International Journal of Pure and Applied Mathematics

Volume 40 No. 1 2007, 11-19

AN APPROXIMATION OF THE GENERALIZED μ-VARIATION IN A PERFECT SET

Benedetto Ballarò

Department of Mathematics and Applications
University of Palermo
Via Archirafi, 34, Palermo, 90123, ITALY
e-mail: ballaro@math.unipa.it

Abstract: We consider a function defined on a closed interval $[a, b]$ with values in a Banach space. We introduce the notion of generalized μ-variation in a perfect set and we establish necessary and sufficient conditions for the existence of an approximation of such variation.

AMS Subject Classification: 49Q20, 26A45
Key Words: Weierstrass-Cesari integration, functions of bounded variation, generalized μ-variation in a perfect set

1. Introduction

Let $f : [a, b] \to X$ be a function from a closed interval $[a, b]$ into a Banach space X. Let $\mu = \int_a^b \varphi dm$ be a weight, with φ a continuous function and m a Lebesgue measure on $[a, b]$ (see [8]); let A be a null measure set in $[a, b]$ and $E = [a, b] - A$ the complement of A in $[a, b]$, it is defined μ-variation (see [2]) of f restricted to the set E the following number (finite or infinite positive):

$$V_\mu(f_E) = \limsup_{\lambda(D)\to 0} \sum_{i=1}^{n} \|\Delta f_E(I_i)\| \frac{\mu(I_i)}{m(I_i)},$$

where $D = \{I_i\}$ is a subdivision of $[a, b]$ in a finite number of not overlapping closed intervals $I_i = [a_i, a_{i+1}]$ for $i = 1, 2, ..., n$, with $a_i, a_{i+1} \in E$, $\lambda(D) = \max_{i=1,...,n}\{a_1 - a, a_{i+1} - a_i, b - a_{n+1}\}$ and $\Delta f_E(I_i) = f_E(a_{i+1}) - f_E(a_i)$ for $i = 1, 2, ..., n$, with f_E a restriction of f to the set E.

Received: July 11, 2007

Moreover, if it exists a set $A \subset [a,b]$, with $m(A)=0$ and such that $V_\mu(f_E) < +\infty$, the function f is said to be of generalized bounded μ-variation in $[a,b]$. In this case it has been called generalized μ-variation of f in $[a,b]$ and it has been identified with $\varphi_\mu(f)$ the value $V_\mu(f_F)$, where F is an specified subset of E and its measure is equal to $b-a$. In [2] has been given a proof that $\varphi_\mu(f)$ is independent from the choice of the set E and moreover it has been showed that for such value can be considered the following equation analogous to that famous one for the classic and generalized variation, see [9]:

$$\varphi_\mu(f) = \lim_{h\to 0^+} \int_a^{b-h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx.$$

In this work we consider a function defined on $[a,b]$ with values into a Banach space X and of generalized bounded μ-variation. Now we can introduce the notion of generalized μ-variation in a perfect set $P \subset [a,b]$, $\varphi_\mu(f,P)$, and we give a collection of necessary and sufficient conditions so that, with reference to the perfect P, the following relation is true:

$$\varphi_\mu(f,P) = \lim_{h\to 0^+} \int_{P_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx.$$

2. Preliminary Remarks

Let $I_0 = [a,b]$ be a closed and bounded interval. We denote with $\{I\}$ the family of all closed intervals contained in I_0, $\boldsymbol{D}_{I_0} = \{D_{I_0}\}$ the family of all subdivisions $D_{I_0} = \{I\}$ of I_0 (mesh) in a finite number of closed intervals with a constraint: no overlap is allowed. $\boldsymbol{E}$ is the family of sets $E \subset I_0$ such that $|E| = |I_0|$, $\{I\}_E$ is the family of all intervals contained in I_0 with bounds in E, and $\boldsymbol{D}_{I_0\cap E} = \{D_{I_0\cap E}\}$ is the family of classes $D_{I_0\cap E}$ constituted from a finite number of consecutive intervals in I_0 and with bounds in E. If will be $D_{I_0\cap E} \equiv \{I_i\}$, with $I_i = [a_{i-1}, a_i]$ for $i = 1,2,...,n$ and $a \le a_0 < a_1 < a_2 < ... < a_{n-1} < a_n \le b$, it places $\lambda(\boldsymbol{D}_{I_0\cap E}) = \max\{|a_0 - a|, |a_i - a_{i-1}|, |b - a_n|\}$ for $i = 1,2,...,n$.

We remind the following definitions and theorems.

Definition 1. We will say that the interval function $\psi(I)$ defined on I_0, nonnegative on I_0, is outer upper semicontinuous (o.u.s.c.) in a set $E \in \boldsymbol{E}$ if for any number $\epsilon > 0$ and for any interval $I \in \{I\}_E$, there exists a number $\delta_\epsilon > 0$

such that for each $J \supset I$ with $J \in \{I\}_E$ and $|J - I| < \delta_\epsilon$ follows that:

$$\psi(J) \leq \psi(I) + \epsilon.$$

Definition 2. It is called “Weierstrass-Cesari upper integral” of the function $\psi(I)$ in I_0 with respect to the family $\boldsymbol{D}_{I_0 \cap E}$ and to the mesh introduced before, the following value (finite or infinite positive):

$$V_E(\psi, I_0) = \limsup_{\lambda(D_{I_0 \cap E}) \to 0} S(\psi, D_{I_0 \cap E}),$$

where:

$$S(\psi, D_{I_0 \cap E}) = \sum_{J \in D_{I_0 \cap E}} \psi(J).$$

Definition 3. It is called “Weierstrass-Cesari integral” of $\psi(I)$ in I_0 with respect to the family $\boldsymbol{D}_{I_0 \cap E}$ and to the mesh λ, the limit (if it exists) (see [10]):

$$W_E(\psi, I_0) = \lim_{\lambda(D_{I_0 \cap E}) \to 0} S(\psi, D_{I_0 \cap E}).$$

Definition 4. An interval function $\psi(I)$ defined on I_0 is said to be of generalized bounded variation (G.B.V.) on I_0 if it is possible to find a set $E \in \boldsymbol{E}$ such that there exists $V_E(\psi, I_0)$ and it is finite.

Definition 5. We will say that an interval function $\psi(I)$ defined on I_0 is Weierstrass-Cesari integrable in the generalized sense on I_0 if it is possible to find a set $E \in \boldsymbol{E}$ such that there exists $W_E(\psi, I_0)$ and it is finite.

Theorem 6. *Let $\psi(I)$ be a nonnegative interval function, Weierstrass-Cesari integrable on I_0. The function $W(\psi, I)$ is o.u.s.c. in I_0 if the $\psi(I)$ is o.u.s.c. in I_0 (see [3]).*

Definition 7. Let $I_1, I_2, ..., I_k$ be a finite system of closed intervals of I_0 such that $I_i \cap I_J = \emptyset$ for $i \neq j$, let $I_1', I_2', ..., I_n', ...$ be a finite or infinite sequence of closed intervals such that $\cup_{i=1}^k I_i \subset \cup_j I_j'$ and let $\psi(I)$ be a nonnegative interval function defined in I_0, it will be said that this function enjoys the property $\boldsymbol{C}$ in I_0 if it is verifying the following inequality:

$$\sum_{i=1}^k \psi(I_i) \leq \sum_j \psi(I_j')$$

for each chosen of the systems $I_1, I_2, ..., I_k$ and $I_1', I_2', ..., I_n', ...$.

The last definition is due to T. Radò (see [4]) and he also shows that if a interval function enjoys the property $\boldsymbol{C}$ on a interval, then it admits an unique completely additive extension to the class of Borel sets.

Theorem 8. *If an interval function on I_0 is:*

— *nonnegative;*

— *finitely additive;*

— *o.u.s.c.,*

then it enjoys the property $\boldsymbol{C}$ and consequently it admits a completely additive extension to the class of Borel sets (see [3]).

Now, with reference to a nonnegative interval function $\psi(I)$ defined on I_0, Weierstrass-Cesari integrable in the generalized sense on it, we consider the family of sets:

$$\boldsymbol{E}_W = \{E \in \boldsymbol{E} : W_E(\psi, I_0) < +\infty\}$$

and for each $E \in \boldsymbol{E}_W$, we consider the set at most countable (see [5], [6], [7] and [1]):

$$N_E = \{c \in I_0 : \lim_{\delta \to 0} \sup_{I \subset [c-\delta, c+\delta]} W_E(\psi, I) > 0\},$$

and we denote with $\boldsymbol{F}_W$ the set:

$$\boldsymbol{F}_W = \{F \subset I_0 : F = E - N_E, E \in \boldsymbol{E}_W\}.$$

It is clear that $|F| = |I_0|$.

Definition 9. We will call "Weierstrass-Cesari integral in generalized sense" of a function $\psi(I)$, which is Weierstrass-Cesari integrable in the generalized sense, the number $W_F(\psi, I_0)$ if it is independent of the choice of the set $E \in \boldsymbol{E}_W$ and we denote it with $W_G(\psi, I_0)$.

Definition 10. We will say that the nonnegative interval function $\psi(I)$ defined on $I_0 = [a, b]$, Weierstrass-Cesari integrable in the generalized sense on it, enjoys the "l_s-property" in a set $F \in \boldsymbol{F}_W$ if for each $[u, v] \in \{I\}$ with $u > a$ exists the limit:

$$l_s(u, v) = \lim_{\substack{(x,y) \to (u^-, v^-) \\ x, y \in F}} \psi([x, y])$$

and moreover $l_s(u, v) = \psi([u, v])$ for each $[u, v] \in \{I\}_F$ with $u > a$.

Taking into account the interval function:

$$\chi(I) = \chi([u, v]) = \begin{cases} l_s(u, v)\,, & \text{if } u > a\,, \\ 0\,, & \text{if } u = a\,, \end{cases}$$

defined in the interval I_0, Ragni (see [5]) proved that if the interval function $\psi(I)$ defined on I_0, nonnegative on it, Weierstrass-Cesari integrable in the generalized sense, it enjoys the l_s-property in a set $F \in \boldsymbol{F}_W$, then the function $\chi(I)$ in I_0 has Weierstrass-Cesari integral, $W(\chi, I_0)$, and for each $I \subset I_0$ we have:

$$W_g(\psi, I_0) = W(\chi, I_0). \tag{1}$$

Furthermore, in the hypothesis that the function $\chi(I)$, associate with the function $\psi(I)$, is o.u.s.c. in I_0, for the 1 and 2 theorems, the function $W(\chi, I)$ admits an unique completely additive extension to the class of Borel sets. Therefore $W_g(\psi, I)$, for equation (1), admits an unique completely additive extension to the class of Borel sets.

Let f be a function defined in the interval $I_0 = [a, b]$ and with values in a Banach space X, let μ be a measure on $[a, b]$ absolutely continuous with respect to Lebesgue measure m, with Radon-Nikodym derivative, $d\mu/dm$, equal to a continuous function $\varphi : [a, b] \to \Re$ almost everywhere in $[a, b]$, with $0 \leq \gamma \leq \varphi(x) \leq \eta$ for x in $[a, b]$ and let $\psi : \{I\}_E \to \Re^+$ be an interval function defined for each $I = [\alpha, \beta] \in \{I\}_E$ by the relation:

$$\psi(I) = \|\Delta f_E(I)\| \frac{\mu(I)}{m(I)},$$

with $\Delta f_E(I) = f_E(\beta) - f_E(\alpha)$, where f_E is the restriction of f to the set E, then we give the following definition.

Definition 11. We will say that the function f defined on $[a, b]$ and with values in X is of bounded μ-variation (B.μ-V.) on E if the $\psi(I)$ is of G.B.V. on I_0, under these conditions we will denote the value $V_E(\psi, I_0)$ with $V_\mu(f_E)$.

3. Our Results

In [2] has been proved that the interval function $V_\mu(f_E, I)$, with $I \in \{I\}$, is continuous on I_0 except on a countable infinity N of points, and let us $F = E - N$, it has been showed that $V_\mu(f_F)$ is independent from the choice of set E on which is considered restrict the function f, from these properties it can be given the following definition.

Definition 12. Given a function $f : [a, b]$ of bounded μ-variation on E, we call "generalized μ-variation" of f on $[a, b]$ the number $V_\mu(f_F)$, and we will denote it with $\varphi_\mu(f)$.

In the case in which the function f is of generalized bounded μ-variation, the interval function ψ is Weierstrass-Cesari integrable on $[a,b]$ and we have:

$$\varphi_\mu(f) = W_g(\psi, I_0),$$

moreover the function ψ enjoys the l_s-property in $[a,b]$ (see [5]). Hence it appears natural to define generalized μ-variation of the function f in a perfect set $P \subset I_0$ the value:

$$\varphi_\mu(f,P) = \varphi_\mu(f) - \varphi_\mu(f, \cup\Delta_i),$$

where we have denoted with $\cup\Delta_i$ the complement of P with respect to the interval I_0, if we suppose that the function $\chi(I)$ associate with the function $\psi(I)$ is o.u.s.c. in $[a,b]$.

Let P be a perfect set contained in $I_0 = [a,b]$, and let I_h be the interval $I_h = \{a \le x \le b-h\}$ and P_h the perfect set $P_h = P \cap I_h$, we can prove the following theorem.

Theorem 13. *Let f be a function defined on I_0, with values into a Banach space X and verifying the above conditions, then the relation:*

$$\varphi_\mu(f,P) = \lim_{h,0^+} \int_{P_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx$$

is verified with respect to a perfect set $P \subset [a,b]$, if and only if the following limit:

$$\lim_{h\to 0^+} \sum_r \int_{\Delta_r \cap I_h} \frac{\psi([x,x+h])}{h}dx$$

there exists and it is finite and moreover:

$$\lim_{h\to 0^+} \sum_r \int_{\Delta_r \cap I_h} \frac{\psi([x,x+h])}{h}dx = \sum_r \varphi_\mu(f,\Delta_r). \tag{2}$$

Proof. To point out that if the intervals Δ_r are of finite number, (2) is clearly satisfied.

We are proving that the condition is necessary. It is clear that:

$$\int_{I_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx = \int_{P_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx + \int_{\cup\Delta_r\cap I_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx. \tag{3}$$

By the hypothesis we have:

$$\lim_{h\to 0^+}\int_{P_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx=\varphi_\mu(f,P)$$

and since:

$$\lim_{h\to 0^+}\int_{I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx=\varphi_\mu(f),$$

then in (3) taking limit as $h\to 0^+$ we have :

$$\lim_{h\to 0^+}\int_{\cup\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx=\varphi_\mu(f)-\varphi_\mu(f,P), \tag{4}$$

but:

$$\int_{\cup\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx=\sum_r\int_{\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx$$

and taking limit in the last relation we get:

$$\lim_{h\to 0^+}\int_{\cup\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx$$

$$=\lim_{h\to 0^+}\sum_r\int_{\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx$$

and from (4) we obtain:

$$\lim_{h\to 0^+}\int_{\cup\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx=\varphi_\mu(f,\cup\Delta_r)$$

$$=\sum_r\varphi_\mu(f,\Delta_r)=\sum_r\lim_{h\to 0^+}\int_{\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx,$$

so:

$$\lim_{h\to 0^+}\sum_r\int_{\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx$$

$$=\sum_r\lim_{h\to 0^+}\int_{\Delta_r\cap I_h}\frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx,$$

that is:

$$\lim_{h\to 0^+}\sum_r\int_{\Delta_r\cap I_h}\frac{\psi([x,x+h])}{h}dx=\sum_r\varphi_\mu(f,\Delta_r).$$

Now, we will prove that the condition is sufficient. We suppose there exists finite the limit:

$$\lim_{h\to 0^+}\sum_r \int_{\Delta_r\cap I_h} \frac{\psi([x,x+h])}{h}dx$$

and moreover that:

$$\lim_{h\to 0^+}\sum_r \int_{\Delta_r\cap I_h} \frac{\psi([x,x+h])}{h}dx = \sum_r \lim_{h\to 0^+}\int_{\Delta_r\cap I_h} \frac{\psi([x,x+h])}{h}dx.$$

Taking account of (3) we have:

$$\int_{I_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx - \int_{\cup\Delta_r\cap I_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx$$

$$= \int_{P_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx,$$

but:

$$\lim_{h\to 0^+}\int_{I_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx = \varphi_\mu(f)$$

and there esists finite the following limit by the hypothesis:

$$\lim_{h\to 0^+}\int_{\cup\Delta_r\cap I_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx,$$

then there exists the limit:

$$\lim_{h\to 0^+}\int_{P_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx$$

and we have:

$$\lim_{h\to 0^+}\int_{P_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx = \varphi_\mu(f) - \varphi_\mu(f,\cup\Delta_r),$$

therefore:

$$\lim_{h\to 0^+}\int_{P_h} \frac{\|f(x+h)-f(x)\|}{h}\varphi(x)dx = \varphi_\mu(f,P).$$

□

References

[1] P. Brandi, A. Salvadori, The nonparametric integral of the calculus of variations as a Weierstrass integral. I. Existence and representation, *J. Math. Anal. Appl.*, **107** (1985), 67-95.

[2] C. Gori Cocchieri, The generalized weighted variation for applications with values in Banach spaces, *Rend. Circ. Mat. Palermo*, **26** (1977), 275-288.

[3] N. Giovannelli, P.G. Reas, On the generalized Weierstrass-Cesari integral in a perfect set, *Atti Accad. Sci. Lett. Arti Palermo*, **38** (1979), 207-218.

[4] T. Radò, *Lenght and Area*, American Mathematical Society press, New York (1948).

[5] M. Ragni, Interval functions and generalized variation, *Rend. Circ. Mat. Palermo*, **26** (1977), 229-250.

[6] M. Ragni, On the classical and generalized Weierstrass integral, *Matematiche*, **34** (1979), 8-20.

[7] M. Ragni, The generalized parametric Weierstrass integral of the calculus of variations, *Matematiche*, **34** (1979), 21-38.

[8] R.L. Schilling, *Measures, Integrals and Matingales*, Cambridge University Press, Cambridge (2006).

[9] C. Vinti, The Weierstrass-Burkill integral, *Atti Sem. Mat. Fis. Univ. Modena*, **18** (1969), 295-316. 10

[10] C. Vinti, Nonlinear integration and Weierstrass integral over a manifold: Connections with theorems on martingales, *Journal of Optimization Theory and Applications*, **41** (1983), 213-237.