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A REMARK ON THE EIGENVALUE ASYMPTOTICS ASSOCIATED WITH SUPERCONDUCTIVITY NEAR CRITICAL TEMPERATURE

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Abstract: We consider the eigenvalue asymptotics for a Shrödinger operator with a magnetic potential associated with the superconductivity near critical temperature. When the magnetic potential is depending on a parameter and the parameter tends to 0, we examine the asymptotics of the first eigenvalue and the corresponding eigenfunction. The result improves Pan [18].

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1. Introduction

In the present paper, we consider the eigenvalue asymptotics for a Schrödinger operator associated with the superconductivity. The superconductivity of the sample in a domain $\Omega \subset \mathbb{R}^3$ under the applied field $\mathbb{H}_{\mathrm{appl}}$ is described by a minimizer $(\psi, \mathbb{A})$ of the Ginzburg-Landau functional

$$\begin{aligned} G[\psi, \mathbb{A}] &= \int_\Omega \{|\xi\nabla\psi - i\gamma\lambda^{-1}\mathbb{A}\psi|^2 + \frac{1}{2}(1-|\psi|^2)^2\}\, dx \\ &\quad + \gamma^2 \int_{\mathbb{R}^3} |\mathrm{curl}\,\mathbb{A} - \mathbb{H}_{\mathrm{appl}}|^2\, dx. \end{aligned}$$

Here ψ is a complex valued function called an order parameter and $\mathbb{A}$ is a real

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valued vector field called a magnetic potential, and the penetration depth λ, the coherence length ξ and γ are positive paremeters depending on materials and temperature. If we put a new parameter $\mu = 1/\xi^2$, μ means, physically,

$$\mu = \frac{1}{\xi^2} = \frac{4m\alpha^2 l^2(T_c - T)}{\hbar T_c},$$

where T is the temperature, T_c is the critical temperature under zero applied field, $\hbar$ is the Plank's constant, l is a typical scale of the sample, m is the electron mass, α is a material constant independent of temperature. The Ginzburg-Landau paremeter κ is defined by $\kappa = \lambda/\xi$. It is well known that if $\kappa > 1/\sqrt{2}$, the sample is of type II and if $0 < \kappa < 1/\sqrt{2}$, the sample is of type I. For these arguments, see Chapman, Howison and Ockendon [4], Du, Gunzburger and Peterson [6], Gunzburger and Ockendon [9], Lu and Pan [15], [16], [17], Helffer and Pan [14].

By a scaling

$$\mathcal{A} = \frac{\gamma\lambda^{-1}}{\xi}\mathbb{A}, \quad \mathcal{H}_{\text{appl}} = \frac{\gamma\lambda^{-1}}{\xi}\mathbb{H}_{\text{appl}},$$

and put $\mathcal{H}_{\text{appl}} = \sigma\boldsymbol{H}$, where $\sigma > 0$ is a parameter which means the intensity of $\mathcal{H}_{\text{appl}}$ and $\mathcal{A} = \sigma\boldsymbol{A}$, the associated energy $G[\psi, \mathbb{A}]/\xi^2$ is written by

$$\mathcal{G}[\psi, \boldsymbol{A}] = \int_\Omega \{|\nabla_{\sigma\boldsymbol{A}}\psi|^2 + \frac{\mu}{2}(1 - |\psi|^2)^2\}\,dx + \frac{\kappa^2\sigma^2}{\mu}\int_{\mathbb{R}^3} |\text{curl}\,\boldsymbol{A} - \boldsymbol{H}|^2\,dx. \tag{1.1}$$

We assume that a given vector field $\boldsymbol{H}(x)$ is smooth and satisfies $\text{div}\,\boldsymbol{H} = 0$ in $\mathbb{R}^3$. Then there exists a unique vector field $\boldsymbol{F}$ such that

$$\text{curl}\,\boldsymbol{F} = \boldsymbol{H}, \quad \text{div}\,\boldsymbol{F} = 0 \quad \text{in } \mathbb{R}^3, \quad \int_\Omega \boldsymbol{F}\,dx = 0. \tag{1.2}$$

In the above and the following, we use the notations for any magnetic potential $\boldsymbol{A}$ and any function ψ, $\nabla_{\boldsymbol{A}}\psi = \nabla\psi - i\boldsymbol{A}\psi$,

$$\nabla^2_{\boldsymbol{A}} = (\nabla - i\boldsymbol{A})^2 = \Delta\psi - i[2\boldsymbol{A}\cdot\nabla\psi + \psi\text{div}\,\boldsymbol{A}] - |\boldsymbol{A}|^2\psi,$$

The minimizers $(\psi, \boldsymbol{A})$ of the functional $\mathcal{G}$ satisfy the following Euler equations, called the Ginzburg-Landau system:

$$\begin{cases} -\nabla^2_{\sigma\boldsymbol{A}}\psi = \mu(1 - |\psi|^2)\psi & \text{in} \quad \Omega, \\ \text{curl}^2(\boldsymbol{A} - \boldsymbol{F}) = \dfrac{\mu}{\sigma\kappa^2}\Im\{\overline{\psi}\nabla_{\sigma\boldsymbol{A}}\psi\}\chi_\Omega & \text{in} \quad \Omega, \\ (\nabla_{\sigma\boldsymbol{A}}\psi)\cdot\boldsymbol{\nu} = 0, \quad [\boldsymbol{\nu}\cdot\boldsymbol{A}] = 0, \quad [\boldsymbol{\nu}\times\text{curl}\,\boldsymbol{A}] = 0 & \text{on} \quad \partial\Omega, \\ \text{curl}\,\boldsymbol{A} \to \boldsymbol{H} & \text{as} \quad |x| \to \infty. \end{cases} \tag{1.3}$$

Here $\boldsymbol{\nu}$ is the unit outward normal vector at the boundary $\partial\Omega$ of Ω, $[\cdot]$ denotes the jump in the enclosed quantity across $\partial\Omega$, and χ_Ω is the characteristic function of Ω.

It is well known that if the applied field is strong, that is to say, if $\sigma > 0$ is large enough, $\mathcal{G}$ has only the trivial minimizer $(0, \boldsymbol{F})$ which corresponds with the normal state. Thus the critical field is defined by

$$H_c(\boldsymbol{H}, \mu, \kappa) = \inf\{\sigma > 0; (0, \boldsymbol{F}) \text{ is a global minimizer of } \mathcal{G}\}.$$

In order to find the asymptotics of H_c as $\mu \to 0$, we must consider the asymptotics of the first eigenvalue of the Schrödinger operator $-\nabla^2_{\varepsilon \boldsymbol{A}}$ with magnetic Neumann condition as $\varepsilon \to 0$. In this paper, we devote only the analysis for the asymptotics of the first eigenvalue and the corresponding eigenfunction of such a linear problem. For the asymptotics of H_c, we will treat in the future work. For the asymptotics as $\varepsilon \to \infty$, there are many articles, for example, see Aramaki [1], [2], Fournais and Helffer [7], Helffer [10], Helffer and Mohamed [11], Helffer and Morame [12], [13].

2. Asymptotics of the Lowest Eigenvalue and the Corresponding Eigenfunction

In this section, we shall consider the asymptotic behavior of the first eigenvalue and the corresponding eigenfunction for a Schrödinger operator.

More precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded, smooth and simply-connected domain and $\boldsymbol{H} = \boldsymbol{H}(x)$ a given smooth vector field in $\mathbb{R}^3$ satisfying

$$\boldsymbol{H}(x) \neq 0 \quad \text{in } \Omega \quad \text{and } \operatorname{div} \boldsymbol{H} = 0 \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

Then there exists a unique vector field $\boldsymbol{F}(x)$ in $\mathbb{R}^3$ such that

$$\operatorname{curl} \boldsymbol{F} = \boldsymbol{H}, \quad \operatorname{div} \boldsymbol{F} = 0 \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad \int_\Omega \boldsymbol{F}\, dx = 0. \tag{2.2}$$

Let $\mu(\varepsilon \boldsymbol{F})$ be the lowest eigenvalue of the Schrödinger operator with magnetic potential under the Neumann boundary condition:

$$\begin{cases} -\nabla^2_{\varepsilon \boldsymbol{F}} \phi = \mu \phi & \text{in } \Omega, \\ (\nabla_{\varepsilon \boldsymbol{F}} \phi) \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

That is to say,

$$\mu(\varepsilon \boldsymbol{F}) = \inf_{\phi \in W^{1,2}(\Omega; \mathbb{C})} \frac{\int_\Omega |\nabla_{\varepsilon \boldsymbol{F}} \phi|^2\, dx}{\|\phi\|^2_{L^2(\Omega)}}. \tag{2.4}$$

Taking (2.2) into consideration, we rewrite (2.3) into the form

$$\begin{cases} -\Delta\phi + 2i\varepsilon \boldsymbol{F}\cdot\nabla\phi + \varepsilon^2|\boldsymbol{F}|^2\phi = \mu\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\boldsymbol{\nu}} - i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

In order to state the main proposition, we give some preliminaries. We define

$$\omega(\boldsymbol{H}) = \inf_{\phi\in W^{1,2}(\Omega)} ⨍_\Omega |\nabla\phi - \boldsymbol{F}|^2\, dx\,, \tag{2.6}$$

where

$$⨍_\Omega f\, dx = \frac{1}{|\Omega|}\int_\Omega f\, dx$$

for any function f. It is clear that $\omega(\boldsymbol{H})$ is achieved by a unique real solution $w_{\boldsymbol{H}} \in W^{1,2}(\Omega)$ satisfying

$$\begin{cases} \Delta w_{\boldsymbol{H}} = 0 & \text{in } \Omega, \\ \frac{\partial w_{\boldsymbol{H}}}{\partial\boldsymbol{\nu}} = \boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

and a normalization $\int_\Omega w_{\boldsymbol{H}}\, dx = 0$. From (2.7), we see that

$$\int_\Omega |\nabla w_{\boldsymbol{H}}|^2\, dx = \int_{\partial\Omega} w_{\boldsymbol{H}}\frac{\partial w_{\boldsymbol{H}}}{\partial\boldsymbol{\nu}} dS = \int_{\partial\Omega} w_{\boldsymbol{H}}\boldsymbol{F}\cdot\boldsymbol{\nu} dS = \int_\Omega \nabla w_{\boldsymbol{H}}\cdot\boldsymbol{F}\, dx.$$

Thus we can write

$$\begin{aligned} \omega(\boldsymbol{H}) &= ⨍_\Omega \{|\nabla w_{\boldsymbol{H}}|^2 - 2\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}} + |\boldsymbol{F}|^2\}\, dx \\ &= ⨍_\Omega \{|\boldsymbol{F}|^2 - |\nabla w_{\boldsymbol{H}}|^2\}\, dx. \end{aligned} \tag{2.8}$$

We see that $\omega(\boldsymbol{H}) > 0$ on Ω. In fact, if $\omega(\boldsymbol{H}) = 0$, we see that $\nabla w_{\boldsymbol{H}} = \boldsymbol{F}$ in Ω. Therefore, we have $\boldsymbol{H} = \operatorname{curl}\boldsymbol{F} = \operatorname{curl}\nabla w_{\boldsymbol{H}} = 0$. This contradict to our hypothesis (2.1) on $\boldsymbol{H}$.

We are in a position to state the main proposition.

Proposition 2.1. *Assume that $\boldsymbol{H}$ is a given vector field satisfying* (2.1) *and $\boldsymbol{F}$ satisfies the condition* (2.2). *Let $\mu(\varepsilon\boldsymbol{F})$ be the lowest eigenvalue of the problem* (2.3) *and ϕ_ε the corresponding eigenfuntion. Then we have*

$$\begin{aligned} \mu(\varepsilon\boldsymbol{F}) &= \varepsilon^2\omega(\boldsymbol{H}) + O(\varepsilon^3), \\ \phi_\varepsilon &= \alpha_\varepsilon + i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}} + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + o(\varepsilon^3) \end{aligned}$$

as $\varepsilon \to 0$, where $\alpha_\varepsilon - 1 = O(\varepsilon)$ and $\beta_\varepsilon - 1 = O(\varepsilon)$ and ϕ_2, ϕ_3 are smooth functions.

Remark 2.2. Pan [18] considered the case where $\boldsymbol{H} = \boldsymbol{n}$, $\boldsymbol{n}$ is a unit constant vector. Our result is an extension to the case of non-constant applied field. Moreover, our result is better than one of [18] with respect to the asymptotics of the eigenfunction.

3. Proof of Proposition 2.1

In this section, we shall devote to the proof of Proposition 2.1.

Step 1. Upper Bound of the Eigenvalue. Choose a test function $\phi = 1 + i\varepsilon w_{\boldsymbol{H}}$. Then we have

$$\mu(\varepsilon \boldsymbol{F}) \leq \frac{\int_\Omega |\nabla_{\varepsilon \boldsymbol{F}} \phi|^2 \, dx}{\int_\Omega |\phi|^2 \, dx}$$
$$= \varepsilon^2 \frac{\int_\Omega \{|\nabla w_{\boldsymbol{H}} - \boldsymbol{F}|^2 + \varepsilon^2 |\boldsymbol{F} w_{\boldsymbol{H}}|^2\} \, dx}{\int_\Omega (1 + \varepsilon^2 |w_{\boldsymbol{H}}|^2) \, dx} = \varepsilon^2 \omega(\boldsymbol{H}) + O(\varepsilon^4).$$

Thus we get the upper bound for $\mu(\varepsilon \boldsymbol{H})$.

Step 2. Proof of Proposition 2.1. Let $\mu(\varepsilon) = \mu(\varepsilon \boldsymbol{F}) = \varepsilon^2 \lambda_\varepsilon$. Since $\{\lambda_\varepsilon\}$ is bounded from Step 1, passing to a subsequence, we may assume that $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$. We shall prove that $\lambda_0 = \omega(\boldsymbol{H})$. Let ϕ_ε be the minimizer of (2.4) satisfying $\|\phi_\varepsilon\|_{L^\infty(\Omega)} = 1$. Then ϕ_ε satisfies the equations

$$\begin{cases} -\nabla^2_{\varepsilon \boldsymbol{F}} \phi_\varepsilon = \mu(\varepsilon) \phi_\varepsilon & \text{in } \Omega, \\ (\nabla_\varepsilon \phi_\varepsilon) \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

From the elliptic estimate (see Gilbarg and Trudinger [8] or Du [5]), for any $\alpha \in (0,1)$, there exists a constant $C(\alpha) > 0$ such that

$$\|\phi_\varepsilon\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\alpha) \quad \text{for small } \varepsilon > 0.$$

Passing to a subsequence, there exists $\phi_0 \in C^{2,\alpha}(\overline{\Omega})$ such that $\phi_\varepsilon \to \phi_0$ in $C^{2,\alpha}(\overline{\Omega})$. Then ϕ_0 satisfies $\|\phi_0\|_{L^\infty(\Omega)} = 1$ and the equations

$$\begin{cases} -\Delta \phi_0 = 0 & \text{in } \Omega, \\ \dfrac{\partial \phi_0}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore since $\phi_0 = \text{const.}$, we may put $\phi_0 = 1$. Thus we have $\phi_\varepsilon \to \phi_0$ in $C^{2,\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$. We put $\psi_\varepsilon = \phi_\varepsilon - \alpha_\varepsilon$, where

$$\alpha_\varepsilon = \fint_\Omega \phi_\varepsilon \, dx.$$

Then we have $\alpha_\varepsilon \to 1$ and $\int_\Omega \psi_\varepsilon \, dx = 0$. It follows from (2.5) that ψ_ε satisfies

$$\begin{cases} -\Delta\psi_\varepsilon - 2i\varepsilon \boldsymbol{F}\cdot\nabla\psi_\varepsilon - \varepsilon^2|\boldsymbol{F}|^2 + \varepsilon^2\lambda_\varepsilon\psi_\varepsilon = \varepsilon^2\alpha_\varepsilon(|\boldsymbol{F}|^2 - \lambda_\varepsilon) & \text{in } \Omega, \\ \frac{\partial\psi_\varepsilon}{\partial\boldsymbol{\nu}} - i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\psi_\varepsilon = \alpha_\varepsilon\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega. \end{cases}$$

Since $\|\nabla_{\varepsilon\boldsymbol{F}}\phi_\varepsilon\|_{L^2(\Omega)} = \varepsilon^2\lambda_\varepsilon\|\phi_\varepsilon\|_{L^2(\Omega)}$, it follows from the Poincaré-Wirtinger inequality (cf. Brezis [3]) that

$$\begin{aligned} \|\psi_\varepsilon\|_{L^2(\Omega)} &= \|\phi_\varepsilon - \alpha_\varepsilon\|_{L^2(\Omega)} \\ &\le C\|\nabla\phi_\varepsilon\|_{L^2(\Omega)} \le C\|\nabla_{\varepsilon\boldsymbol{F}}\phi_\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\boldsymbol{F}\phi_\varepsilon\|_{L^2(\Omega)} \le C_1\varepsilon. \end{aligned}$$

Therefore, by the theory of elliptic operators, for any integer $k \ge 2$, there exist constants C_k and $C(k)$ such that

$$\begin{aligned} \|\psi_\varepsilon\|_{W^{k,2}(\Omega)} &\le C_k\{\varepsilon^2\||\boldsymbol{F}|^2\|_{W^{k-2,2}(\Omega)} + \varepsilon\|\boldsymbol{F}\cdot\boldsymbol{\nu}\|_{W^{k-3/2,2}(\partial\Omega)} + \|\psi_\varepsilon\|_{L^2(\Omega)}\} \\ &\le C(k)\varepsilon. \end{aligned}$$

Thus by the Sobolev imbedding theorem, we have $\|\psi_\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon$ with a constant $C > 0$ independent of ε. Therefore if we put

$$\phi_1^\varepsilon = \frac{\psi_\varepsilon}{\varepsilon} = \frac{\phi_\varepsilon - \alpha_\varepsilon}{\varepsilon}, \tag{3.2}$$

clearly we have $\|\phi_1^\varepsilon\|_{L^\infty(\Omega)} \le C$ and $\int_\Omega \phi_1^\varepsilon \, dx = 0$. Moreover, ϕ_1^ε satisfies the following equations:

$$\begin{cases} \Delta\phi_1^\varepsilon - 2i\varepsilon\boldsymbol{F}\cdot\nabla\phi_1^\varepsilon - \varepsilon^2(|\boldsymbol{F}|^2 - \lambda_\varepsilon)\phi_1^\varepsilon = \varepsilon\alpha_\varepsilon(|\boldsymbol{F}|^2 - \lambda_\varepsilon) & \text{in } \Omega, \\ \frac{\partial\phi_1^\varepsilon}{\partial\boldsymbol{\nu}} - i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\phi_1^\varepsilon = i\alpha_\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

By the Hölder estimate (cf. [8]), for any $\alpha \in (0,1)$ there exist constants $C > 0$ and $C(\alpha) > 0$ independent of ε such that

$$\|\phi_1^\varepsilon\|_{C^{2,\alpha}(\overline{\Omega})} \le C(\|\boldsymbol{F}\cdot\boldsymbol{\nu}\|_{C^{1,\alpha}(\partial\Omega)} + \varepsilon\||\boldsymbol{F}|^2 - \lambda\|_{C^{0,\alpha}(\Omega)}) \le C(\alpha).$$

Therefore passing to a subsequence, we may assume that $\phi_1^\varepsilon \to \phi_1$ in $C^{2,\alpha}(\overline{\Omega})$. Thus ϕ_1 saitsfies

$$\begin{cases} \Delta\phi_1 = 0 & \text{in } \Omega, \\ \frac{\partial\phi_1}{\partial\boldsymbol{\nu}} = i\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{in } \partial\Omega \end{cases}$$

and $\int_\Omega \phi_1 \, dx = 0$. From the uniqueness of (2.7) under the normalization we see that $\phi_1 = iw_{\boldsymbol{H}}$.

Let su write

$$\phi_1^\varepsilon = i\beta_\varepsilon w_{\boldsymbol{H}} + \varepsilon\phi_2^\varepsilon\,, \tag{3.4}$$

where

$$\beta_\varepsilon = -i\frac{\int_\Omega w_{\boldsymbol{H}}\phi_1^\varepsilon\,dx}{\int_\Omega |w_{\boldsymbol{H}}|^2\,dx}.$$

Then we see that

$$\int_\Omega w_{\boldsymbol{H}}\phi_2^\varepsilon\,dx = 0, \quad \int_\Omega \phi_2^\varepsilon\,dx = 0.$$

By (2.5), ϕ_2^ε satisfies the equations

$$\begin{cases} \Delta\phi_2^\varepsilon - 2i\varepsilon\boldsymbol{F}\cdot\nabla\phi_2^\varepsilon - \varepsilon^2(|\boldsymbol{F}|^2-\lambda_\varepsilon)\phi_2^\varepsilon = f_\varepsilon & \text{in } \Omega, \\ \frac{\partial\phi_2^\varepsilon}{\partial\boldsymbol{\nu}} - i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\phi_2^\varepsilon = (i\delta_\varepsilon - \beta_\varepsilon w_{\boldsymbol{H}})\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where

$$f_\varepsilon = -2\beta_\varepsilon\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}} + i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}}(|\boldsymbol{F}|^2-\lambda_\varepsilon) + \alpha_\varepsilon(|\boldsymbol{F}|^2-\lambda_\varepsilon)$$

and $\delta_\varepsilon = (\alpha_\varepsilon - \beta_\varepsilon)/\varepsilon$. Since $\phi_1^\varepsilon \to iw_{\boldsymbol{H}}$ in $C^{2,\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$, we see that $\beta_\varepsilon \to 1$, and moreover, since $\lambda_\varepsilon \to \lambda_0$, $f_\varepsilon \to -2\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}} + |\boldsymbol{F}|^2 - \lambda_0$ uniformly as $\varepsilon \to 0$. Now we claim that there exists $C > 0$ independent of ε such that

$$|\delta_\varepsilon| \le C \tag{3.6}$$

for small ε. In fact, suppose that the claim (3.6) is not true. Then passing to a subsequence, we may assume that $\delta_\varepsilon \to \infty$ as $\varepsilon \to 0$. We define functions $\xi_\varepsilon = \phi_2^\varepsilon/\delta_\varepsilon$. Then we see that

$$\int_\Omega \xi_\varepsilon\,dx = 0, \quad \int_\Omega w_{\boldsymbol{H}}\xi_\varepsilon\,dx = 0. \tag{3.7}$$

From (3.5), we get the equations for ξ_ε :

$$\begin{cases} \Delta\xi_\varepsilon - 2i\varepsilon\boldsymbol{F}\cdot\nabla\xi_\varepsilon - \varepsilon^2(|\boldsymbol{F}|^2-\lambda_\varepsilon)\xi_\varepsilon = \frac{f_\varepsilon}{\delta_\varepsilon} & \text{in } \Omega, \\ \frac{\partial\xi_\varepsilon}{\partial\boldsymbol{\nu}} - i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\xi_\varepsilon = (i - \frac{\beta_\varepsilon}{\delta_\varepsilon}w_{\boldsymbol{H}})\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

We consider two cases.

Case 1. $\|\xi_\varepsilon\|_{L^2(\Omega)} \le C$ for small ε. Applying the elliptic estimate for (3.8), for any integer $k > 2$ there exists a constant $C(k) > 0$ such that $\|\xi_\varepsilon\|_{W^{k,2}(\Omega)} \le C(k)$. Therefore from the Sobolev Imbedding Theorem, for any $\alpha \in (0,1)$, there exists a constant $C(\alpha) > 0$ such that $\|\xi_\varepsilon\|_{C^{2,\alpha}(\overline{\Omega})} \le C(\alpha)$. Thus passing to a subsequence, we have $\xi_\varepsilon \to \xi_0$ in $C^{2,\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$. Then letting $\varepsilon \to 0$ in the equations (3.8), ξ_0 satisfies

$$\begin{cases} \Delta\xi_0 = 0 & \text{in } \Omega, \\ \frac{\partial\xi_0}{\partial\boldsymbol{\nu}} = i\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega \end{cases}$$

and $\int_\Omega \xi_0\,dx = 0$. Therefore it follows from (2.7) that $\xi_0 = iw_{\boldsymbol{H}}$. This contradicts to the second equation in (3.7). Thus Case 1 cannot happen.

Case 2. $\|\xi_\varepsilon\|_{L^2(\Omega)}$ is not bounded. In this case, passing to a subsequence, we may assume that $\|\xi_\varepsilon\|_{L^2(\Omega)} = C_\varepsilon \to \infty$ as $\varepsilon \to 0$. Then $\widetilde{\xi}_\varepsilon = \xi_\varepsilon / C_\varepsilon$ saitsfies

$$\begin{cases} \Delta\widetilde{\xi}_\varepsilon - 2i\varepsilon \boldsymbol{F}\cdot\nabla\widetilde{\xi}_\varepsilon - \varepsilon^2(|\boldsymbol{F}|^2 - \lambda_\varepsilon)\widetilde{\xi}_\varepsilon = \frac{f_\varepsilon}{\delta_\varepsilon C_\varepsilon} & \text{in } \Omega, \\ \frac{\partial\widetilde{\xi}_\varepsilon}{\partial\boldsymbol{\nu}} - i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\widetilde{\xi}_\varepsilon = \left(\frac{i}{C_\varepsilon} - \frac{\beta_\varepsilon}{C_\varepsilon\delta_\varepsilon} w_{\boldsymbol{H}}\right)\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega. \end{cases}$$

By the elliptic estimate and the Sobolev Imbedding Theorem, passing to a subsequence, we have $\widetilde{\xi}_\varepsilon \to \widetilde{\xi}_0$ in $C^{2,\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$ and

$$\|\widetilde{\xi}_0\|_{L^2(\Omega)} = 1, \quad \int_\Omega \widetilde{\xi}_0\, dx = 0. \tag{3.9}$$

Thus $\widetilde{\xi}_0$ satisfies

$$\begin{cases} \Delta\widetilde{\xi}_0 = 0 & \text{in } \Omega, \\ \frac{\partial\widetilde{\xi}_0}{\partial\boldsymbol{\nu}} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

This contradicts to (3.9). Thus Case 2 also cannot happen. Therefore, we have proved the claim (3.6).

Since it follows from (3.6) that δ_ε is bounded, passing to a subsequence, we may assume that $\lim_{\varepsilon\to 0}\delta_\varepsilon = \delta_0$.

We return to (3.5). We claim that there exists a constant $C > 0$ independent of ε such that

$$\|\phi_2^\varepsilon\|_{L^2(\Omega)} \le C. \tag{3.10}$$

In fact, if the claim were not hold, passing to a subsequence, we may assume that $\|\phi_2^\varepsilon\|_{L^2(\Omega)} = d_\varepsilon \to \infty$ as $\varepsilon \to 0$. We put $\widetilde{\phi}_2^\varepsilon = \phi_2^\varepsilon / d_\varepsilon$. Then we note that $\|\widetilde{\phi}_2^\varepsilon\|_{L^2(\Omega)} = 1$. By the same arguments as in Case 2, we see that $\widetilde{\phi}_2^\varepsilon \to \widetilde{\phi}_0$ in $C^{2,\alpha}(\overline{\Omega})$. Therefore, $\widetilde{\phi}_0$ satisfies

$$\begin{cases} \Delta\widetilde{\phi}_0 = 0 & \text{in } \Omega, \\ \frac{\partial\widetilde{\phi}_0}{\partial\boldsymbol{\nu}} = 0 & \text{on } \partial\Omega. \end{cases}$$

and $\int_\Omega \widetilde{\phi}_0\, dx = 0$. This contradicts to the fact that $\|\widetilde{\phi}_2^\varepsilon\|_{L^2(\Omega)} = 1$.

Thus as in Case 2, $\{\phi_2^\varepsilon\}$ is bounded in $C^{2,\alpha}(\overline{\Omega})$ and passing to a subsequence, we may assume that $\phi_2^\varepsilon \to \phi_2$ in $C^{2,\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$. Thus we see that ϕ_2 satisfies the equations

$$\begin{cases} \Delta\phi_2 = -2\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}} + |\boldsymbol{F}|^2 - \lambda_0 & \text{in } \Omega, \\ \frac{\partial\phi_2}{\partial\boldsymbol{\nu}} = (i\delta_0 - w_{\boldsymbol{H}})\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Integrating the first equation in (3.11), using the second equation, we have

$$\begin{aligned}\lambda_0|\Omega| &= \int_\Omega(-2\boldsymbol{F}\cdot w_{\boldsymbol{H}}+|\boldsymbol{F}|^2-\Delta\phi_2)\,dx\\ &= \int_\Omega|\boldsymbol{F}|^2\,dx-\int_{\partial\Omega}\{2\boldsymbol{F}\cdot\boldsymbol{\nu}w_{\boldsymbol{H}}+(i\delta_0-w_{\boldsymbol{H}})\boldsymbol{F}\cdot\boldsymbol{\nu}\}\,dS\\ &= \int_\Omega|\boldsymbol{F}|^2\,dx-\int_{\partial\Omega}w_{\boldsymbol{H}}\frac{\partial w_{\boldsymbol{H}}}{\partial\boldsymbol{\nu}}\,dS\\ &= \int_\Omega(|\boldsymbol{F}|^2-|\nabla w_{\boldsymbol{H}}|^2)\,dx\\ &= \omega(\boldsymbol{H})|\Omega|.\end{aligned}$$

Thus we have $\lambda_0=\omega(\boldsymbol{H})$. And so, $\lambda_\varepsilon\to\lambda_0$ as $\varepsilon\to 0$. At this point, we get, as $\varepsilon\to 0$,

$$\begin{cases}\mu(\varepsilon)=\varepsilon^2\lambda_\varepsilon=\varepsilon^2(\omega(\boldsymbol{H})+o(1)),\\ \phi_\varepsilon=\alpha_\varepsilon+i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}}+\varepsilon^2\phi_2+o(\varepsilon^2).\end{cases}\tag{3.12}$$

Now we return to (3.1). If we put $\varphi_\varepsilon=(\phi_\varepsilon-\phi_0)/\varepsilon$ with $\phi_0=1$, φ_ε satisfies

$$\begin{cases}\Delta\varphi_\varepsilon=2i\boldsymbol{F}\cdot\varepsilon\nabla\varphi+(|\boldsymbol{F}|^2-\lambda_\varepsilon)\varepsilon^2\varphi+\varepsilon|\boldsymbol{F}|^2+\varepsilon\lambda_\varepsilon & \text{in } \Omega,\\ \frac{\partial\varphi}{\partial\boldsymbol{\nu}}=i\boldsymbol{F}\cdot\boldsymbol{\nu}\varepsilon\varphi+i\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega.\end{cases}\tag{3.13}$$

Since the right hand side of the first equation in (3.13) is uniformly bounded in $L^2(\Omega)$ and the second also uniformly bounded in $H^{1/2}(\partial\Omega)$, it follows from the boot strap method that for any integer $k>0$, $\{\varphi_\varepsilon\}$ is uniformly bounded in $W^{k,2}(\Omega)$, and so by the Sobolev Imbedding Theorem, we see that $\{\varphi_\varepsilon\}$ is uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$ for any $\alpha\in(0,1)$. In particular, $\{\varphi_\varepsilon\}$ is bounded, that is to say, $(\phi_\varepsilon-1)/\varepsilon$ is uniformly bounded with respect to ε. Since $(\psi_\varepsilon-\alpha_\varepsilon)/\varepsilon$ is bounded, we see that $(\alpha_\varepsilon-1)/\varepsilon$ is bounded. Thus we have $\alpha_\varepsilon-1=O(\varepsilon)$ as $\varepsilon\to 0$. Since $\delta_\varepsilon=(\alpha_\varepsilon-\beta_\varepsilon)/\varepsilon$ is bouded, we also see that $\beta_\varepsilon-1=O(\varepsilon)$ as $\varepsilon\to 0$. If we take (3.11) of (3.5), we get the equations

$$\begin{cases}\Delta(\phi_2^\varepsilon-\phi_2)-2i\varepsilon\boldsymbol{F}\cdot\nabla\phi_2^\varepsilon-\varepsilon^2(|\boldsymbol{F}|^2-\lambda_\varepsilon)\phi_2^\varepsilon\\ \quad=-2(\beta_\varepsilon-1)\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}}+i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}}(|\boldsymbol{F}|^2-\lambda_\varepsilon)\\ \quad\;+(\alpha_\varepsilon-1)(|\boldsymbol{F}|^2-\lambda_\varepsilon)-(\lambda_\varepsilon-\lambda_0) & \text{in } \Omega,\\ \frac{\partial(\phi_2^\varepsilon-\phi_0)}{\partial\boldsymbol{\nu}}-i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\phi_2^\varepsilon=(i(\delta_\varepsilon-\delta_0)-(\beta_\varepsilon-1)w_{\boldsymbol{H}})\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega.\end{cases}\tag{3.14}$$

Integrating (3.14) over Ω, we have

$$\begin{aligned}&\{(\lambda_\varepsilon-\lambda_0)+(\alpha_\varepsilon-1)\}|\Omega|\\ &\quad=\int_\Omega\{2i\varepsilon\boldsymbol{F}\nabla\phi_2^\varepsilon+\varepsilon^2(|\boldsymbol{F}|^2-\lambda_\varepsilon)\phi_2^\varepsilon-\Delta(\phi_2^\varepsilon-\phi_2)-2(\beta_\varepsilon-1)\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}}\\ &\quad\;+i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}}(|\boldsymbol{F}|^2-\lambda_\varepsilon)+(\alpha_\varepsilon-1)|\boldsymbol{F}|^2\}\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \{2i\varepsilon \boldsymbol{F}\nabla\phi_2^\varepsilon + \varepsilon^2(|\boldsymbol{F}|^2-\lambda_\varepsilon)\phi_2^\varepsilon - 2(\beta_\varepsilon-1)\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}} \\
&\quad + i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}}(|\boldsymbol{F}|^2-\lambda_\varepsilon) + (\alpha-1)|\boldsymbol{F}|^2\}\,dx \\
&\quad - \int_{\partial\Omega} \big[i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\phi_2^\varepsilon + \{(i(\delta_\varepsilon-\delta_0)-(\beta_\varepsilon-1)\}\boldsymbol{F}\cdot\boldsymbol{\nu}\big]\,dS \\
&= \int_\Omega \{2i\varepsilon \boldsymbol{F}\nabla\phi_2^\varepsilon + \varepsilon^2(|\boldsymbol{F}|^2-\lambda_\varepsilon)\phi_2^\varepsilon - 2(\beta_\varepsilon-1)\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}} \\
&\quad + i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}}(|\boldsymbol{F}|^2-\lambda_\varepsilon) + (\alpha-1)|\boldsymbol{F}|^2\}\,dx \\
&\quad - \int_{\partial\Omega} i\varepsilon\phi_2^\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\,dS.
\end{aligned}
$$

Here we used the relation

$$\int_{\partial\Omega}\boldsymbol{F}\cdot\boldsymbol{\nu}\,dS = \int_\Omega \operatorname{div}\boldsymbol{F}\,dx = 0.$$

Since the last term is of $O(\varepsilon)$, we see that $\lambda_\varepsilon - \lambda_0 = O(\varepsilon)$ as $\varepsilon\to 0$.

We return to the equation (3.14). If we put $\phi_3^\varepsilon = (\phi_2^\varepsilon-\phi_2)/\varepsilon$, ϕ_3^ε satisfies the equations

$$
\begin{cases}
\Delta\phi_3^\varepsilon - 2i\varepsilon\boldsymbol{F}\cdot\nabla\phi_3^\varepsilon - \varepsilon^2(|\boldsymbol{F}|^2-\lambda_\varepsilon)\phi_3^\varepsilon \\
\quad = 2i\boldsymbol{F}\cdot\nabla\phi_2 + \varepsilon(|\boldsymbol{F}|^2-\lambda_\varepsilon)\phi_2 - 2\frac{\beta_\varepsilon-1}{\varepsilon}\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}} \\
\quad\quad + i\beta_\varepsilon w_{\boldsymbol{H}}(|\boldsymbol{F}|^2-\lambda_\varepsilon) + \frac{\alpha_\varepsilon-1}{\varepsilon}(|\boldsymbol{F}|^2-\lambda_\varepsilon) - \frac{\lambda_\varepsilon-\lambda_0}{\varepsilon} & \text{in } \Omega, \\
\frac{\partial\phi_3^\varepsilon}{\partial\boldsymbol{\nu}} - i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\phi_3^\varepsilon = (i\frac{\delta_\varepsilon-\delta_0}{\varepsilon} - \frac{\beta_\varepsilon-1}{\varepsilon}w_{\boldsymbol{H}})\boldsymbol{F}\cdot\boldsymbol{\nu} + i\boldsymbol{F}\cdot\boldsymbol{\nu}\phi_2 & \text{on } \partial\Omega.
\end{cases}
\tag{3.15}
$$

We claim that $\{\gamma_\varepsilon = (\delta_\varepsilon-\delta_0)/\varepsilon\}$ is bounded. In fact, if not, we may assume that $\gamma_\varepsilon\to\infty$ as $\varepsilon\to 0$. We put $\widetilde{\phi}_3^\varepsilon = \phi_3^\varepsilon/\gamma_\varepsilon$, $\widetilde{\phi}_3^\varepsilon$ satisfies the equations :

$$
\begin{cases}
\Delta\widetilde{\phi}_3^\varepsilon - 2i\varepsilon\boldsymbol{F}\cdot\nabla\widetilde{\phi}_3^\varepsilon - \varepsilon^2(|\boldsymbol{F}|^2-\lambda_\varepsilon)\widetilde{\phi}_3^\varepsilon \\
\quad = \frac{2i}{\gamma_\varepsilon}\boldsymbol{F}\cdot\nabla\phi_2 + \frac{\varepsilon}{\gamma_\varepsilon}(|\boldsymbol{F}|^2-\lambda_\varepsilon)\phi_2 - 2\frac{1}{\gamma_\varepsilon}\frac{\beta_\varepsilon-1}{\varepsilon}\boldsymbol{F}\cdot\nabla w_{\boldsymbol{H}} \\
\quad\quad + i\frac{\beta_\varepsilon}{\gamma_\varepsilon}w_{\boldsymbol{H}}(|\boldsymbol{F}|^2-\lambda_\varepsilon) + \frac{1}{\gamma_\varepsilon}\frac{\alpha_\varepsilon-1}{\varepsilon}(|\boldsymbol{F}|^2-\lambda_\varepsilon) - \frac{1}{\gamma_\varepsilon}\frac{\lambda_\varepsilon-\lambda_0}{\varepsilon} & \text{in } \Omega, \\
\frac{\partial\widetilde{\phi}_3^\varepsilon}{\partial\boldsymbol{\nu}} - i\varepsilon\boldsymbol{F}\cdot\boldsymbol{\nu}\widetilde{\phi}_3^\varepsilon = (i - \frac{1}{\gamma_\varepsilon}\frac{\beta_\varepsilon-1}{\varepsilon}w_{\boldsymbol{H}})\boldsymbol{F}\cdot\boldsymbol{\nu} + i\frac{1}{\gamma_\varepsilon}\boldsymbol{F}\cdot\boldsymbol{\nu}\phi_2 & \text{on } \partial\Omega.
\end{cases}
\tag{3.16}
$$

As above arguments, we see that $\widetilde{\phi}_3^\varepsilon$ is uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$ for any $\alpha\in(0,1)$. Passing to a subsequence, we may assume that $\widetilde{\phi}_3^\varepsilon\to\phi_3$ in $C^{2,\alpha}(\overline{\Omega})$ as $\varepsilon\to 0$. Therefore, ϕ_3 satisfies

$$
\begin{cases}
\Delta\phi_3 = 0 & \text{in } \Omega, \\
\frac{\partial\phi_3}{\partial\boldsymbol{\nu}} = i\boldsymbol{F}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega.
\end{cases}
$$

Moreover, ϕ_3 satisfies $\int_\Omega\phi_3\,dx = 0$ and $\int_\Omega w_{\boldsymbol{H}}\phi_3\,dx = 0$. This leads to a contradiction as above.

Since γ_ε is bounded, we may assume that $\gamma_\varepsilon\to\gamma_0$ as $\varepsilon\to 0$. By (3.16), we see that $\{\phi_3^\varepsilon\}$ is uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$. Passing to a subsequence, we may assume that $\phi_3^\varepsilon \to \phi_3$ in $C^{2,\alpha}(\overline{\Omega})$. Thus we can write

$$\begin{aligned}\phi_\varepsilon &= \alpha_\varepsilon + i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}} + \varepsilon^2\phi_2^\varepsilon \\ &= \alpha_\varepsilon + i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}} + \varepsilon^2\phi_2 + \varepsilon^3\phi_3^\varepsilon \\ &= \alpha_\varepsilon + i\varepsilon\beta_\varepsilon w_{\boldsymbol{H}} + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + o(\varepsilon^3)\end{aligned}$$

and $\mu(\varepsilon) = \varepsilon^2\omega(\boldsymbol{H}) + O(\varepsilon^3)$ as $\varepsilon \to 0$. This completes the proof of Proposition 2.1. □

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