m x) \right\|_Y^p \leq \frac{\theta^p}{2^p} \sum_{j=1}^{m-1} \frac{4^{prj}}{2^p j} ||x||_X^{2pr} \tag{2.39}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.39) that the sequence $\left\{ \frac{1}{2^n} g(2^n x) \right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{ \frac{1}{2^n} g(2^n x) \right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$\begin{aligned} H(x) &= \lim_{n\to\infty} 2^n g(x/2^n)\,, \quad \text{for all } x \in X\,, \\ &= \lim_{n\to\infty} 2^n [f(x/2^n) - f(0)]\,, \quad \text{for all } x \in X\,, \\ &= \lim_{n\to\infty} 2^n f(x/2^n)\,, \quad \text{for all } x \in X. \end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

(G) *We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebra, associated to the additive mapping of second form*

$$f(x + y) + f(x - y) = 2f(y)\,. \tag{E_7}$$

Now we define

$$D\ f_7(x, y) = f(x + y) + f(x - y) - 2f(y).$$

Theorem 2.13. *Let $r > \frac{1}{2}$ and θ be a positive real numbers and let $f : X \to Y$ satisfying (2.2) such that*

$$\|D\ f_7(x, y)\|_Y \leq \theta ||x||_X^r ||y||_X^r \tag{2.40}$$

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that

$$\|f(x) + f(0) - H(x)\|_Y \leq \frac{\theta}{(4^{pr} - 2^p)^{1/p}} ||x||_X^{2r} \tag{2.41}$$

for all $x \in X$.

Proof. Setting $y = x$ in (2.40), we get

$$||f(2x) - f(0) - 2f(x)||_Y \leq \theta ||x||_X^{2r}. \tag{2.42}$$

Replacing x by $x/2$, we get

$$||f(x) - f(0) - 2f(x/2)||_Y \leq \frac{\theta}{4^r}||x||_X^{2r}$$
$$||g(x) - 2g(x/2)||_Y \leq \frac{\theta}{4^r}||x||_X^{2r},$$

where $g(x) = f(x) + f(0)$.

Replacing x by $\frac{x}{2^l}$ and multipling by 2^l, we get

$$\left\|2^l g\left(\frac{x}{2^l}\right) - 2^{l+1} g\left(\frac{x}{2^{l+1}}\right)\right\|_Y \leq \frac{\theta 2^l}{4^r 4^{lr}}||x||_X^{2r}.$$

Since Y is a p-Banach algebra, we have

$$\left\|2^l g\left(\frac{x}{2^l}\right) - 2^{l+1} g\left(\frac{x}{2^{l+1}}\right)\right\|_Y^p \leq \frac{\theta^p 2^{pl}}{4^{pr} 4^{prl}}||x||_X^{2r}.$$
$$\Rightarrow \quad \left\|2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right)\right\|_Y^p \leq \frac{\theta^p}{4^{pr}} \sum_{j=1}^{m-1} \frac{2^{pj}}{4^{prj}}||x||_X^{2pr}, \tag{2.43}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.43) that the sequence $\left\{2^n g(\frac{x}{2^n})\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{2^n g(\frac{x}{2^n})\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$\begin{aligned} H(x) &= \lim_{n\to\infty} 2^n g(x/2^n), \quad \text{for all } x \in X, \\ &= \lim_{n\to\infty} 2^n [f(x/2^n) + f(0)], \quad \text{for all } x \in X, \\ &= \lim_{n\to\infty} 2^n f(x/2^n), \quad \text{for all } x \in X. \end{aligned}$$

It follows from (2.40) that

$$\begin{aligned} \|D\ H(x,y)\|_Y &\leq \lim_{n\to\infty} 2^n \left\|D\ f_7\left(\frac{x}{2^n}, \frac{x}{2^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} \frac{\theta 2^n}{4^{nr}}||x||_X^r ||y||_X^r = 0 \end{aligned}$$

for all $x, y \in X$. So

$$H(x+y) + H(x-y) = 2H(y)$$

for all $x, y \in X$. Moreover, letting $l = 0$ and allowing the limit $m \to \infty$ in (2.43), we get the inequality (2.41).

By the same reasoning as in the proof of Theorem of [28], the mapping $H : X \to Y$ is $\mathbb{R}$-linear. It follows from (2.2) that

$$\begin{aligned} \|H(x_1x_2) - H(x_1)H(x_2)\|_Y &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x_1x_2}{2^n 2^n}\right) - f\left(\frac{x_1}{2^n}\right) f\left(\frac{x_2}{2^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}}||x_1||_X^r ||x_2||_X^r = 0 \end{aligned}$$

for all $x, y \in X$. So $H(x_1x_2) = H(x_1)H(x_2)$ for all $x_1, x_2 \in X$.

Now let $T : X \to Y$ be another additive mapping satisfying (2.41). Then we have

$$\begin{aligned}\|H(x) - T(x)\|_Y &= 2^n \left\| f\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_Y \\ &\le 2^n K\left[\left\|H\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_Y + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_Y\right] \\ &\le \frac{2^{n+1}K\theta}{4^{nr}\left(4^{pr} - 2^p\right)^{\frac{1}{p}}}||x||_X^{2r}\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. We can conclude that $H(x) = T(x)$ for all $x \in X$. This proves the uniqueness of H. Thus the mapping $H : X \to Y$ is a unique homomorphism satisfying (2.41). □

Theorem 2.14. *Let $r < \frac{1}{2}$ and θ be a positive real numbers and let $f : X \to Y$ be a mapping satisfying (2.40) and (2.2). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\|_Y \le \frac{\theta}{(2^p - 4^{pr})^{1/p}}||x||_X^{2r} \tag{2.44}$$

for all $x \in X$.

Proof. It follows from (2.42) that

$$||f(x) + \frac{f(0)}{2} - \frac{1}{2}f(2x)||_Y \le \frac{\theta}{2}||x||_X^{2r}$$
$$||g(x) - \frac{1}{2}g(2x)||_Y \le \frac{\theta}{2}||x||_X^{2r}$$

where $g(x) = f(x) + f(0)$ for all $x \in X$.

Replacing x by $2^l x$ and dividing by 2^l, we get

$$\left\|\frac{1}{2^l}g(2^l x) - \frac{1}{2^{l+1}}g(2^{l+1}x)\right\|_Y \le \frac{\theta 4^{rl}}{2\ 2^l}||x||_X^{2r}.$$

Since Y is a p-Banach algebra, we have

$$\left\|\frac{1}{2^l}g(2^l x) - \frac{1}{2^{l+1}}g(2^{l+1}x)\right\|_Y^p \le \frac{\theta^p 4^{prl}}{2^p 2^{lp}}||x||_X^{2pr}.$$

$$\Rightarrow \quad \left\|\frac{1}{2^l}g(2^l x) - \frac{1}{2^m}g(2^m x)\right\|_Y^p \le \frac{\theta^p}{2^p}\sum_{j=1}^{m-1}\frac{4^{prj}}{2^p j}||x||_X^{2pr} \tag{2.45}$$

for all non-negative integers m and l with $l < m$ and for all $x \in X$. It follows from (2.45) that the sequence $\left\{\frac{1}{2^n}g(2^n x)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{2^n}g(2^n x)\right\}$ converges. So one can define the mapping $H : X \to Y$ by

$$\begin{aligned} H(x) &= \lim_{n\to\infty} 2^n g(x/2^n)\,, \quad \text{for all } x \in X \\ &= \lim_{n\to\infty} 2^n [f(x/2^n) + f(0)]\,, \quad \text{for all } x \in X \\ &= \lim_{n\to\infty} 2^n f(x/2^n)\,, \quad \text{for all } x \in X. \end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Isomorphisms between Quasi-Banach Algebra

Through this section, assume that X is a quasi-Banach algebra with quasi-norm $||.||_X$ and unit e and that Y is a p-Banach algebra with p-norm $||.||_Y$ and unit e'. Let K be the modulus of concavity of $||.||_Y$.

(A) *We investigate isomorphisms between quasi-Banach algebras, associated to the general form of Cauchy functional equation (E_1).*

Theorem 3.1. *Let $r > \frac{1}{k}$, $k \in N \geq 2$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.1) such that*

$$f(x_1x_2) = f(x_1)f(x_2) \tag{3.1}$$

for all $x_1, x_2 \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} k^n f\left(\frac{e}{k^n}\right) = e'$, then the mapping $f : X \to Y$ is an isomorphism.

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.1, there exists a homomorphisms $H : X \to Y$ satisfying (2.3). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in X$. It follows from (3.1) that

$$\begin{aligned} H(x) =& H(ex) = \lim_{n\to\infty} k^n f\left(\frac{ex}{k^n}\right) = \lim_{n\to\infty} k^n f\left(\frac{e}{k^n}.x\right) \\ &= \lim_{n\to\infty} k^n f\left(\frac{e}{k^n}\right) f(x) = e'f(x) = f(x) \end{aligned}$$

for all $x \in X$. So the bijective mapping $f : X \to Y$ is an isomorphism. □

Theorem 3.2. *Let $r < \frac{1}{k}$, $k \in N \geq 2$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.1) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} \frac{1}{k^n} f(k^n e) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping

$f : X \to Y$ satisfies (2.2). By Theorem 2.2, there exists a homomorphisms $H : X \to Y$ satisfying (2.6). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x)$$

for all $x \in X$.It follows from (3.1) that

$$\begin{aligned} H(x) =& H(ex) = \lim_{n\to\infty} \frac{1}{k^n} f(ek^n x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n e.x) \\ =& \lim_{n\to\infty} \frac{1}{k^n} f(k^n e) f(x) = e' f(x) = f(x) \end{aligned}$$

for all $x \in X$. So the bijective mapping $f : X \to Y$ is an isomorphism. □

(B) *We investigate isomorphisms between quasi-Banach algebras, associated to the Jensen functional equation (E_2).*

Theorem 3.3. *Let $r < 1/3$ and θ be positive real numbers, let $f : X \to B$ be a bijective mapping with $f(0) = 0$ satisfying (2.8) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} \frac{1}{4^n} f(4^n e) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in A$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.3, there exists a homomorphisms $H : X \to Y$ satisfying (2.9). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{4^n} f(4^n x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Theorem 3.4. *Let $r > 1/3$ and θ be positive real numbers, let $f : X \to B$ be a bijective mapping with $f(0) = 0$ satisfying (2.8) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} 4^n f\left(\frac{e}{4^n}\right) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.4, there exits a homomorphisms $H : X \to Y$ satisfying (2.12). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{4^n}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

(C) *We investigate isomorphisms between quasi-Banach algebras, associated to the Cauchy-Jensen functional equation (E_3).*

Theorem 3.5. *Let $r > 1/3$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.14) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.5, there exists a homomorphisms $H : X \to Y$ satisfying (2.15). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.6. *Let $r < 1/3$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.14) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.6, there exists a homomorphisms $H : X \to Y$ satisfying (2.18). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

(D) *We investigate isomorphisms between quasi-Banach algebras, associated to the Apollonius-type additive functional equation (E_4).*

Theorem 3.7. *Let $r > 1/3$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.20) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} 4^n f\left(\frac{e}{4^n}\right) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.7, there exists a homomorphisms $H : X \to B$ satisfying (2.21). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{4^n}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.8. *Let $r < 1/3$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.20) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} 4^n f\left(\frac{e}{4^n}\right) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.8, there exists a homomorphisms $H : X \to Y$ satisfying (2.24). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{4^n} f(4^n x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

(E) *We investigate isomorphisms between quasi-Banach algebras, associated to the additive type functional equation (E_5).*

Theorem 3.9. *Let $r > 1/2$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.26) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} 3^n f\left(\frac{e}{3^n}\right) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.9, there exists a homomorphisms $H : X \to Y$ satisfying (2.27). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.10. *Let $r < 1/2$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.26) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} \frac{1}{3^n} f(3^n e) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.10, there exists a homomorphisms $H : X \to Y$ satisfying (2.27). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

(F) *We investigate isomorphisms between quasi-Banach algebras, associated to the additive mapping of first form (E_6).*

Theorem 3.11. *Let $r > \frac{1}{2}$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.34) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.11, there exists a homomorphisms $H : X \to Y$ satisfying (2.35). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.12. *Let $r < 1/2$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.34) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.12, there exists a homomorphisms $H : X \to Y$ satisfying (2.38). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

(G) *We investigate isomorphisms between quasi-Banach algebras, associated to the additive mapping of second form (E_7).*

Theorem 3.13. *Let $r > 1/2$ and θ be positive real numbers, let $f : X \to Y$ be X bijective mapping satisfying (2.40) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.13, there exists a homomorphisms

$H : X \to Y$ satisfying (2.41). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.14. *Let $r < 1/2$ and θ be positive real numbers, let $f : X \to Y$ be a bijective mapping satisfying (2.40) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : X \to Y$ is an isomorphism.*

Proof. Since $f(x_1x_2) - f(x_1)f(x_2) = 0$ for all $x_1, x_2 \in X$, the mapping $f : X \to Y$ satisfies (2.2). By Theorem 2.14, there exists a homomorphisms $H : X \to Y$ satisfying (2.44). The mapping $H : X \to Y$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \quad \text{for all} \quad x \in X.$$

The rest of the proof is similar to the proof of Theorem 3.2. □

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