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RELATIVE DERIVATIVES AND THE ECONOMIC SCIENCE

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Abstract: "Relative derivative", $(a/b)(dy/dx)$, is a generalization of the ordinary derivative for $a = b = 1$, and that of "elasticity" in economics for $a = x$ and $b = y$; the role of (a, b) is twofold: scaling and removing units. In a previous publication, this author applied relative derivatives to Taylor series in n variables and the fundamental theorem of calculus. Nature does not come with unit labels; rather, it is characterized by proportionalities, which are precisely what relative derivatives reveal – the geometric invariance of the Ricci scalar curvature R for example. The economic science has a pronounced feature, viz., the indeterminacy of the units of variables. This paper shows how relative derivatives, by revealing proportionalities, streamline the mathematical logic of economics and integrate all the building blocks therein, and at the same time change the mode of analysis from qualitative to quantitative. To the extent that many fields share the same interests as in economics of (post) optimization and equilibrium analysis yet also the same problem of unit specifications, this paper provides an illustration of how relative derivatives can be applied to fruitful theoretical derivations by four fundamental examples in economics.

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1. Introduction

This paper applies relative derivatives (see [4]), $\left(\frac{\xi_i}{\lambda}\frac{\partial}{\partial x_i}\right)_{i=1}^{n}$, to the economic

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science; we show that this unit-free approach connects all the key theoretical components in a tight logical framework.

The central theme of the economic science is optimization, which in fact also manifests itself in nature, as in the geodesics for gravitational motions or optimal metrics for manifolds, among many others. In economic theory, a consumer seeks to purchase goods and services to maximize his/her psychological satisfaction level, but only subject to what he/she has to offer to the market in exchange, and this constrained maximization leads to the demand function $D :$ prices $\mathbf{p} \in \mathbb{R}_+^n \longmapsto$ quantities $\mathbf{q} \in \mathbb{R}_+^n$. Producers, on the other hand, seek to maximize their firms' earnings by producing the optimal quantities in the least costly way, resulting in the supply function $S : \mathbf{p} \longmapsto \mathbf{q}$; if $D(\mathbf{p}^*) = S(\mathbf{p}^*)$ for some $\mathbf{p}^*$, then an equilibrium over the economy is achieved; otherwise, price dynamics readjusts S to D. However, the worldwide economic depression of the 1930s revealed that the interest rate, which is the price of current consumption relative to future consumption, failed to equate capital demand to capital supply because of the latter's functional dependence on people's incomes, and thus economics branched out into a macro study of national income. Whether a study of the original micro price theory or the later macro income theory, there are three major analytical methodologies: statics for determining the equilibrium states, comparative statics for comparing two equilibrium states due to parametric perturbations, and dynamics for studying the motions of the economic variables (for an application of relative derivatives to comparative dynamics, see [5]). Yet the simple fact is that the mathematical logic remains qualitative in nature due to the sheer difficulties faced in evaluating the involved derivatives – which would require a complete specification of the contained units. This great drawback of ordinary derivatives in economic analyses has led researchers to the following tactics: (1) appealing to specialized functional forms, (2) investigating the properties of signed matrices, (3) employing lattice algebra in functional analyses (see, e.g., [6, 7]), and (4) adding expedient assumptions during derivations, e.g., assuming dynamic stability so as to settle the signs of the expressions and make qualitative comparative static predictions. All the while, however, the device of elasticity – $\left(\frac{x}{y}\frac{\partial y}{\partial x}\right)$ – has coexisted with the derivative $\left(\frac{\partial y}{\partial x}\right)$, except that $\left(\frac{x}{y}\frac{\partial y}{\partial x}\right)$ was not generalized into $\left(\frac{a}{b}\frac{\partial y}{\partial x}\right)$ (cf. [4]).

Section 2 below will begin with a proposition that re-expresses a standard perturbation of a constrained optimization in terms of relative derivatives; then Example 1 will reformulate the comparative statics of consumer theory by relative derivatives, which effectively converts the nonquantifiable incremental psychological utility $\frac{\partial u}{\partial x_1}$ to $\left(\frac{x_1}{u}\frac{\partial u}{\partial x_1}\right)$, and directly deduces the conclusion of the sensitivity of demand to price, $\left(\frac{p_1}{D}\frac{\partial D}{\partial p_1}\right)$; Example 2 will analogously treat the producer theory, where relative derivatives deduce the key measurement of "elasticity of substitution" between inputs – a unit-free construct that has been separated from the derivative-driven mainline logic – directly from the initial setup of cost minimization; Example 3 will build upon Example 2 to set up a simple but complete system of general equilibrium growth of an economy via relative derivatives, which integrates the main economic theory of optimization with the following isolated major topics: long-run economic growth, price inflation, and labor employment, business cycles, and income distribution; finally Example 4 will apply relative derivatives to price dynamics, where we will replace $\frac{dp_i}{dt}$ with $\frac{dp_i}{dt}/p_i^*$ and linearize the differential equation to equal to a product of two relative-derivative expressed matrices, viz., that of the speeds of price adjustments and that of the excess demands over supplies, and as such one can simulate the matrices to compute the eigenvalues and determine the stability of the system or perhaps even to solve the differential equations numerically, in comparison with the paradigm of putting aside the matrix of speeds of price adjustments and solely examining the signs of the various excess demands. All in all these four examples will show how relative derivatives can integrate the economic science as a whole.

2. Relative Derivatives

Notation 1. Consider $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r$ with $n \in [2, \infty)$, $m \in [1, \infty)$, and $r \in [1, n-1]$; let $\mathbf{a} \equiv (a_\mu)_{\mu=1}^m \in \mathbb{R}^m$, $\mathbf{c} \equiv (c_k)_{k=1}^r \in \mathbb{R}^r$, and

$$f, h_{\langle k \rangle} \in C^3 (\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}) \tag{2.1}$$

be given, and consider the following constrained parametric optimization

$$\text{[Problem Max]}:$$

$$\operatorname*{Max}_{\mathbf{x} \equiv (x_i)_{i=1}^n \in \mathbb{R}^n} f(\mathbf{x}, \mathbf{a}) \tag{2.2}$$

$$\text{subject to } \left(h_{\langle k \rangle}(\mathbf{x}, \mathbf{a})\right)_{k=1}^r = \mathbf{c}. \tag{2.3}$$

Now $\forall i = 1, \cdots, n$ $\forall \mu = 1, \cdots, m$ $\forall\, k = 1, \cdots, r$, let $\xi_i > 0$, $\alpha_\mu > 0$, $\gamma_k > 0$, $\phi > 0$ be given. Define:

$$f_i^{\prime\odot} := \frac{\xi_i}{\phi}\frac{\partial f}{\partial x_i},\ f_\mu^{\prime\odot} := \frac{\alpha_\mu}{\phi}\frac{\partial f}{\partial a_\mu}, \tag{2.4}$$

$$h'^{\odot}_{\langle k\rangle i} := \frac{\xi_i}{\gamma_k}\frac{\partial h_{\langle k\rangle}}{\partial x_i},\ h'^{\odot}_{\langle k\rangle \mu} := \frac{\alpha_\mu}{\gamma_k}\frac{\partial h_{\langle k\rangle}}{\partial a_\mu}, \tag{2.5}$$

$$f''^{\odot}_{ij} := \frac{\xi_i\xi_j}{\phi}\frac{\partial^2 f}{\partial x_j \partial x_i},\ f''^{\odot}_{i\mu} := \frac{\alpha_\mu \xi_i}{\phi}\frac{\partial^2 f}{\partial a_\mu \partial x_i}, \tag{2.6}$$

$$h''^{\odot}_{\langle k\rangle ij} := \frac{\xi_i\xi_j}{\gamma_k}\frac{\partial^2 h_{\langle k\rangle}}{\partial x_j \partial x_i},\ h''^{\odot}_{\langle k\rangle i\mu} := \frac{\alpha_\mu \xi_i}{\gamma_k}\frac{\partial^2 h_{\langle k\rangle}}{\partial a_\mu \partial x_i}, \tag{2.7}$$

and

$$L(\mathbf{x},\mathbf{a},\lambda) := f(\mathbf{x},\mathbf{a}) - \sum_{k=1}^{r}\lambda_k \cdot \left(h_{\langle k\rangle}(\mathbf{x},\mathbf{a}) - c_k\right); \tag{2.8}$$

set

$$\lambda_k^{\odot} := \frac{\lambda_k}{(\phi/\gamma_k)},\ \forall k = 1,\cdots,r, \tag{2.9}$$

$$l'^{\odot}_i(\mathbf{x},\mathbf{a},\lambda) := f'^{\odot}_i(\mathbf{x},\mathbf{a}) - \sum_{k=1}^{r}\lambda_k^{\odot}\cdot h'^{\odot}_{\langle k\rangle i}(\mathbf{x},\mathbf{a}),\ \forall i = 1,\cdots,n, \tag{2.10}$$

$$l'^{\odot}_{\lambda_k}(\mathbf{x},\mathbf{a},\lambda) := \frac{(\phi/\gamma_k)}{\phi}\frac{\partial L}{\partial \lambda_k} = \left(\frac{1}{\gamma_k}\right)\frac{\partial L}{\partial \lambda_k},\ \text{and} \tag{2.11}$$

$$l''^{\odot}_{ij}(\mathbf{x},\mathbf{a},\lambda) := f''^{\odot}_{ij}(\mathbf{x},\mathbf{a}) - \sum_{k=1}^{r}\lambda_k^{\odot}\cdot h''^{\odot}_{\langle k\rangle ij}(\mathbf{x},\mathbf{a}),\ \forall \{i,j\}\subset\{1,\cdots,n\}, \tag{2.12}$$

$$= \left(\frac{\xi_i\xi_j}{\phi}\right)\left[f''_{ij}(\mathbf{x},\mathbf{a}) - \sum_{k=1}^{r}\lambda_k\cdot h''_{\langle k\rangle ij}(\mathbf{x},\mathbf{a})\right]$$

$$\equiv \left(\frac{\xi_i\xi_j}{\phi}\right)\cdot l''_{ij}(\mathbf{x},\mathbf{a},\lambda). \tag{2.13}$$

Proposition 1. *Under the above Notation 1, let $\mathbf{a}^* \in \mathbb{R}^m$; assume that $\exists\ \mathbf{p}^* \equiv (\mathbf{x}^*,\mathbf{a}^*,\lambda^*) \in \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^r$ such that*

$$\left(l'^{\odot}_i(\mathbf{p}^*)\right)_{i=1}^{n} = \mathbf{0} \in \mathbb{R}^n, \tag{2.14}$$

$$\left(l'^{\odot}_{\lambda_k}(\mathbf{p}^*)\right)_{k=1}^{r} = \mathbf{0} \in \mathbb{R}^r, \tag{2.15}$$

and $\forall q = r+1,\cdots,n$

$$(-1)^q \det H^{\odot}_{q+r} > 0, \tag{2.16}$$

where

$$H_{q+r}^{\odot} := \begin{pmatrix} l_{11}^{''\odot} & \cdots & l_{1q}^{''\odot} & -h_{\langle 1\rangle 1}^{'\odot} & \cdots & -h_{\langle r\rangle 1}^{'\odot} \\ & \vdots & & & \vdots & \\ l_{q1}^{''\odot} & \cdots & l_{qq}^{''\odot} & -h_{\langle 1\rangle q}^{'\odot} & \cdots & -h_{\langle r\rangle q}^{'\odot} \\ -h_{\langle 1\rangle 1}^{'\odot} & \cdots & -h_{\langle 1\rangle q}^{'\odot} & & & \\ & \vdots & & & \mathbf{O}_{r\times r} & \\ -h_{\langle r\rangle 1}^{'\odot} & \cdots & -h_{\langle r\rangle q}^{'\odot} & & & \end{pmatrix}_{(q+r)\times(q+r),\mathbf{p}^*}. \tag{2.17}$$

Then (1) $f(\mathbf{x}^*, \mathbf{a}^*)$ solves [Problem Max], and (2) at $\mathbf{p}^*$

$$\text{(a)} \begin{pmatrix} \left(\frac{\alpha_\mu}{\xi_i}\frac{\partial x_i}{\partial a_\mu}\right)_{n\times m} \\ \left(\frac{\alpha_\mu}{\phi/\gamma_k}\frac{\partial \lambda_k}{\partial a_\mu}\right)_{r\times m} \end{pmatrix} = \left(H_{n+r}^{\odot}\right)^{-1} \begin{pmatrix} \left(-f_{i\mu}^{''\odot} + \sum_{k=1}^r \lambda_k^{*\odot} \cdot h_{\langle k\rangle i\mu}^{''\odot}\right)_{n\times m} \\ \left(h_{\langle k\rangle \mu}^{'\odot}\right)_{r\times m} \end{pmatrix}, \tag{2.18}$$

$$\text{and (b) } \frac{df}{\phi} = \sum_{\mu=1}^m \left(f_\mu^{'\odot} - \sum_{k=1}^r \lambda_k^{*\odot} \cdot h_{\langle k\rangle \mu}^{'\odot}\right)\left(\frac{da_\mu}{\alpha_\mu}\right). \tag{2.19}$$

Proof. $\left(l_i^{'\odot}(\mathbf{p}^*)\right)_{i=1}^n \equiv \left(\frac{\xi_i}{\phi}\frac{\partial f}{\partial x_i} - \sum_{k=1}^r \frac{\lambda_k}{\phi/\gamma_k} \cdot \frac{\xi_i}{\gamma_k}\frac{\partial h_{\langle k\rangle}}{\partial x_i}\right)_{i=1}^n = \mathbf{0} \in \mathbb{R}^n \iff$

$$\left(\frac{\partial f}{\partial x_i} - \sum_{k=1}^r \lambda_k \cdot \frac{\partial h_{\langle k\rangle}}{\partial x_i}\right)_{i=1}^n = \mathbf{0}; \tag{2.20}$$

$$\left(l_{\lambda_k}^{'\odot}(\mathbf{p}^*)\right)_{k=1}^r = \mathbf{0} \in \mathbb{R}^r \iff \left(\frac{\partial L}{\partial \lambda_k}\right)_{k=1}^r = \mathbf{0}; \tag{2.21}$$

$\forall q = r+1, \cdots, n$

$$H_{q+r}^{\odot} = \frac{1}{\phi}\operatorname{diag}\left(\xi_1, \cdots, \xi_q, \frac{\phi}{\gamma_1}, \cdots, \frac{\phi}{\gamma_r}\right) \circ \begin{pmatrix} l_{11}^{''} & \cdots & l_{1q}^{''} & -h_{\langle 1\rangle 1}^{'} & \cdots & -h_{\langle r\rangle 1}^{'} \\ & \vdots & & & \vdots & \\ l_{q1}^{''} & \cdots & l_{qq}^{''} & -h_{\langle 1\rangle q}^{'} & \cdots & -h_{\langle r\rangle q}^{'} \\ -h_{\langle 1\rangle 1}^{'} & \cdots & -h_{\langle 1\rangle q}^{'} & & & \\ & \vdots & & & \mathbf{O}_{r\times r} & \\ -h_{\langle r\rangle 1}^{'} & \cdots & -h_{\langle r\rangle q}^{'} & & & \end{pmatrix}$$

$$\circ \operatorname{diag}\left(\xi_1, \cdots, \xi_q, \frac{\phi}{\gamma_1}, \cdots, \frac{\phi}{\gamma_r}\right)$$
$$\equiv \frac{1}{\phi} D_{q+r} H_{q+r} D_{q+r}, \tag{2.22}$$

so that

$$\det H_{q+r}^{\odot} = \left(\frac{\xi_1 \cdots \xi_q}{\gamma_1 \cdots \gamma_r}\right)^2 \phi^{r-q} \det H_{q+r}, \tag{2.23}$$

and thus

$$(-1)^q \det H_{q+r}^{\odot} > 0 \iff (-1)^q \det H_{q+r} > 0. \tag{2.24}$$

Equations (2.20), (2.21), and (2.24) imply that (see, e.g., [3, Corollary 1]): (1) $f(\mathbf{x}^*, \mathbf{a}^*)$ solves [Problem Max], and (2) at $\mathbf{p}^*$

$$\text{(a)} \quad \begin{pmatrix} \left(\frac{\partial x_i}{\partial a_\mu}\right)_{n\times m} \\ \left(\frac{\partial \lambda_k}{\partial a_\mu}\right)_{r\times m} \end{pmatrix} = (H_{n+r})^{-1} \begin{pmatrix} \left(-f''_{i\mu} + \sum_{k=1}^r \lambda_k^* \cdot h''_{\langle k\rangle i\mu}\right)_{n\times m} \\ \left(h'_{\langle k\rangle \mu}\right)_{r\times m} \end{pmatrix}, \tag{2.25}$$

so that

$$\begin{pmatrix} \left(\frac{\alpha_\mu}{\xi_i}\frac{\partial x_i}{\partial a_\mu}\right)_{n\times m} \\ \left(\frac{\alpha_\mu}{\phi/\gamma_k}\frac{\partial \lambda_k}{\partial a_\mu}\right)_{r\times m} \end{pmatrix} \tag{2.26}$$
$$= D_{n+r}^{-1} \begin{pmatrix} \left(\frac{\partial x_i}{\partial a_\mu}\right)_{n\times m} \\ \left(\frac{\partial \lambda_k}{\partial a_\mu}\right)_{r\times m} \end{pmatrix} \operatorname{diag}(\alpha_1, \cdots, \alpha_m)$$
$$= \left[D_{n+r}^{-1} (H_{n+r})^{-1} \left(\frac{1}{\phi} D_{n+r}\right)^{-1}\right]$$
$$\circ \left[\left(\frac{1}{\phi} D_{n+r}\right) \begin{pmatrix} \left(-f''_{i\mu} + \sum_{k=1}^r \lambda_k^* \cdot h''_{\langle k\rangle i\mu}\right)_{n\times m} \\ \left(h'_{\langle k\rangle \mu}\right)_{r\times m} \end{pmatrix} \operatorname{diag}(\alpha_1, \cdots, \alpha_m)\right]$$
$$= \left(H_{n+r}^{\odot}\right)^{-1} \begin{pmatrix} \left(-f''^{\odot}_{i\mu} + \sum_{k=1}^r \lambda_k^{*\odot} \cdot h''^{\odot}_{\langle k\rangle i\mu}\right)_{n\times m} \\ \left(h'^{\odot}_{\langle k\rangle \mu}\right)_{r\times m} \end{pmatrix},$$

and (b)

$$df = \sum_{\mu=1}^m \left(f'_\mu - \sum_{k=1}^r \lambda_k^* \cdot h'_{\langle k\rangle \mu}\right) da_\mu, \tag{2.27}$$

so that

$$\frac{df}{\phi} = \sum_{\mu=1}^{m} \left(\frac{\alpha_\mu}{\phi} f'_\mu - \sum_{k=1}^{r} \frac{\lambda_k^*}{\phi/\gamma_k} \cdot \frac{\alpha_\mu}{\gamma_k} h'_{\langle k \rangle \mu} \right) \frac{da_\mu}{\alpha_\mu} \tag{2.28}$$

$$= \sum_{\mu=1}^{m} \left(f'^{\odot}_\mu - \sum_{k=1}^{r} \lambda_k^{*\odot} \cdot h'^{\odot}_{\langle k \rangle \mu} \right) \left(\frac{da_\mu}{\alpha_\mu} \right). \tag{2.29}$$

Remark 1. $\left(h_{\langle k \rangle}(\mathbf{x}, \mathbf{a})\right)_{k=1}^{r} = \mathbf{c} \Longleftrightarrow -\left(h_{\langle k \rangle}(\mathbf{x}, \mathbf{a})\right)_{k=1}^{r} = -\mathbf{c}$. Since $\mathbf{c}$ does not enter the matrix $H_{q+r}^{\odot}$ in equation (2.17), $\forall k \in \{1, \cdots, r\}$ the presentations of $\left(-h'^{\odot}_{\langle k \rangle \cdot}\right)_{1 \times q}$ and $\left(-h'^{\odot}_{\langle k \rangle \cdot}\right)_{q \times 1}$ in $H_{q+r}^{\odot}$ can be alternatively shown as $\left(h'^{\odot}_{\langle k \rangle \cdot}\right)_{1 \times q}$ and $\left(h'^{\odot}_{\langle k \rangle \cdot}\right)_{q \times 1}$ respectively. As such, the problem of

$$\min f(\mathbf{x}, \mathbf{a}) \equiv \max \left(-f(\mathbf{x}, \mathbf{a})\right) \tag{2.30}$$

has the following second-order condition in comparison with inequality (2.16):

$$0 < (-1)^q \det \left(-H_{q+r}^{\odot}\right) = (-1)^{2q+r} \det \left(H_{q+r}^{\odot}\right) = (-1)^r \det \left(H_{q+r}^{\odot}\right) \quad \forall q = r+1, \cdots, n. \tag{2.31}$$

Example 1. (Consumer Maximization) Given the prices $\mathbf{p} \equiv (p_i)_{i=1}^{n} \in \mathbb{R}_+^n$ of products $\mathbf{x} \equiv (x_i)_{i=1}^{n}$ and a fixed household income $y > 0$, a consumer seeks to purchase the optimal $\mathbf{x}^*$ to achieve the greatest psychological utility $u(\mathbf{x}^*) \equiv u^*$, $u \in C^3(\mathbb{R}^n, \mathbb{R})$, i.e.,

$$\max_{\mathbf{x} \in \mathbb{R}_+^n} u(\mathbf{x}) \quad \text{subject to } \mathbf{px} = y, \tag{2.32}$$

which can be reformulated as

$$L(\mathbf{x}, \mathbf{p}, \lambda) := u(\mathbf{x}) - \lambda \cdot (\mathbf{px} - y). \tag{2.33}$$

Assume that $u(\mathbf{x}^*) \equiv u^*$ is a local unique solution; then relative to $(p_i)_{i=1}^{n}$, $(x_i^*)_{i=1}^{n}$, y, and u^*, denoting by

$$u''^{\odot}_{ij} := \frac{x_i^* x_j^*}{u^*} \frac{\partial^2 u}{\partial x_j \partial x_i}, \text{ and } \lambda^{*\odot} := \frac{\lambda^*}{(u^*/y)}, \tag{2.34}$$

we have at $u(\mathbf{x}^*) \equiv u^*$ (cf. e.g., [2], to verify that all the results as derived from relative derivatives reduce to the regular expressions by a simple substitution of 1 into all the relative values):

$$\begin{pmatrix} \left(\frac{p_\mu}{x_i^*}\frac{\partial x_i}{\partial p_\mu}\right)_{n\times n} \\ \left(\frac{p_\mu}{(u^*/y)}\frac{\partial \lambda}{\partial p_\mu}\right)_{1\times n} \end{pmatrix} \tag{2.35}$$

$$= \begin{pmatrix} u_{11}^{''\odot} & \cdots & u_{1n}^{''\odot} & -\frac{p_1 x_1^*}{y} \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1}^{''\odot} & \cdots & u_{nn}^{''\odot} & -\frac{p_n x_n^*}{y} \\ -\frac{p_1 x_1^*}{y} & \cdots & -\frac{p_n x_n^*}{y} & 0 \end{pmatrix}_{(n+1)\times(n+1)}^{-1}$$

$$\circ \begin{pmatrix} \frac{p_1 x_1^*}{y}\lambda^\odot & \mathbf{0}_{1\times(n-2)} & 0 \\ \mathbf{0}_{(n-2)\times 1} & \ddots & \mathbf{0}_{(n-2)\times 1} \\ 0 & \mathbf{0}_{1\times(n-2)} & \frac{p_n x_n^*}{y}\lambda^\odot \\ \frac{p_1 x_1^*}{y} & \cdots & \frac{p_n x_n^*}{y} \end{pmatrix}_{(n+1)\times n}.$$

Thus the well-known price elasticity of demand, $\frac{p_\mu}{x_i^*}\frac{\partial x_i}{\partial p_\mu}$, is directly deduced from the initial setup of maximizing psychological utility u, and by the relative-derivative approach u has become quantifiable in terms of unit-free fractional changes.

Example 2. (Producer Minimization) Given the prices $\mathbf{w} \equiv (w_i)_{i=1}^n \in \mathbb{R}_+^n$ of production inputs $\mathbf{x} \equiv (x_i)_{i=1}^n$ and a decided production quantity $y > 0$, a producer seeks to purchase the optimal input quantities $\mathbf{x}^*$ to achieve the least total manufacturing cost $\mathbf{w} \cdot \mathbf{x}^* \equiv c^*$ in accordance with the available technology $T(\mathbf{x}) = y$, $T \in C^3(\mathbb{R}^n, \mathbb{R})$, i.e.,

$$\min_{\mathbf{x}\in\mathbb{R}_+^n} \mathbf{w}\cdot\mathbf{x}^* \text{ subject to } T(\mathbf{x}) = y, \tag{2.36}$$

which can be reformulated as

$$L(\mathbf{x}, \mathbf{w}, \lambda) := \mathbf{w}\cdot\mathbf{x} - \lambda\cdot(T(\mathbf{x}) - y). \tag{2.37}$$

Assume that $\mathbf{w} \cdot \mathbf{x}^* \equiv c^*$ is a local unique solution; then relative to $(w_i)_{i=1}^n$, $(x_i^*)_{i=1}^n$, y, and c^*, denoting by

$$T_{ij}^{''\odot} := \frac{x_i^* x_j^*}{y}\frac{\partial^2 T}{\partial x_j \partial x_i}, \text{ and } \lambda^{*\odot} := \frac{\lambda^*}{(c^*/y)}, \tag{2.38}$$

we have at $\mathbf{w} \cdot \mathbf{x}^* \equiv c^*$:

$$\begin{pmatrix} \left(\frac{w_\mu}{x_i^*}\frac{\partial x_i}{\partial w_\mu}\right)_{n\times n} \\ \left(\frac{w_\mu}{(c^*/y)}\frac{\partial \lambda}{\partial w_\mu}\right)_{1\times n} \end{pmatrix} \tag{2.39}$$

$$= \begin{pmatrix} -\lambda^{*\odot}T''^{\odot}_{11} & \cdots & -\lambda^{*\odot}T''^{\odot}_{1n} & -T'^{\odot}_1 \\ \vdots & \vdots & \vdots & \vdots \\ -\lambda^{*\odot}T''^{\odot}_{n1} & \cdots & -\lambda^{*\odot}T''^{\odot}_{nn} & -T'^{\odot}_n \\ -T'^{\odot}_1 & \cdots & -T'^{\odot}_n & 0 \end{pmatrix}^{-1}_{(n+1)\times(n+1)}$$

$$\circ \begin{pmatrix} -\frac{w_1 x_1^*}{c^*} & \mathbf{0}_{1\times(n-2)} & 0 \\ \mathbf{0}_{(n-2)\times 1} & \ddots & \mathbf{0}_{(n-2)\times 1} \\ 0 & \mathbf{0}_{1\times(n-2)} & -\frac{w_n x_n^*}{c^*} \\ 0 & \cdots & 0 \end{pmatrix}_{(n+1)\times n}$$

$$\equiv H^{-1}_{(n+1)\times(n+1)} R_{(n+1)\times n}, \tag{2.40}$$

which in essence is the same as equation (2.35). To continue to derive input substitutability, we note that as a first-order condition of the optimization problem, we have $\forall i = 1, \cdots, n$

$$\lambda^* = \frac{w_i}{\frac{\partial T}{\partial x_i}}; \tag{2.41}$$

thus, $\forall \{i,j\} \subset \{1, \cdots, n\} \quad \lambda^{*\odot}T''^{\odot}_{ij} = \frac{y}{c^*}\lambda^* \cdot \frac{x_i^* x_j^*}{y}\frac{\partial^2 T}{\partial x_j \partial x_i} = \frac{w_i x_i^*}{c^*} \cdot \frac{x_j^*}{\partial T/\partial x_i} \cdot \frac{\partial}{\partial x_j}\left(\frac{\partial T}{\partial x_i}\right)$; denoting by

$$\left(\frac{\partial T}{\partial x_i}\right)'^{\odot}_j := \frac{x_j^*}{\partial T/\partial x_i} \cdot \frac{\partial}{\partial x_j}\left(\frac{\partial T}{\partial x_i}\right), \tag{2.42}$$

we then have

$$\lambda^{*\odot}T''^{\odot}_{ij} = \frac{w_i x_i^*}{c^*} \cdot \left(\frac{\partial T}{\partial x_i}\right)'^{\odot}_j \tag{2.43}$$

$$= \frac{w_j x_j^*}{c^*} \cdot \left(\frac{\partial T}{\partial x_j}\right)'^{\odot}_i, \tag{2.44}$$

so that

$$\left(\frac{c^*}{w_i x_i^*}\right) \cdot \lambda^{*\odot}T''^{\odot}_{ij} = \left(\frac{\partial T}{\partial x_j}\right)'^{\odot}_i; \tag{2.45}$$

also, $\left(\frac{c^*}{w_i x_i^*}\right) \cdot T_i^{'\odot} = \left(\frac{c^*}{w_i x_i^*}\right) \cdot \frac{x_i^*}{y} \frac{\partial T}{\partial x_i} = \frac{c^*}{y} \cdot \left(\frac{1}{w_i} \frac{\partial T}{\partial x_i}\right) = \frac{c^*}{y\lambda^*} = \lambda^{\odot -1}$. Consequently,

$$\begin{aligned} & H^{-1}_{(n+1)\times(n+1)} R_{(n+1)\times n} && (2.46) \\ & = H^{-1}_{(n+1)\times(n+1)} \circ \operatorname{diag}\left(\frac{c^*}{w_1 x_1^*}, \cdots, \frac{c^*}{w_n x_n^*}, 0\right)^{-1} \\ & \circ \operatorname{diag}\left(\frac{c^*}{w_1 x_1^*}, \cdots, \frac{c^*}{w_n x_n^*}, 0\right) \circ R_{(n+1)\times n} \\ & = \begin{pmatrix} \left(\frac{\partial T}{\partial x_1}\right)_1^{'\odot} & \cdots & \left(\frac{\partial T}{\partial x_1}\right)_n^{'\odot} & \lambda^{\odot -1} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\partial T}{\partial x_n}\right)_1^{'\odot} & \cdots & \left(\frac{\partial T}{\partial x_n}\right)_n^{'\odot} & \lambda^{\odot -1} \\ T_1^{'\odot} & \cdots & T_n^{'\odot} & 0 \end{pmatrix}_{(n+1)\times(n+1)} \\ & \circ \begin{pmatrix} 1 & \mathbf{0}_{1\times(n-2)} & 0 \\ \mathbf{0}_{(n-2)\times 1} & \ddots & \mathbf{0}_{(n-2)\times 1} \\ 0 & \mathbf{0}_{1\times(n-2)} & 1 \\ 0 & \cdots & 0 \end{pmatrix}_{(n+1)\times n} \\ & \equiv \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \vdots & \vdots \\ s_{n1} & \cdots & s_{nn} \\ s_{n+1,1} & \cdots & s_{n+1,n} \end{pmatrix}_{(n+1)\times n} \equiv S_{(n+1)\times n}; && (2.47) \end{aligned}$$

i.e., by equations (2.39), (2.40), (2.46), and (2.47), we have:

$$\begin{pmatrix} \left(\frac{w_\mu}{x_i^*} \frac{\partial x_i}{\partial w_\mu}\right)_{n\times n} \\ \left(\frac{w_\mu}{(c^*/y)} \frac{\partial \lambda}{\partial w_\mu}\right)_{1\times n} \end{pmatrix} = S_{(n+1)\times n}, \tag{2.48}$$

or

$$\begin{pmatrix} \frac{dx_1}{x_1^*} \\ \vdots \\ \frac{dx_n}{x_n^*} \\ \frac{d\lambda}{(c^*/y)} \end{pmatrix} = S_{(n+1)\times n} \begin{pmatrix} \frac{dw_1}{w_1} \\ \vdots \\ \frac{dw_n}{w_n} \end{pmatrix}. \tag{2.49}$$

Now

$$\text{set } w_n \equiv 1, \tag{2.50}$$

$$\text{so that } \frac{dw_n}{w_n} \equiv 0, \tag{2.51}$$

$$\text{and } \frac{dw_i}{w_i} \equiv \frac{d\left(\frac{w_i}{w_n}\right)}{\left(\frac{w_i}{w_n}\right)} \quad \forall i = 1, \cdots, n-1. \tag{2.52}$$

Then

$$\begin{pmatrix} \frac{dx_1}{x_1^*} \\ \vdots \\ \frac{dx_{n-1}}{x_{n-1}^*} \\ 0 \end{pmatrix}_{n\times 1} - \begin{pmatrix} \frac{dx_n}{x_n^*} \\ \vdots \\ \frac{dx_n}{x_n^*} \\ 0 \end{pmatrix}_{n\times 1} \tag{2.53}$$

$$= \left[\begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \vdots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix} - \begin{pmatrix} s_{n1} & \cdots & s_{nn} \\ \vdots & \vdots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix}\right] \begin{pmatrix} \frac{dw_1}{w_1} \\ \vdots \\ \frac{dw_{n-1}}{w_{n-1}} \\ 0 \end{pmatrix}_{n\times 1}$$

$$= \begin{pmatrix} s_{11} - s_{n1} \\ \vdots \\ s_{n-1,1} - s_{n1} \\ 0 \end{pmatrix} \frac{dw_1}{w_1} + \cdots + \begin{pmatrix} s_{1,n-1} - s_{n,n-1} \\ \vdots \\ s_{n-1,n-1} - s_{n,n-1} \\ 0 \end{pmatrix} \frac{dw_{n-1}}{w_{n-1}},$$

or $\forall \{i, j\} \subset \{1, \cdots, n\}$

$$\begin{aligned} &\frac{dx_i}{x_i^*} - \frac{dx_j}{x_j^*} \\ &= \left(\frac{dx_i}{x_i^*} - \frac{dx_n}{x_n^*}\right) - \left(\frac{dx_j}{x_j^*} - \frac{dx_n}{x_n^*}\right) \\ &= (s_{i1} - s_{j1}) \frac{dw_1}{w_1} + \cdots + (s_{i,n-1} - s_{j,n-1}) \frac{dw_{n-1}}{w_{n-1}}; \end{aligned} \tag{2.54}$$

in particular,

$$\begin{aligned} &\frac{dx_i}{x_i^*} - \frac{dx_n}{x_n^*} \\ &= (s_{ii} - s_{ni}) \left(\frac{dw_i}{w_i} - \frac{dw_n}{w_n}\right) \\ &+ \sum_{j \in \{1,\cdots,n-1\} - \{i\}} (s_{ij} - s_{nj}) \left(\frac{dw_j}{w_j} - \frac{dw_n}{w_n}\right), \end{aligned} \tag{2.55}$$

i.e.,

$$s_{ii} - s_{ni} \equiv \sigma_{x_i x_n w_i} \tag{2.56}$$

measures the fractional change in the optimal input ratio $\left(\frac{x_i^*}{x_n^*}\right)$ due to a one-percent increase in their price ratio $\left(\frac{w_i}{w_n}\right)$, i.e., $\sigma_{x_i x_n w_i}$:= "the elasticity of substitution between x_i and x_n." As such, relative derivatives readily derives $\sigma_{x_i x_n w_i}$ from the initial setup of cost minimization (for the common practice of assuming the "CES" functions – i.e., $T(x_1, x_2)$ with a constant elasticity of substitution $\sigma_{x_1 x_2 w_1}$, see, e.g., [1]).

Example 3. Consider an aggregation of national production into one single composite product y, which is produced by capital K, labor L, and raw materials T in

$$y = f(K, L, T); \tag{2.57}$$

then the growth of y is

$$\frac{\frac{dy}{dt}}{y} \equiv \dot{y}^{\odot} = \eta_K \dot{K}^{\odot} + \eta_L \dot{L}^{\odot} + \eta_T \dot{T}^{\odot} + \dot{A}^{\odot}, \tag{2.58}$$

where $\eta_K \equiv f_K^{'\odot} := \frac{K\frac{\partial f}{\partial K}}{y}$ = (the capital share of y) > 0, $\eta_L \equiv f_L^{'\odot} := \frac{L\frac{\partial f}{\partial L}}{y}$ = (the labor share of y) > 0, and $\eta_T \equiv f_T^{'\odot} := \frac{T\frac{\partial f}{\partial T}}{y}$ = (the land owners' share of y) > 0 can be considered as "semi-parameters" – fixed in the short run as technological parameters but vary in the long run due to income redistribution; $\dot{A}^{\odot}$ is a given technology growth parameter. Given money supply M as set by the monetary authorities, there is an associate price level P, so that $M^{'\odot} - P^{'\odot}$ = money growth−inflation rate = effective money growth, which for equilibrium growth must be balanced by the demand for currencies $l(y, r)$ resulting from the growth of y or changes in the interest rates r, i.e.,

$$\dot{M}^{\odot} - \dot{P}^{\odot} = l_o + l_y \dot{y}^{\odot} + l_r \dot{r}^{\odot}, \tag{2.59}$$

where $l_y > 0$, $l_r < 0$, and l_o are given behavioral parameters. Similarly, equilibrium growth requires savings $s(y, r)$ and investments $i(r)$ grow at the same rate, i.e.,

$$s_o + s_r \dot{r}^{\odot} + s_y \dot{y}^{\odot} = \dot{s}^{\odot} = \dot{i}^{\odot} = i_o + i_r \dot{r}^{\odot}, \tag{2.60}$$

where $s_r, s_y > 0$, and $i_r < 0$, are given behavioral parameters; $s_o > 0$ if there is a positive growth in the sum of government taxes and foreign imports; $i_o > 0$ if there is a positive growth in the sum of self-initiated private investments, government expenditures, and exports to foreign countries). As investments $i = \frac{dK}{dt}$ contribute to capital accumulation, we have

$$\dot{i}^{\odot} = \dot{K}^{\odot} + J, \tag{2.61}$$

where J is a given parameter taking into account of capital depreciation and the discrepancies in

$$\frac{\frac{d^2K}{dt^2}}{\frac{dK}{dt}} - \frac{\frac{dK}{dt}}{K}. \tag{2.62}$$

Yet from the previous Example 2 the demand for capital K must be in optimal ratios to labor L and raw materials T in accordance with the interest rate r, the wage rate w, and the price of raw materials or "rent" v in effect, i.e.,

$$\dot{K}^{\odot} = \dot{L}^{\odot} + \sigma_{KLw} \cdot \left(\dot{w}^{\odot} - \dot{r}^{\odot}\right) + \sigma_{KLv} \cdot \left(\dot{v}^{\odot} - \dot{r}^{\odot}\right) + k_o, \text{ and} \tag{2.63}$$

$$\dot{K}^{\odot} = \dot{T}^{\odot} + \sigma_{KTw} \cdot \left(\dot{w}^{\odot} - \dot{r}^{\odot}\right) + \sigma_{KTv} \cdot \left(\dot{v}^{\odot} - \dot{r}^{\odot}\right) + t_o, \tag{2.64}$$

where $\sigma_{KLw} > 0, \sigma_{KTv} > 0$, σ_{KLv}, and σ_{KTw} are given technological parameters; k_o, and t_o represent shifts in technology toward using more K if positive. The supplies of L and T are positively related to their prices w and v; thus,

$$\dot{L}^{\odot} = L_o + L_w \cdot \dot{w}^{\odot}, \text{ and} \tag{2.65}$$

$$\dot{T}^{\odot} = T_o + T_v \cdot \dot{v}^{\odot}, \tag{2.66}$$

where $L_w, T_v > 0$ are given behavioral parameters; $L_o > 0$ would mean a growth of the labor force due to population growth and $T_o < 0$ would mean a depletion of natural resources. In this way, relative derivatives readily model a system of general equilibrium economic growth, where the eight key macroeconomic variables – inflation rate $\dot{P}^{\odot}$, economic growth rate $\dot{y}^{\odot}$, effective growth rate of interest $\dot{r}^{\odot}$, capital growth rate $\dot{K}^{\odot}$, effective growth rate of wage $\dot{w}^{\odot}$, labor employment growth rate $\dot{L}^{\odot}$, effective growth rate of rent $\dot{v}^{\odot}$, and raw materials growth rate $\dot{T}^{\odot}$ – are solutions of the following eight simultaneous linear equations:

$$s_o + s_r\dot{r}^{\odot} + s_y\dot{y}^{\odot} = i_o + i_r\dot{r}^{\odot} = \dot{i}^{\odot}, \tag{2.67}$$

$$\dot{i}^{\odot} = \dot{K}^{\odot} + J, \tag{2.68}$$

$$\dot{M}^{\odot} - \dot{P}^{\odot} = l_o + l_y\dot{y}^{\odot} + l_r\dot{r}^{\odot}, \tag{2.69}$$

$$\dot{y}^{\odot} = \eta_K\dot{K}^{\odot} + \eta_L\dot{L}^{\odot} + \eta_T\dot{T}^{\odot} + \dot{A}^{\odot}, \tag{2.70}$$

$$\dot{K}^{\odot} = \dot{L}^{\odot} + \sigma_{KLw} \cdot \left(\dot{w}^{\odot} - \dot{r}^{\odot}\right) + \sigma_{KLv} \cdot \left(\dot{v}^{\odot} - \dot{r}^{\odot}\right) + k_o, \tag{2.71}$$

$$\dot{K}^{\odot} = \dot{T}^{\odot} + \sigma_{KTw} \cdot \left(\dot{w}^{\odot} - \dot{r}^{\odot}\right) + \sigma_{KTv} \cdot \left(\dot{v}^{\odot} - \dot{r}^{\odot}\right) + t_o, \tag{2.72}$$

$$\dot{L}^{\odot} = L_o + L_w \cdot \dot{w}^{\odot}, \tag{2.73}$$

$$\text{and } \dot{T}^{\odot} = T_o + T_v \cdot \dot{v}^{\odot}. \tag{2.74}$$

Equations (2.67) and (2.68) equate capital supply to capital demand; equation (2.69) equates the given money supply to money demand; equation (2.70) specifies the production technology; equations (2.71) and (2.72) yield the demand for labor and raw materials, and equations (2.73) and (2.74) give the supplies of labor and raw materials. In this way, economic growth, inflation, etc. can be solved for and simulated by varying all the parameter values.

Example 4. Consider the dynamics of a general equilibrium of n markets as cleared by the prices $\mathbf{p}(t) \equiv (p_i(t))_{i=1}^n$, where the demand D_i for goods i and the supply S_i are functions of $\mathbf{p}(t)$; i.e., $\forall i = 1, \cdots, n$

$$\dot{p}_i(t) = \rho_i\left(D_i(\mathbf{p}(t)) - S_i(\mathbf{p}(t))\right), \tag{2.75}$$
$$\text{where } \rho_i \in C^3(\mathbb{R}, \mathbb{R}),\ \rho_i(0) = 0, \text{ and } \rho_i' > 0.$$

Assume that at $\mathbf{p}^*$, $\forall i = 1, \cdots, n$ $D_i(\mathbf{p}^*) = S_i(\mathbf{p}^*) = Q_i^*$; then linearization at $\mathbf{p}^*$ yields

$$\dot{p}_i(t) \simeq \rho_i'(0) \cdot \sum_{j=1}^n \left(\frac{\partial D_i}{\partial p_j} - \frac{\partial S_i}{\partial p_j}\right)\left(p_j(t) - p_j^*\right), \tag{2.76}$$

or

$$\begin{pmatrix} \frac{\dot{p}_1(t)}{p_1^*} \\ \vdots \\ \frac{\dot{p}_n(t)}{p_n^*} \end{pmatrix} \tag{2.77}$$
$$\simeq \operatorname{diag}\left(\frac{Q_1^*}{p_1^*}\rho_1'(0), \cdots, \frac{Q_n^*}{p_n^*}\rho_n'(0)\right)$$
$$\circ \begin{pmatrix} \frac{p_1^*}{Q_1^*}\frac{\partial D_1}{\partial p_1} - \frac{p_1^*}{Q_1^*}\frac{\partial S_1}{\partial p_1} & \cdots & \frac{p_n^*}{Q_1^*}\frac{\partial D_1}{\partial p_n} - \frac{p_n^*}{Q_1^*}\frac{\partial S_1}{\partial p_n} \\ \vdots & \vdots & \vdots \\ \frac{p_1^*}{Q_n^*}\frac{\partial D_n}{\partial p_1} - \frac{p_1^*}{Q_n^*}\frac{\partial S_n}{\partial p_1} & \cdots & \frac{p_n^*}{Q_n^*}\frac{\partial D_n}{\partial p_n} - \frac{p_n^*}{Q_n^*}\frac{\partial S_n}{\partial p_n} \end{pmatrix} \circ \begin{pmatrix} \frac{p_1(t)-p_1^*}{p_1^*} \\ \vdots \\ \frac{p_n(t)-p_n^*}{p_n^*} \end{pmatrix}.$$

As such, relative derivatives lend themselves to a direct computation of the dynamic stability of the system by examining the eigenvalues of the matrix product

$$\operatorname{diag}\left(\frac{Q_1^*}{p_1^*}\rho_1^{'}\left(0\right),\cdots,\frac{Q_n^*}{p_n^*}\rho_n^{'}\left(0\right)\right)$$
$$\circ\begin{pmatrix} \frac{p_1^*}{Q_1^*}\frac{\partial D_1}{\partial p_1}-\frac{p_1^*}{Q_1^*}\frac{\partial S_1}{\partial p_1} & \cdots & \frac{p_n^*}{Q_1^*}\frac{\partial D_1}{\partial p_n}-\frac{p_n^*}{Q_1^*}\frac{\partial S_1}{\partial p_n} \\ \vdots & \vdots & \vdots \\ \frac{p_1^*}{Q_n^*}\frac{\partial D_n}{\partial p_1}-\frac{p_1^*}{Q_n^*}\frac{\partial S_n}{\partial p_1} & \cdots & \frac{p_n^*}{Q_n^*}\frac{\partial D_n}{\partial p_n}-\frac{p_n^*}{Q_n^*}\frac{\partial S_n}{\partial p_n} \end{pmatrix}. \quad (2.78)$$

3. Concluding Remark

This paper has made theoretical economic analyses quantitative by the construct of the relative derivative. To the extent that the study of equilibrium is common to many fields which at the same time are encumbered with a multitude of measuring units, our work here serves as an illustration of how relative derivatives can be used in the general domain of science.

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