International Journal of Pure and Applied Mathematics
Volume 40 No. 1 2007, 1-10

SUPERLINEAR CONVERGENCE OF A FAMILY OF TWO–STEP STEFFENSEN–TYPE METHODS FOR GENERALIZED EQUATIONS

Saïd Hilout
Department of Applied Mathematics and Computation
Faculty of Sciences and Technics
BP 523, Béni–Mellal, 23000, MOROCCO
e-mail: said_hilout@yahoo.fr

Abstract: We approximate solution of generalized equations in a Banach spaces. We provide sufficient conditions for the local convergence of a family of two–step Steffensen–type algorithms using a fixed points theorem of multi-applications, the usual Hölder continuous condition and the Aubin continuous concept. This study follows the recent result of two–step Steffensen's method related to the resolution of nonlinear equations (see [1]).

AMS Subject Classification: 47H04, 65K10
Key Words: Steffensen's method, set–valued map, generalized equation, Aubin continuity, divided difference

1. Introduction

Some problems from mathematical economics, complementarity, mathematical programming, optimal control, variational inequalities and other fields can be represented in the form of "abstract" generalized equations, see [21, 22]:

$$0 \in f(x) + G(x)\,, \tag{1}$$

Received: July 10, 2007

where f is a continuous function from X into Y and G is a set–valued map from X to the subsets of Y with closed graph and X, Y are two Banach spaces.

For the case of nonlinear equations ($G \equiv 0$ in (1)), Argyros [6] presents a semilocal convergence analysis of the secant method for nonsmooth operator f under a flexible point–based approximation of f. In finite dimensional spaces, Robinson [21, 22] studied the stability of solutions set of (1) under certain hypotheses and proved a general form of the implicit function theorem. The problem of approximating locally unique solution of (1) in infinite dimensional spaces is treated by Dontchev [9] and Argyros [3, 4, 5] using Newton's method. A third iterative method is presented in [11].

For approximating locally the unique solution x^* of (1), we consider the two–step Steffensen–type method given by ($k = 0, 1, 2, \dots$)

$$\left\{ \begin{array}{l} x_0 \text{ is given as starting point}\,, \\ 0 \in f(x_k) + [g_1(x_k), g_2(x_k); f]\ (y_k - x_k) + G(y_k)\,, \\ 0 \in f(y_k) + [g_1(x_k), g_2(x_k); f]\ (x_{k+1} - y_k) + G(x_{k+1})\,, \end{array} \right. \tag{2}$$

where g_i $(i = 1, 2)$ are a continuous functions from a neighborhood D of x^* into X and $[g_1(x_k), g_2(x_k); f]$ is a first order divided difference of f on the points $g_1(x_k)$ and $g_2(x_k)$.

For the case of nonlinear equations ($G \equiv 0$ in (1)), the algorithm (2) is reduced to two–step Stenffensen's method (due to Amat and Busquier [1]) in the form

$$\left\{ \begin{array}{l} x_0 \text{ is given as starting point}\,, \\ y_k = x_k - [g_1(x_k), g_2(x_k); f]^{-1}\ f(x_k)\,, \\ x_{k+1} = y_k - [g_1(x_k), g_2(x_k); f]^{-1}\ f(y_k)\,. \end{array} \right. \tag{3}$$

A study of convergence of a family of iterative processes (3) is presented in [1] using one and twice differentiability of g_1, g_2 and f respectively on a neighborhood of solution.

We give a new secant–type method for solving (1) in [17] under suitable conditions on first order divided difference and the Aubin continuity property. In [14], we provide sufficient conditions for the local convergence of a new Steffensen–type methods to solve (1). We also consider a combination of this method with Newton–type method proposed in [17]. We consider in [15] an uniparametric Newton–Steffensen–type method for solving perturbed generalized equations and we deduce an improved convergence analysis of a Regula–Falsi–type method. A convergence analysis of a family of Steffensen–type algorithms is developed in [16].

In this paper, we propose a family of two–step Steffensen–type algorithms procedure for solving (1). The idea of this work rises from works of Amat and

Busquier [1] and Argyros [2].

The paper is organized as follows. In Section 2, we collect a basic definitions and recall a fixed–points theorem [10], which is the main tool to prove the existence and the convergence of the sequence (2). We prove in Section 3 the existence of a sequence satisfying (2) and we show that is locally superlinear convergent. Finally, we give an illustrative example.

2. Preliminaries

Let us begin with some standard notations that will be used throughout this paper. We let $\mathcal{Z}$ be a metric space endowed with the metric ρ. For $A \subset \mathcal{Z}$, we denote by dist $(x, A) = \inf\{\rho(x, y),\ y \in A\}$ the distance from a point x to A. The excess e from A to the set $C \subset \mathcal{Z}$ is given by $e(A, C) = \sup\{\text{dist}(x, A),\ x \in C\}$. Let $\Lambda : X \rightrightarrows Y$ be a set–valued map, we denote by $gph\,\Lambda = \{(x, y) \in X \times Y,\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y) = \{x \in X,\ y \in \Lambda(x)\}$ is the inverse of Λ. We call $B_r(x)$ the closed ball centered at x with radius r. ∇f denotes the Fréchet derivative of f. Finally, the norm in the Banach spaces X and Y are both denoted by $\| \,.\, \|$ and $\mathcal{L}(X, Y)$ is the space of bounded and linear operators from X to Y.

Definition 1. A set–valued Λ is said to be pseudo–Lipschitz around $(x_0, y_0) \in gph\,\Lambda$ with modulus M if there exist constants a and b such that

$$e(\Lambda(y') \cap B_a(y_0), \Lambda(y'')) \leq M \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0). \tag{4}$$

For more details and applications of this concept see for example [7, 19, 8, 10, 18, 20] and the references given there.

Definition 2. An operator $[.,.;f]$ belonging $\mathcal{L}(X, Y)$ is called the first order divided difference of the function f on the points x and y in X $(x \neq y)$ if both of the following properties hold:

1. $[x, y; f](y - x) = f(y) - f(x)$.
2. If f is Fréchet differentiable at x then $[x, x; f] = \nabla f(x)$.

Definition 3. (see [13]) Let Ω be open subset of X, we say that the operator $[.,.;f]$ is $(\nu,\, p)$–Hölder continuous in Ω, where $\nu \geq 0$ and $p \in [0, 1]$ if the following inequality hold

$$\parallel [x, y; f] - [u, v; f] \parallel \leq\ \nu(\parallel x - u \parallel^p + \parallel y - v \parallel^p), \text{ for all } x, y \in \Omega.$$

As the main tool of our analysis we will use the following lemmas. The first is the fixed points theorem. The second gives an equivalence of pseudo–Lipschitzness for inverses of two set–valued mappings.

Lemma 4. (see [10]) *Let $(Z, \| \, . \, \|)$ be a Banach space, let ϕ a set–valued map from Z into the closed subsets of Z, let $\eta_0 \in Z$ and let r and λ be such that $0 \leq \lambda < 1$ and*

(a) dist $(\eta_0, \phi(\eta_0)) \leq r(1-\lambda)$,

(b) $e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \leq \lambda \ \| x_1 - x_2 \|, \ \ \forall x_1, x_2 \in B_r(\eta_0)$.

Then ϕ has a fixed–point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

Lemma 5. (see [17]) *The set–valued mapping $(f+G)^{-1}$ is pseudo–Lipschitz around $(0, x^*)$ if and only if the set–valued mapping $(f(x^*)+G(.))^{-1}$ is pseudo–Lipschitz around $(0, x^*)$.*

We suppose that, for every distinct points x and y in a open neighborhood V of x^*, there exists a first order divided difference of f at these points. We will make the following assumptions:

($\mathcal{H}0$) For $i = 1, 2$; the function g_i is α_i–Lipschitz from V into V; $\alpha_i \in [0, 1)$ and $g_i(x^*) = x^*$.

($\mathcal{H}1$) $[., .; f]$ is (ν, p)–Hölder continuous in V.

($\mathcal{H}2$) The set–valued map $(f+G)^{-1}$ is M–pseudo–Lipschitz around $(0, x^*)$.

($\mathcal{H}3$) For all $x, y \in V$, we have $|||[x, y; f]||| \leq \kappa$ and $M\kappa < 1$.

Before stating the main result on this study, we need to introduce some notations. First, for $k \in I\!N$ and (y_k), (x_k) defined in (2), let us define the set–valued mappings Q, ψ_k, $\phi_k : X \rightrightarrows Y$ by the following

$$Q(.) := f(x^*) + G(.); \quad \psi_k(.) := Q^{-1}(Z_k(.)); \quad \phi_k(.) := Q^{-1}(W_k(.)) \,, \tag{5}$$

where Z_k and W_k are defined from X to Y by

$$\begin{aligned} Z_k(x) &:= f(x^*) - f(y_k) - [g_1(x_k), g_2(x_k); f](x - y_k) \,, \\ W_k(x) &:= f(x^*) - f(x_k) - [g_1(x_k), g_2(x_k); f](x - x_k) \,. \end{aligned} \tag{6}$$

3. Convergence Analysis

In this section, we will be concerned with the existence of the sequences (y_n) and (x_n) defined by (2) and the convergence analysis of (x_n) to the solution x^* of (1) under the previous assumptions. The main result of this study is as follows.

Theorem 6. *We suppose that assumptions ($\mathcal{H}0$)–($\mathcal{H}3$) are satisfied. For every constant* $C > \frac{M\nu([1+\alpha_1]^p+\alpha_2^p)}{1-M\kappa}$*, one can find* $\delta > 0$ *such that for every starting point* x_0 *in* $B_\delta(x^*)$ *(*x_0 *and* x^* *distinct), there exists a sequence* (x_k) *defined by (2) which satisfies*

$$\| x_{k+1} - x^* \| \leq C \| x_k - x^* \|^{p+1} . \tag{7}$$

The proof of Theorem 6 is by induction on k. We need to give two lemmas. In the first, we prove the existence of starting point y_0 for x_0 in V. In the second, we state a result which the starting point (x_0, y_0). Let us mention that y_0 and x_1 are a fixed points of ϕ_0 and ψ_0, respectively if and only if $0 \in f(x_0)+[g_1(x_0), g_2(x_0); f](y_0-x_0)+G(y_0)$ and $0 \in f(y_0)+[g_1(x_0), g_2(x_0); f](x_1-y_0) + G(x_1)$, respectively.

Proposition 7. *Under the assumptions of Theorem 6, one can find* $\delta > 0$ *such that for every starting point* x_0 *in* $B_\delta(x^*)$ *(*x_0 *and* x^* *distinct), the set–valued map* ϕ_0 *has a fixed point* y_0 *in* $B_\delta(x^*)$ *satisfying*

$$\| y_0 - x^* \| \leq C \| x_0 - x^* \|^{p+1} . \tag{8}$$

Proof of Proposition 7. By hypothesis ($\mathcal{H}2$) and Lemma 5 there exist positive numbers M, a and b such that

$$e(Q^{-1}(y') \cap B_a(x^*), Q^{-1}(y'')) \leq M \| y' - y'' \|, \ \forall y', y'' \in B_b(0). \tag{9}$$

Fix $\delta > 0$ such that

$$\delta < \min \left\{ a \ ; \ \sqrt[p+1]{\frac{b}{4 \ \nu \ ([1+\alpha_1]^p + ([1+\alpha_2]^p)}} \ ; \ \frac{1}{\sqrt[p]{C}} \ ; \ \frac{b}{2\kappa} \right\}. \tag{10}$$

The main idea of the proof of Proposition 7 is to show that both assertions (a) and (b) of Lemma 4 hold; where $\eta_0 := x^*$, ϕ is the function ϕ_0 defined in (5) and where r and λ are numbers to be set. According to the definition of the excess e, we have

$$\text{dist}\,(x^*, \phi_0(x^*)) \leq e\Big(Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*) \Big). \tag{11}$$

Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct) we have

$$\| W_0(x^*) \| = \| f(x^*) - f(x_0) - [g_1(x_0), g_2(x_0); f](x^* - x_0) \| .$$

By Definition 2 and assumptions ($\mathcal{H}0$)–($\mathcal{H}1$) we deduce

$$\begin{aligned}\| W_0(x^*) \| =& \| \Big([x_0,x^*;f]-[g_1(x_0),g_2(x_0);f]\Big)(x^*-x_0) \| \\ &\leq \| [x_0,x^*;f]-[g_1(x_0),g_2(x_0);f] \| \; \| x^*-x_0 \| \\ &\leq \nu \; (\| x_0-g_1(x_0) \|^p + \| x^*-g_2(x_0) \|^p) \;\; \| x^*-x_0 \| \\ &\leq \nu \; ([1+\alpha_1]^p+\alpha_2^p) \;\; \| x^*-x_0 \|^{p+1} \; . \qquad (12)\end{aligned}$$

Then (10) yields, $W_0(x^*) \in B_b(0)$.

Using (9) we have

$$\begin{aligned} e\Big(Q^{-1}(0)\cap B_\delta(x^*),\phi_0(x^*)\Big) &= e\Big(Q^{-1}(0)\cap B_\delta(x^*),Q^{-1}[W_0(x^*)]\Big) \\ &\leq M \; \nu \; ([1+\alpha_1]^p+\alpha_2^p) \;\; \| x^*-x_0 \|^{p+1} \; . \qquad (13)\end{aligned}$$

By the inequality (11), we get

$$\operatorname{dist}(x^*,\phi_0(x^*)) \quad \leq \quad M \; \nu \; ([1+\alpha_1]^p+\alpha_2^p) \;\; \| x^*-x_0 \|^{p+1} \; . \qquad (14)$$

Since $C(1-M\kappa) > M \; \nu \; ([1+\alpha_1]^p+\alpha_2^p)$, there exists $\lambda \in [M\kappa, 1[$ such that $C(1-\lambda) \geq M \; \nu \; ([1+\alpha_1]^p+\alpha_2^p)$ and

$$\operatorname{dist}(x^*,\phi_0(x^*)) \leq C \; (1-\lambda) \;\; \| x_0-x^* \|^{p+1} \; . \qquad (15)$$

By setting $r := r_0 = C \;\; \| x_0-x^* \|^{p+1}$ we can deduce from the inequality (15) that the assertion (a) in Lemma 4 is satisfied.

Now, we show that condition (b) of Lemma 4 is satisfied.

By (10) we have $r_0 \leq \delta \leq a$ and moreover for $x \in B_\delta(x^*)$ we have

$$\begin{aligned}\| W_0(x) \| =& \| f(x^*)-f(x_0)-[g_1(x_0),g_2(x_0);f](x-x_0) \| \\ &\leq \| f(x^*)-f(x) \| + \| f(x)-f(x_0)-[g_1(x_0),g_2(x_0);f](x-x_0) \| \\ &\leq \| f(x^*)-f(x) \| + \| [x_0,x;f]-[g_1(x_0),g_2(x_0);f] \| \; \| x-x_0 \| \; . \qquad (16)\end{aligned}$$

Using the assumptions $(\mathcal{H}0)$–$(\mathcal{H}1)$ and $(\mathcal{H}3)$ we obtain

$$\begin{aligned}\| W_0(x) \| \leq& \kappa \;\; \| x^*-x \| + \nu \; (\| x_0-g_1(x_0) \|^p + \| x-g_2(x_0) \|^p) \;\; \| x-x_0 \| \\ &\leq \kappa \;\; \| x^*-x \| + \nu \; \Big((\| x_0-x^* \| + \| x^*-g_1(x_0) \|)^p \\ &\quad + (\| x-x^* \| + \| x^*-g_2(x_0) \|)^p \Big) \;\; \| x-x_0 \| \\ &\leq \kappa \; \delta + \; \nu \; ([1+\alpha_1]^p + ([1+\alpha_2]^p) \; \delta^p \; (2\delta)\end{aligned}$$

$$= \kappa\ \delta + 2\ \nu\ ([1+\alpha_1]^p + ([1+\alpha_2]^p)\ \delta^{p+1}\,. \quad (17)$$

Then by (10) we deduce that for all $x \in B_\delta(x^*)$ we have $W_0(x) \in B_b(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$ we have

$$e(\psi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq e(\phi_0(x') \cap B_\delta(x^*), \phi_0(x'')),$$

which yields by (9)

$$\begin{aligned} e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) &\leq M\ \ \| W_0(x') - W_0(x'') \| \\ &\leq M\ \ \| [g_1(x_0), g_2(x_0); f] \| \, \| x'' - x' \|\ . \quad (18) \end{aligned}$$

Using $(\mathcal{H}3)$ and the fact that $\lambda \geq M\kappa$, we obtain

$$e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq M\ \kappa \, \| x'' - x' \| \leq \lambda\ \ \| x'' - x' \| \quad (19)$$

and thus condition (b) of Lemma 4 is satisfied. Since both conditions of Lemma 4 are fulfilled, we can deduce the existence of a fixed point $y_0 \in B_{r_0}(x^*)$ for the map ϕ_0. This finishes the proof of Proposition 7. □

Proposition 8. *Under the assumptions of Theorem 6, one can find $\delta > 0$ such that for every starting point x_0 in $B_\delta(x^*)$ and y_0 given by Proposition 7 (x_0 and x^* distinct), the set–valued map ψ_0 has a fixed point x_1 in $B_\delta(x^*)$ satisfying*

$$\| x_1 - x^* \| \leq C \, \| x_0 - x^* \|^{p+1}\ . \quad (20)$$

Idea of the Proof of Proposition 8. The proof of Proposition 8 is the same one as that of Proposition 7. The choise of δ is the same one given by (10). The inequality (11) is valid if we replace ϕ_0 by ψ_0. Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct) we have

$$\| Z_0(x^*) \| = \| f(x^*) - f(y_0) - [g_1(x_0), g_2(x_0); f](x^* - y_0) \|\ .$$

By Definition 2 and assumptions $(\mathcal{H}0)$–$(\mathcal{H}1)$ we deduce

$$\begin{aligned} \| Z_0(x^*) \| &= \| \Big([y_0, x^*; f] - [g_1(x_0), g_2(x_0); f]\Big)(x^* - y_0) \| \\ &\leq \| [y_0, x^*; f] - [g_1(x_0), g_2(x_0); f] \| \| x^* - y_0 \| \\ &\leq \nu(\| y_0 - g_1(x_0) \|^p + \| x^* - g_2(x_0) \|^p) \, \| x^* - y_0 \|\ . \quad (21) \end{aligned}$$

By Proposition 7 and (10) we have

$$\| Z_0(x^*) \| \leq C\nu\Big((C \| x_0 - x^* \|^{p+1} + \alpha_1 \| x_0 - x^* \|)^p + \alpha_2^p\Big) \| x^* - x_0 \|^{p+1}$$
$$\leq \nu([1+\alpha_1]^p + \alpha_2^p) \| x^* - x_0 \|^{p+1} \,. \quad (22)$$

Then (10) yields, $Z_0(x^*) \in B_b(0)$. Setting $r := r_0 = C \ \| x_0 - x^* \|^{p+1}$, we can deduce from the assertion (a) in Lemma 4 is satisfied.

By (10) we have $r_0 \leq \delta \leq a$ and moreover for $x \in B_\delta(x^*)$ we have

$$\begin{aligned} \| Z_0(x) \| &= \| f(x^*) - f(y_0) - [g_1(x_0), g_2(x_0); f](x - y_0) \| \\ &\leq \| f(x^*) - f(x) \| + \| f(x) - f(y_0) - [g_1(x_0), g_2(x_0); f](x - y_0) \| \\ &\leq \| f(x^*) - f(x) \| + \| [y_0, x; f] - [g_1(x_0), g_2(x_0); f] \| \ \| x - y_0 \| \,. \end{aligned} \quad (23)$$

Using the assumptions $(\mathcal{H}0)$–$(\mathcal{H}1)$ and $(\mathcal{H}3)$, Proposition 7 and (10) we obtain

$$\| Z_0(x) \| \quad \leq \quad \kappa \ \delta + 2 \ \nu \ ([1+\alpha_1]^p + ([1+\alpha_2]^p) \ \delta^{p+1} \,. \quad (24)$$

A slight change in the end of proof of Proposition 7 shows that the condition (b) of Lemma 4 is satisfied. The existence of a fixed point $x_1 \in B_{r_0}(x^*)$ for the map ψ_0 is ensured. This finishes the proof of Proposition 8. □

Proof of Theorem 6. Keeping $\eta_0 = x^*$ and setting $r := r_k = C \ \| x^* - x_k \|^{p+1}$, the application of Proposition 7 and Proposition 8 to the map ϕ_k and ψ_k respectively gives the existence of a fixed points y_k and x_{k+1} for ϕ_k and ψ_k respectively which is an elements of $B_{r_k}(x^*)$. This last fact implies the inequality (7), which is the desired conclusion. □

Remark 9. The sequence (y_n) given by algorithm (2) is also super–linearly convergent to a solution x^* of (1).

Illustrative Example. (see [21]) Let K be a convex set in $\mathbb{R}^n$, P is a topological space and φ is a function from $P \times K$ to $\mathbb{R}^n$, the "perturbed" variational inequality problem consists of seeking k_0 in K such that

$$\text{For each } k \in K, \quad (\varphi(p, k_0); k - k_0) \ \geq \ 0 \,, \quad (25)$$

where $(.;.)$ is the usual scalar product on $\mathbb{R}^n$ and p is fixed parameter in P. Let $\mathcal{I}_K$ be a convex indicator function of K and ∂ denotes the subdifferential operator. Then the problem (25) is equivalent to problem

$$0 \in \varphi(p, k_0) + \mathcal{H}(k_0) \quad (26)$$

with $\mathcal{H} = \partial \mathcal{I}_K$. $\mathcal{H}$ is also called the normal cone of K. The "perturbed" variational inequality problem (25) is equivalent to (26) which is a generalized equation in the form (1). Consequently, we can approximate the solution k_0 of (25) using our method (2).

References

[1] S. Amat, S. Busquier, Convergence and numerical analysis of a family of two-step Steffensen's methods, *Comput. and Math. with Appl.*, **49** (2005), 13-22.

[2] I.K. Argyros, A new convergence theorem for Steffensen's method on Banach spaces and applications, *Southwest J. of Pure and Appl. Math.*, **01** (1997), 23-29.

[3] I.K. Argyros, On the solution of generalized equations using m ($m \geq 2$) Fréchet differential operators, *Comm. Appl. Nonlinear Anal.*, **09** (2002), 85-89.

[4] I.K. Argyros, Results on the solution of generalized equations, *Comm. Appl. Nonlinear Anal.*, **09** (2002), 103-107.

[5] I.K. Argyros, On the approximation of strongly regular solutions for generalized equations, *Comm. Appl. Nonlinear Anal.*, **12** (2005), 97-107.

[6] I.K. Argyros, On the secant method for solving nonsmooth equations, *J. Math. Anal. Appl.*, **322** (2006), 146-157.

[7] J.P. Aubin, Lipschitz behavior of solutions to convex minimization problems, *Mathematics of Operations Research*, **9** (1984), 87-111.

[8] J.P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[9] A.L. Dontchev, Uniform convergence of the Newton method for Aubin continuous maps, *Serdica Math. J.*, **22** (1996), 385-398.

[10] A.L. Dontchev, W.W. Hager, An inverse function theorem for set-valued maps, *Proc. Amer. Math. Soc.*, **121** (1994), 481-489.

[11] M.H. Geoffroy, S. Hilout, A. Piétrus, Stability of a cubically convergent method for generalized equations, *Set-Valued Anal.*, **14** (2006), 41-54.

[12] M.A. Hernández, The Newton method for operators with Hölder continuous first derivative, *J. Optim. Theory Appl.*, **109** (2001), 631-648.

[13] M.A. Hernández, M.J. Rubio, Semilocal convergence of the secant method under mild convergence conditions of differentiability, *Comput. and Math. with Appl.*, **44** (2002), 277-285.

[14] S. Hilout, Steffensen-type methods on Banach spaces for solving generalized equations, *Adv. Nonlinear Var. Ineq.*, **10** (2007), 105-113.

[15] S. Hilout, An uniparametric Newton-Steffensen-type methods for perturbed generalized equations, *Adv. Nonlinear Var. Ineq.*, **10** (2007), 115-124.

[16] S. Hilout, Convergence analysis of a family of Steffensen-type methods for solving generalized equations, *J. Math. Anal. Appl.* (2007), To Appear.

[17] S. Hilout, A. Piétrus, A semilocal convergence of a secant-type method for solving generalized equations, *Positivity*, **10** (2006), 673-700.

[18] B.S. Mordukhovich, Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis, *Trans. Amer. Math. Soc.*, **343** (1994), 609-657.

[19] R.T. Rockafellar, Lipschitzian properties of multifunctions, *Nonlinear Analysis*, **9** (1984), 867-885.

[20] R.T. Rockafellar, R. J-B. Wets, *Variational Analysis*, A Series of Comprehensives Studies in Mathematics, Springer, **317** (1998).

[21] S.M. Robinson, Generalized equations and their solutions, Part I: Basic theory, *Math. Programming Study*, **10** (1979), 128-141.

[22] S.M. Robinson, Generalized equations and their solutions, Part II: Applications to nonlinear programming, *Math. Programming Study*, **19** (1982), 200-221.