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STRUCTURE PROPERTIES OF THE CONE BEAM TOMOGRAPHY MATRIX IN CYLINDRICAL COORDINATES

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Abstract: Iterative regularization methods and algebraic techniques are very powerful in the solution of complex tomographic problems because of better noise tolerance and possibility of accurate modeling of physical phenomena. The main drawback of using algebraic routines in practical situations are high computational complexity and prohibitive storage requirements. In this work we show how the representation of the object by means of an alternative coordinate system, such as the cylindric system, gives raise to highly sparse block structured matrices. The structural properties of the projection matrix allow us to identify a minimum number of elements that must be computed and stored in order to operate on the whole matrix by means of efficient methods based on matrix vector products.

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1. Introduction

The three dimensional tomography problem deals with the recovery of volume information from line integrals. The nature of the information retrieved depends on the quality of the data measured during the scanning process. Different

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tomographic problems have their own model equations and solution algorithms, see [6].

The cone beam three dimensional tomography problem can be formalized by the Fredholm integral equation:

$$g(s) = \int_{\Omega} k(r,s) f(r) dr \,, \tag{1}$$

where $\Omega \in \mathbb{R}^3$ is a closed compact set, f represents the unknown object whose projections g are measured at points s. In the case of X rays transmission tomography, the values g consist of a set of projections of the object f obtained rotating the X rays source around the object on a circular orbit.

Current cone beam reconstruction algorithms are based either on inversion of the 3D Radon Transform or on the solution of the linear system obtained discretizing the ill–posed problem (1) or a regularized modification. In the first approach, modified filtered backprojections are applied, see [3], [2]. Difficulties arise when limited projection set is used and, moreover, the physical phenomena taking place during the acquisition phase are not always suitably modeled.

In the second case, algebraic reconstruction methods (see [5]), as well as iterative regularization methods (see [4], [7]) are applied to the linear system obtained discretizing (1). This approach has reportedly a number of advantages over the filtered backprojections such as better noise tolerance and possibility of accurate modeling of physical phenomena. So far the main drawback of using algebraic routines in practical situations are high computational complexity and prohibitive storage requirements.

In the case of some specific tomographic problems, such as Single Photon Emission Tomography (SPECT), with coarse vertical resolution, the object is reconstructed in horizontal independent slices with moderate distortions of the overall result, see [8]. One way to treat efficiently the three dimensional SPECT problem is that of computing and storing only a subset of the projection matrix relative to the vertical direction, see [1].

In the case of three dimensional X rays cone beam tomographic problem the storage of entire projection matrix would be impossible for even medium size problems. The main purpose of this work to investigate the structural properties of the projection matrix which is obtained by dicretizing the object function f with a cylindric coordinate system. This choice allows optimal exploiting of the invariance properties of the scanning process, obtaining a projection matrix which has a highly structured block form. Each block can be splitted into two antidiagonal nonzero submatrices whose elements are obtained by row and column permutations. The structural properties of the projection matrix allow us to identify a minimum number of elements that must be computed and stored in order to operate on the whole matrix. With this analysis it is easy to write basic linear algebra algorithms that have low computational complexity and storage requirements. These modules can be suitably used by every algorithm that requires a projection (projection matrix times vector product) or backprojection (transpose projection matrix times vector product) step.

In Section 2 the coordinates systems are introduced and the properties of the projection of single points are proved together with the kernel analytic equation. In Section 3 the matrix of the linear system is derived an its structural properties are proved. The structure of the matrix is analyzed and some practical consequences are reported in Section 4.

2. Mathematical Model

In this section we define the mathematical model of the data acquisition process. We consider a cartesian coordinate system (O, x, y, z) whose origin O is placed in correspondence of the rotation center of the X rays source (S_r), in such a way that the orbit described by S_r around the object is a circumference of radius ρ_S on the plane $z = 0$. For a fixed value of the rotation angle θ we define a projection plane T_θ orthogonal to the xy plane at distance ρ_T from the origin O. A system of cartesian coordinates (O_T, u, v) is defined on T_θ in such a way that v is parallel to the z axis, u lies on the xy plane and the coordinates of O_T in the (O, x, y, z) system are: $(\rho_T \cos\theta, \rho_T \sin\theta, 0)$. The source S_r is placed at angle $\theta + \pi$ and has coordinates: $(-\rho_S \cos\theta, -\rho_S \sin\theta, 0)$ (see Figure 1).

We describe the object points cylindrical coordinates (σ, φ, τ) such that:

$$\varphi \in [0, 2\pi] \quad \sigma \in [0, \rho_s], \quad \tau \in [-\rho_s, \rho_s]$$

$$\text{and} \quad x = \sigma\cos\varphi, \quad y = \sigma\sin\varphi, \quad z = \tau\,. \tag{2}$$

We assume that

$$\rho_s < \min(\rho_S, \rho_T)\,, \tag{3}$$

so that the projection source and planes T_θ do not cross the object.

With this coordinate change, the integral equation (1) assumes the following

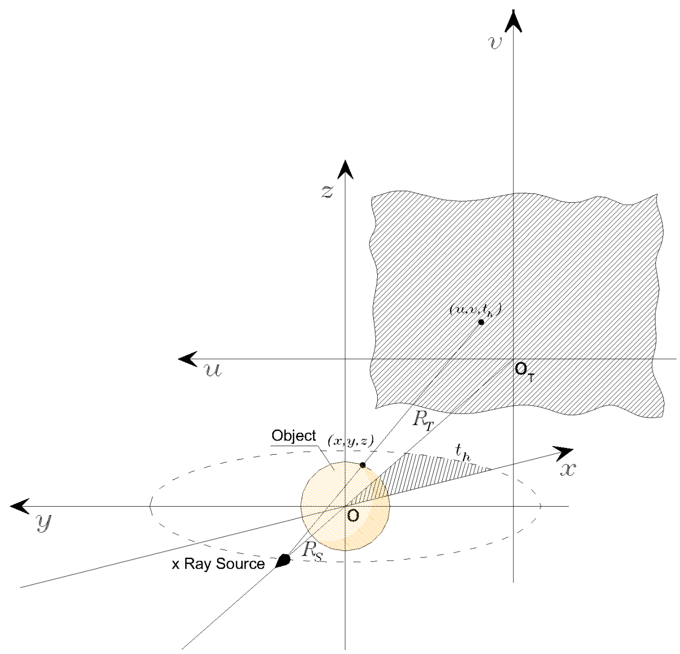

Figure 1: Coordinate systems of the object and projection plane with source rotation angle $t_h \equiv \theta$. In the figure $R_S \equiv S_r$

expressions:

$$g(u, v, \theta) = \int_{-\rho_s}^{\rho_s} \int_0^{2\pi} \int_0^{\rho_s} k_c(\sigma, \varphi, \tau, u, v, \theta) F(\sigma, \varphi, \tau) d\sigma d\varphi d\tau \;, \qquad (4)$$

where $F(\sigma, \varphi, \tau) = f(\sigma \cos \varphi, \sigma \sin \varphi, \tau)\sigma^2$ and k_c is the kernel function whose analytic expression is obtained in Theorem 2 in such a way that each measured value $g(u, v, \theta)$ represents the integral along the projection ray departing from S_r and hitting the projection plane T_θ at position (u, v). In this case only the geometrical system response is taken into account while other physical phenomena such as attenuation or scattering are left for future developments.

The values of (u, v) for the projection of the point (σ, φ, τ) are defined in the following theorem.

Theorem 1. *Let us define the functions*

$$Y_u : (\sigma, \varphi, \tau, \theta) \longrightarrow u_\theta, \qquad Y_v : (\sigma, \varphi, \tau, \theta) \longrightarrow v_\theta$$

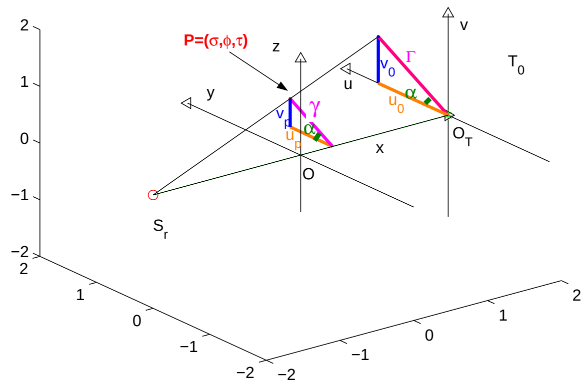

Figure 2: Projection of the point (σ, ϕ, τ) on T_0 $(\theta = 0)$

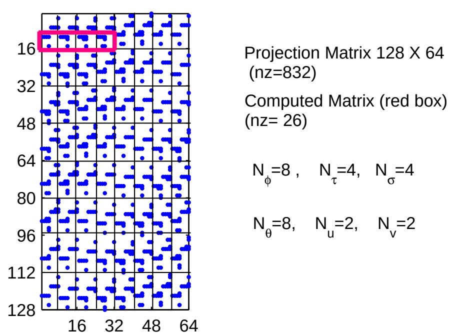

Figure 3: Plot of the non zero elements (nz) of the whole projection matrix with respect to the sub matrices A_j, $j = 1, \ldots, N_\varphi/2$

that map the object point (σ, φ, τ) *into its cone beam projection coordinates* (u_θ, v_θ) *on the plane* T_θ*, then*

$$u_\theta = \frac{\rho_S + \rho_T}{\rho_S + \sigma\cos(\varphi + \theta)}\sigma\sin(\varphi + \theta), \qquad v_\theta = \frac{\rho_S + \rho_T}{\rho_S + \sigma\cos(\varphi + \theta)}\tau\,. \tag{5}$$

The following properties hold:

(i) $Y_u(\sigma,\varphi,\tau,\theta) = Y_u(\sigma,\varphi+\theta,\tau,0)$, $Y_v(\sigma,\varphi,\tau,\theta) = Y_v(\sigma,\varphi+\theta,\tau,0)$;

(ii) $Y_u(\sigma,\varphi,\tau,\theta) = -Y_u(\sigma,2\pi-(\varphi+\theta),\tau,0)$;

(iii) $Y_v(\sigma,\varphi,\tau,\theta) = -Y_v(\sigma,\varphi+\theta,-\tau,0)$.

Proof. Observing Figure 2 we can see that, in the case $\theta = 0$, the coordinates (u_0, v_0) of the projection of the point $P = (\rho,\varphi,\tau)$ onto T_0 are given by:

$$u_0 = \Gamma\cos\alpha\,, \quad v_0 = \Gamma\sin\alpha. \tag{6}$$

Let u_p, v_p be the coordinates of the point P onto the plane containing P and parallel T_0, then:

$$u_p = \gamma\cos\alpha\,, \quad v_p = \gamma\sin\alpha\,. \tag{7}$$

Substituting (7) in (6) we obtain

$$u_0 = \frac{\Gamma}{\gamma}u_p\,, \quad v_0 = \frac{\Gamma}{\gamma}v_p\,. \tag{8}$$

Since the plane T_0 is parallel to the plane holding the directions u_p, v_p the following proportion holds:

$$\frac{\Gamma}{\gamma} = \frac{\rho_S+\rho_T}{\rho_S+\sigma\cos\varphi}. \tag{9}$$

Substituting (9) into (8) and expressing u_p, v_p in cylindric coordinates, we have:

$$u_0 = \Delta\cdot\sigma\sin\varphi\,, \quad v_0 = \Delta\cdot\tau\,, \quad \Delta = \frac{\rho_S+\rho_T}{\rho_S+\sigma\cos\varphi}\,. \tag{10}$$

When $\theta > 0$ the projection onto T_θ is equivalent to projecting $(\sigma_\theta,\varphi_\theta,\tau_\theta)$ onto T_0, where:

$$\sigma_\theta = \sigma, \quad \varphi_\theta = \varphi+\theta, \quad \tau_\theta = \tau\,, \tag{11}$$

then applying the relations (11) into (10) we obtain (5).

In order to prove the properties (i)-(ii)-(iii) we observe that, from the assumptions (3):

$$\frac{\rho_S+\rho_T}{\rho_S+\sigma\cos(\varphi+\theta)} > 0\,.$$

Then *(iii)* follows immediately.

The property (i) follows from (5). In the case (ii), we have:

$$Y_u(\sigma,2\pi-(\varphi+\theta),\tau,0) = \frac{\rho_S+\rho_T}{\rho_S+\sigma\cos(2\pi-(\varphi+\theta))}\sigma\sin(2\pi-(\varphi+\theta))$$

since $\sin(2\pi-(\varphi+\theta)) = -\sin(\varphi+\theta)$ and $\cos(2\pi-(\varphi+\theta)) = \cos(\varphi+\theta)$ we have:

$$Y_u(\sigma, 2\pi - (\varphi + \theta), \tau, 0) = -\frac{\rho_S + \rho_T}{\rho_S + \sigma\cos(\varphi + \theta)}\sigma\sin(\varphi + \theta)$$
$$= -Y_u(\sigma, \varphi, \tau, \theta)\,. \quad \square$$

Theorem 2. *The kernel of the integral equation (4) is given by:*

$$k_c(\sigma, \varphi, \tau, u, v, \theta) = \delta\left(\frac{\sigma\cos(\varphi + \theta) + \rho_S}{\rho_T + \rho_S} - 2\frac{\sigma\sin(\varphi + \theta)}{u} + \frac{\tau}{v}\right), \tag{12}$$

where δ is the Dirac impulsive function

$$\delta(w) = \begin{cases} 1\,, & \text{if } w = 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Proof. We start by deriving the equation of the projection ray departing from S_r and hitting the projection plane T_θ at position (u, v). If $\theta = 0$ than each point u, v of the plane T_θ is described in the (O, x, y, z) system by the coordinates (ρ_T, u, v). The equation of the line starting from the source $(-\rho_S, 0, 0)$ and hitting the plane T_θ at (ρ_T, u, v) is:

$$\frac{x + \rho_S}{\rho_T + \rho_S} - 2\frac{y}{u} + \frac{z}{v} = 0\,.$$

Changing to the cylindric coordinate system (2) we have:

$$\frac{\rho_S + \sigma\cos\varphi}{\rho_T + \rho_S} - 2\frac{\sigma\sin\varphi}{u} + \frac{\tau}{v} = 0\,.$$

When $\theta \neq 0$ we use the previous relation with the rotated coordinates (11), getting:

$$\frac{\rho_S + \sigma\cos(\varphi + \theta)}{\rho_T + \rho_S} - 2\frac{\sigma\sin(\varphi + \theta)}{u} + \frac{\tau}{v} = 0\,. \tag{13}$$

Now we can define the integral of the object function $\sigma^2 F(\sigma, \varphi, \tau)$ along the line (13) by means of the impulse Dirac function as given in (12). $\square$

3. Structure of the Projection Matrix

In this section we analyze the structure and properties of the matrices obtained discretizing the integral equation (4). Rotating the source S_r of angles

$$\theta_n = (n - 1) \cdot \Delta_\theta, \quad \Delta_\theta = \frac{2\pi}{N_\theta}, \quad n = 1, \ldots, N_\theta\,, \tag{14}$$

and sampling the projection plane into a grid of $(2 \cdot N_u) \times (2 \cdot N_v)$ points of coordinates (u_ℓ, v_m):

$$\begin{aligned} u_\ell &= \ell \cdot \Delta_u,\ \ \ell = -N_u + 1, \ldots, N_u, \\ v_m &= m \cdot \Delta_v,\ \ m = -N_v + 1, \ldots, N_v\,, \end{aligned} \tag{15}$$

we obtain the discrete observations $g_{\ell,m,n}$. The object is discretized by means of a grid in the cylindric coordinates $(\sigma_i, \varphi_j, \tau_k)$ where

$$\begin{aligned} &\sigma_i = i \cdot \Delta_\sigma\,, \quad \Delta_\sigma = \rho_S / N_\sigma\,, \quad i = 0, \ldots, N_\sigma\,, \\ &\varphi_j = j \cdot \Delta_\varphi\,, \quad \Delta_\varphi = 2\pi / N_\phi\,, \quad j = 0, \ldots, N_\phi\,, \\ &\tau_k = k \cdot \Delta_\tau\,, \quad \Delta_\tau = \rho_S / q_z\,, \quad k = -q_z, \ldots, q_z\,. \end{aligned} \tag{16}$$

Discretizing the integral equation (4) in the points (14), (15) and (16) we obtain the following linear system:

$$g_{\ell,m,n} = \sum_{j=1}^{N_\phi} \sum_{k=-q_z+1}^{q_z} \sum_{i=1}^{N_\sigma} \mathcal{K}_{\ell,m,n}^{i,j,k} F(\sigma_i, \varphi_j, z_k), \qquad \begin{aligned} &\ell = -N_u + 1, \ldots, N_u, \\ &m = -N_v + 1, \ldots, N_v, \\ &n = 1, \ldots, N_\theta, \end{aligned} \tag{17}$$

where:

$$\mathcal{K}_{\ell,m,n}^{i,j,k} = \int_{\varphi_{j-1}}^{\varphi_j} \int_{\tau_{k-1}}^{\tau_k} \int_{\sigma_{i-1}}^{\sigma_i} \sigma^2 \cdot \delta \left(\frac{\sigma \cos(\varphi + \theta_n) + \rho_S}{\rho_T + \rho_S} - 2 \frac{\sigma \sin(\varphi + \theta_n)}{u_\ell} + \frac{\tau}{v_n} \right) d\sigma d\tau d\varphi.$$

Collecting the elements $\mathcal{K}_{\ell,m,n}^{i,j,k}$ into a matrix of $4 \cdot N_u \cdot N_v \cdot N_\theta$ rows and $N_\phi \cdot N_\tau \cdot N_\sigma$ columns, we can write (17) as:

$$\mathcal{K}\mathbf{f} = \mathbf{g}\,,$$

where $\mathbf{g}$ stores the values of $g_{\ell,m,n}$ and $\mathbf{f}$ is relative to the values of $F(\sigma_i, \varphi_j, \tau_k)$.

Let us suppose to organize the elements of $\mathcal{K}$ into $N_\theta \times N_\phi$ blocks called $\mathcal{K}_{n,j}$ each with $4 \cdot N_u \cdot N_v$ rows and $N_\rho \cdot N_\tau$ columns. We can prove the following properties.

Theorem 3. *If $\Delta_\varphi = \pi / q_\varphi$, $N_\varphi = 2 \cdot q_\varphi$, $q_\varphi \in \mathbb{N}$ and $N_\theta = N_\varphi$ with $\Delta_\varphi = \Delta_\theta$, then the matrix $\mathcal{K}$ can be written as follows:*

$$\mathcal{K} = \begin{pmatrix} K_1 & K_2 & K_3 & \ldots & K_{N_\varphi - 1} & K_{N_\varphi} \\ K_2 & K_3 & \ddots & K_{N_\varphi - 1} & K_{N_\varphi} & K_1 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ K_{N_\varphi - 1} & K_{N_\varphi} & K_1 & K_2 & \ddots & K_{N_\varphi - 2} \\ K_{N_\varphi} & K_1 & K_2 & K_3 & \ldots & K_{N_\varphi - 1} \end{pmatrix}, \tag{18}$$

where $K_j = \mathcal{K}_{1,j}$, $(j = 1, \dots, N_\varphi)$.

Proof. Let us write $\mathcal{K}^{i,j,k}_{\ell,m,n}$ with the following change of variable:

$$\varphi = (j-1)\cdot\Delta_\varphi + t\Delta_\varphi, \quad t \in [0,1],$$

$$\mathcal{K}^{i,j,k}_{\ell,m,n} = \int_0^1 \int_{\tau_{k-1}}^{\tau_k} \int_{\sigma_{i-1}}^{\sigma_i} \sigma^2 \delta \left(\frac{\sigma\cos((j-1+t)\cdot\Delta_\varphi + \theta_n) + \rho_S}{\rho_T + \rho_S} - 2\frac{\sigma\sin((j-1+t)\cdot\Delta_\varphi + \theta_n)}{u_\ell} + \frac{\tau}{v_n} \right) d\sigma d\tau d\varphi \,. \quad (19)$$

By the hypotheses we have that $\theta_n = (n-1)\Delta_\theta = (n-1)\Delta_\varphi$ then having $N_\varphi \cdot \Delta_\varphi = 2\pi$, we can state the following relations:

$$\begin{aligned}\cos((j-1+t)\cdot\Delta_\varphi + \theta_n) &= \cos(((j+n-1)-1)\cdot\Delta_\varphi + t\Delta_\varphi) \\ &= \cos(((j+n-N_\varphi-1)-1)\cdot\Delta_\varphi + t\Delta_\varphi)\,, \quad (20)\end{aligned}$$

and

$$\begin{aligned}\sin((j-1+t)\cdot\Delta_\varphi + \theta_n) &= \sin(((j+n-1)-1)\cdot\Delta_\varphi + t\Delta_\varphi) \\ &= \sin(((j+n-N_\varphi-1)-1)\cdot\Delta_\varphi + t\Delta_\varphi)\,. \quad (21)\end{aligned}$$

Substituting in (19) we can conclude that

$$\begin{aligned}\mathcal{K}^{i,j,k}_{\ell,m,n} &= \int_0^1 \int_{\tau_{k-1}}^{\tau_k} \int_{\sigma_{i-1}}^{\sigma_i} \sigma^2 \delta \left(\frac{\sigma\cos(((j+n-N_\varphi-1)-1)\cdot\Delta_\varphi + t\Delta_\varphi + \rho_S}{\rho_T + \rho_S} \right. \\ &\quad \left. - 2\frac{\sigma\sin(((j+n-N_\varphi-1)-1)\cdot\Delta_\varphi + t\Delta_\varphi)}{u_\ell} + \frac{\tau}{v_n} \right) d\sigma d\tau d\varphi \\ &= \mathcal{K}^{i,\bar{j},k}_{\ell,m,1}\,, \quad (22)\end{aligned}$$

where:

$$\bar{j} = \begin{cases} j+n-1\,, & \text{if } j+n-1 \le N_\varphi\,, \\ j+n-1-N_\varphi\,, & \text{otherwise}\,. \end{cases}$$

In fact we have that if $j+n-1 \ge N_\varphi + 1$ having

$$N_\varphi + 1 \le j+n-1 \le 2N_\varphi + 1\,,$$

then

$$1 \le j+n-1-N_\varphi \le N_\varphi - 1\,.$$

Observing that $K_{n,j} = \mathcal{K}_{1,\bar{j}}$ we obtain the structure in (18). □

The hypotheses $\Delta_\varphi = \Delta_\theta$ and $N_\varphi = N_\theta$ may be too restrictive in practical situations. Since Δ_θ and N_θ are given tomographic machine dependent parameters, we observe how different values of Δ_φ and N_φ modify the structure of the projection matrix.

Corollary 4. *If $\Delta_\varphi = \Delta_\theta/\lambda$ with $\lambda \in \mathbb{N}$ and $\lambda > 1$, then $N_\varphi = \lambda N_\theta$ and*

$$\mathcal{K} = \begin{pmatrix} H_1 & H_2 & H_3 & \dots & H_{N_\theta-1} & H_{N_\theta} \\ H_2 & H_3 & \iddots & H_{N_\theta-1} & H_{N_\theta} & H_1 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ H_{N_\theta-1} & H_{N_\theta} & H_1 & H_2 & \iddots & H_{N_\theta-2} \\ H_{N_\theta} & H_1 & H_2 & H_3 & \dots & H_{N_\theta-1} \end{pmatrix}, \tag{23}$$

where

$$H_j = \begin{pmatrix} \mathcal{K}_{1,(j-1)\cdot\lambda+1} & \cdots & \mathcal{K}_{1,j\cdot\lambda} \end{pmatrix} \quad j = 1, \dots, N_\theta\,. \tag{24}$$

Proof.

$$N_\varphi = \frac{2\pi}{\Delta_\varphi} = \lambda \frac{2\pi}{\Delta_\theta} = \lambda N_\theta\,.$$

Since $\theta_n = (n-1)\Delta_\theta = \lambda(n-1)\Delta_\varphi$ substituting in (19) we have that:

$$(j-1+t)\Delta_\varphi + \theta_n = (j + \lambda(n-1) - 1)\Delta_\varphi + t\Delta_\varphi\,,$$

then:

$$\mathcal{K}^{i,j,k}_{\ell,m,n} = \mathcal{K}^{i,\bar{j},k}_{\ell,m,1}$$

where:

$$\bar{j} = \begin{cases} j + \lambda(n-1)\,, & \text{if } j + \lambda(n-1) \le N_\varphi\,, \\ j - N_\varphi + \lambda(n-1)\,, & \text{otherwise}\,. \end{cases}$$

In fact if $j + \lambda(n-1) \ge N_\varphi - 1$, having

$$N_\varphi + 1 \le j + \lambda(n-1) \le N_\varphi + \lambda(N_\theta - 1)\,,$$

we have that:

$$1 \le j - N_\varphi + \lambda(n-1) \le N_\varphi - \lambda\,.$$

Assigning the elements $\mathcal{K}^{i,\bar{j},k}_{\ell,m,1}$ to N_θ blocks as in (24) we obtain the structure given in (23). □

Corollary 5. *If $\Delta_\varphi = \lambda \cdot \Delta_\theta$ with $N_\theta/\lambda \in \mathbb{N}$ and $\lambda > 1$, then $N_\varphi = N_\theta/\lambda$*

and

$$\mathcal{K} = \begin{pmatrix} Q_1 & Q_2 & Q_3 & \dots & Q_{N_\varphi - 1} & Q_{N_\varphi} \\ Q_2 & Q_3 & \ddots & Q_{N_\varphi - 1} & Q_{N_\varphi} & Q_1 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ Q_{N_\varphi - 1} & Q_{N_\varphi} & Q_1 & Q_2 & \ddots & Q_{N_\varphi - 2} \\ Q_{N_\varphi} & Q_1 & Q_2 & Q_3 & \dots & Q_{N_\varphi - 1} \end{pmatrix}, \tag{25}$$

where

$$Q_j = \begin{pmatrix} \mathcal{K}_{1,j} \\ \mathcal{K}_{2,j} \\ \vdots \\ \mathcal{K}_{\lambda,j} \end{pmatrix}, \quad j = 1, \dots, N_\varphi . \tag{26}$$

Proof.

$$N_\varphi = \frac{2\pi}{\Delta_\varphi} = \frac{2\pi}{\lambda \Delta_\theta} = \frac{N_\theta}{\lambda} .$$

Let us write the quotient q and remainder r of the division of $n-1$ by λ:

$$n - 1 = q \cdot \lambda + r , \quad r < \lambda ,$$

then

$$\begin{aligned}(j - 1 + t)\Delta_\varphi + \theta_n &= (j - 1 + t)\Delta_\varphi + (n-1)\Delta_\theta \\ &= (j - 1 + t)\Delta_\varphi + (q \cdot \lambda + r)\Delta_\theta = (j + q + t - 1)\Delta_\varphi + r\Delta_\theta . \end{aligned} \tag{27}$$

Substituting in (19) we have

$$\mathcal{K}^{i,j,k}_{\ell,m,n} = \mathcal{K}^{i,\overline{j},k}_{\ell,m,\overline{n}} ,$$

where $\overline{n} = r + 1$ and

$$\overline{j} = \begin{cases} j + q , & \text{if } j + q \le N_\varphi , \\ j + q - N_\varphi , & \text{otherwise} . \end{cases}$$

Assigning the elements $\mathcal{K}^{i,\overline{j},k}_{\ell,m,\overline{n}}$ to the N_φ blocks as in (26) we obtain the structure given in (25). □

It is possible to prove some more structure properties of the matrices K_j once a suitable row and column ordering is assigned. Let us suppose that the points (u_ℓ, v_m) of the projection plane are organized following a row lexicographic order starting from $\ell = -N_u + 1$ and $m = -N_v + 1$ and that object points are sorted using the following indexing function:

$$(j-1)\cdot N_\tau \cdot N_\varphi + (q_z - k - 1)\cdot N_\sigma + i\,, \qquad \begin{cases} i = 1,\ldots,N_\sigma\,, \\ j = 1,\ldots,N_\varphi\,, \\ k = 1,\ldots,N_\tau\,. \end{cases} \tag{28}$$

Theorem 6. *If*

$$\Delta_\tau = \frac{\rho_s}{q_\tau}\,, \quad N_\tau = 2\cdot q_\tau\,, \quad q_\tau \in \mathbb{N}\,,$$

then each matrix K_j in (18) has the following block structure:

$$K_j = \begin{pmatrix} 0 & B_j \\ A_j & 0 \end{pmatrix}\,, \tag{29}$$

where A_j, B_j have $(2N_u)\cdot N_v$ rows and $N_\rho \cdot q_\tau$ columns. The elements of B_j are obtained by column and row permutation of the elements of A_j.

Proof. We prove that the only nonzero elements on each matrix K_j are those relative to the blocks A_j and B_j.

For the column ordering (28) the first $N_\tau/2\cdot N_\sigma$ columns of each block K_j are relative to the projection of elements $(\sigma_i, \varphi_j, \tau_k)$, where $\tau_k > 0$. Then, using (5), the projection coordinates (u,v) have $v > 0$. The discrete value v_m in (15) is positive if $m \geq 1$. Now, using the row lexicographic ordering, the first $2\cdot N_u \cdot N_v$ rows of K_j are relative to indices (ℓ, m), where $-N_u + 1 \leq \ell \leq N_u$ and $-N_v + 1 \leq m \leq 0$ and therefore do not contain any projection value. On the other hand, the projection values relative to the the first $N_\tau/2\cdot N_\sigma$ columns of each K_j are stored in the rows

$$2\cdot N_u \cdot N_v + 1, \ldots, 4\cdot N_u \cdot N_v$$

relative to the indices (ℓ, m), where $-N_u + 1 \leq \ell \leq N_u$ and $1 \leq m \leq N_v$. We identify this submatrix of $2\cdot N_u\cdot N_v$ rows and $N_\tau/2\cdot N_\sigma$ columns as A_j. Following an analogous argumentation we can prove that the projection values relative to the last $N_\tau/2\cdot N_\sigma$ columns of K_j are stored in the first $2\cdot N_u\cdot N_v$ rows identified as the block B_j.

We prove that B_j contains the same values of A_j with row and column permutation. The elements of A_j are given by $\mathcal{K}^{i,j,k}_{\ell,m,1}$ with $m \leq 1$ and $k \leq 1$.

Changing the τ integration variable:

$$\tau = (k-1)\cdot\Delta_\tau + t\cdot\Delta_\tau, \quad t \in [0,1]\,,$$

we have that:

$$\mathcal{K}^{i,j,k}_{\ell,m,1} = \int_{\varphi_{j-1}}^{\varphi_j} \int_0^1 \int_{\sigma_{i-1}}^{\sigma_i} \sigma^2 \delta \left(\frac{\sigma \cos\varphi + \rho_S}{\rho_T + \rho_S} - 2\frac{\sigma \sin\varphi}{u_\ell} + \frac{(k-1)\cdot\Delta_\tau + t\cdot\Delta_\tau}{v_m} \right) d\sigma dt d\varphi \,. \quad (30)$$

Considering the relation:

$$\frac{(k-1)\cdot\Delta_\tau + t\cdot\Delta_\tau}{v_m} = \frac{-((k-1)\cdot\Delta_\tau + t\cdot\Delta_\tau)}{-v_m}$$

using (15) we have: $-v_m = v_{-m+1}$. Then

$$-((k-1)\cdot\Delta_\tau + t\cdot\Delta_\tau) = ((-k+1)-1)\cdot\Delta_\tau + \overline{t}\cdot\Delta_\tau), \quad \overline{t} \in [0,1]$$

with $\overline{t} = 1-t$. Substituting these expressions into the integral we have:

$$\mathcal{K}^{i,j,k}_{\ell,m,1} = \int_{\varphi_{j-1}}^{\varphi_j} \int_0^1 \int_{\sigma_{i-1}}^{\sigma_i} \sigma^2 \delta \left(\frac{\sigma \cos\varphi + \rho_S}{\rho_T + \rho_S} - 2\frac{\sigma \sin\varphi}{u_\ell} + \frac{(\overline{k}-1)\cdot\Delta_\tau + \overline{t}\cdot\Delta_\tau}{v_{-m+1}} \right) d\sigma d\overline{t} d\varphi \,, \quad (31)$$

where $\overline{t} = 1-t$ and $\overline{k} = 1-k$. We conclude that each element of A_j, is equal to the correspondent element of B_j obtained changing the indices (m,k) into $\overline{m} = -m+1$ and $\overline{k}$ since

$$\mathcal{K}^{i,j,k}_{\ell,m,1} = \mathcal{K}^{i,j,\overline{k}}_{\ell,\overline{m},1} \,,$$

where $m \geq 1$, $k \geq 1$, $\overline{m} \leq 0$ and $\overline{k} \leq 0$. □

Theorem 7. *With the assumptions of Theorems 3 – 6, the elements of the matrices $K_{N_\varphi - j}$ in (18) are obtained as permutation of the matrices K_j $j = 1, \ldots, N_\varphi/2$*

Proof. Using the relation (19) with $n = 1$ and $j = 1, \ldots, N_\varphi/2$ we observe that:

$$\frac{\sigma \sin((j-1+t)\cdot\Delta_\varphi + \theta_1)}{u_\ell} = \frac{-\sigma \sin((j-1+t)\cdot\Delta_\varphi)}{-u_\ell} \,.$$

From (15) follows $-u_\ell = u_{-\ell+1}$. Since $\theta_1 = 0$ we have:

$$-\sin((j-1+t)\cdot\Delta_\varphi) = \sin(2\pi + (1-j-t)\cdot\Delta_\varphi)$$

and

$$\cos((j-1+t)\cdot\Delta_\varphi) = \cos(2\pi + (1-j-t)\cdot\Delta_\varphi) \,.$$

Now having $2\pi = N_\varphi \cdot \Delta_\varphi$ we can write the previous expressions as:

$$\sin((N_\varphi - j + 1)\cdot\Delta_\varphi + \overline{t}\cdot\Delta_\varphi)$$

and

$$\cos((N_\varphi - j + 1) \cdot \Delta_\varphi + \overline{t} \cdot \Delta_\varphi)\,,$$

where $\overline{t} = 1 - t$. We have that each element of K_j for $1 \le j \le N_\varphi/2$ is equal to the correspondent element of $K_{\overline{j}}$ where $\overline{j} = N_\varphi - j + 1$ and $\overline{j} > N_\varphi/2$:

$$\mathcal{K}^{i,j,k}_{\ell,m,1} = \int_0^1 \int_{\tau_{k-1}}^{\tau_k} \int_{\sigma_{i-1}}^{\sigma_i} \sigma^2 \delta \left(\frac{\sigma \cos((\overline{j} - 1) \cdot \Delta_\varphi + \overline{t} \cdot \Delta_\varphi) + \rho_S}{\rho_T + \rho_S} \right. \\ \left. -2 \frac{\sigma \sin((\overline{j} - 1) \cdot \Delta_\varphi + \overline{t} \cdot \Delta_\varphi)}{u_\ell} + \frac{\tau}{v_n} \right) d\sigma d\tau d\varphi = \mathcal{K}^{i,\overline{j},k}_{-\ell+1,m,1}\,. \quad (32)$$

The column indices of the matrices K_j and $K_{\overline{j}}$ do not change since they depend on i and k. The rows of $K_{\overline{j}}$ are obtained by substituting ℓ with $-\ell + 1$ in the row index of K_j. Since the m index does not change, we have that $A_{\overline{j}}$ and $B_{\overline{j}}$ are obtained by row permutation of the matrices of A_j and B_j respectively. □

4. Conclusions

The main consequence of analysis performed in the previous sections is that it is possible to describe the whole projection matrix $\mathcal{K}$ in terms of the sub matrices named A_j, $1 \le j \le N_{\varphi/2}$ with great savings in term of computational work and storage occupation. In Figure 3 it is shown the whole projection matrix in a small size example with respect to the elements that must be computed (red box) in order to perform the projection/backprojection steps using the properties proved in Theorems 3, 6 and 7.

Using the result of Theorem 6, the total number of non zero elements can be obtained by counting the number of elements of each block K_j:

$$\mathcal{N}(K_j) = 2 N_u N_v N_\sigma N_\tau$$

and then multiplying by the number of blocks, i.e.:

$$\mathcal{N}(\mathcal{K}) = N_\theta N_\varphi \mathcal{N}(K_j)\,.$$

When $N_\varphi = N_\theta$, only $1 \le j \le N_{\varphi/2}$ blocks A_j are computed and stored, we have that the number of computed elements is:

$$\mathcal{N}(A_j) = \frac{\mathcal{N}(K_j)}{2} \frac{N_\varphi}{2}\,.$$

The number of saved elements is given by:

$$\mathcal{N}(\mathcal{K}) - \mathcal{N}(A_j) = N_\varphi \left(N_\theta - \frac{1}{4} \right) \mathcal{N}(K_j) \, .$$

As a measurement of the efficiency we use the following ratio:

$$E = \frac{\mathcal{N}(\mathcal{K}) - \mathcal{N}(A_j)}{\mathcal{N}(A_j)} = \frac{\left(N_\theta - \frac{1}{4}\right)}{N_\varphi} = 1 - \frac{1}{4N_\theta} \, .$$

This value is much closer to the optimal value for even small size problems.

We observe that the matrix $\mathcal{K}$ can be written in block Toeplitz form by means of permutation matrix Π:

$$\Pi\mathcal{K} = \begin{pmatrix} K_1 & K_2 & K_3 & \dots & K_{N_\varphi - 1} & K_{N_\varphi} \\ K_{N_\varphi} & K_1 & K_2 & K_3 & \dots & K_{N_\varphi - 1} \\ K_{N_\varphi - 1} & K_{N_\varphi} & K_1 & K_2 & \ddots & K_{N_\varphi - 2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ K_2 & K_3 & \ddots & K_{N_\varphi - 1} & K_{N_\varphi} & K_1 \end{pmatrix} , \qquad (33)$$

where

$$\Pi = \begin{pmatrix} I & \\ & \Upsilon \end{pmatrix}$$

and Υ has $N_\theta - 1$ non zero identity blocks of order $N_\sigma \cdot N_\tau$ on its antidiagonal position:

$$\Upsilon = \begin{pmatrix} & & & & I \\ & & & I & \\ & & \vdots & & \\ & I & & & \\ I & & & & \end{pmatrix} .$$

Our final observation concerns the possibility to devise structure properties with more general kernels. It happens often, in tomographic problems, that physical phenomena taking place during the acquisition process, may change slightly the position of the projection points. In this case, a Gaussian probability function is introduced instead of the Dirac geometric response. It is reasonable to argue that if the variance depends only in the source point distance then similar structure properties can be proved. Anyway, once the analytic kernel is obtained, the same analysis can be performed to investigate the structure of the projection matrix.

References

[1] P. Boccacci, P. Bonetto, P. Calvini, A.R. Formiconi, A simple model for the efficient correction of collimator blur in 3d spect imaging, *Inverse Problems*, **15**, No. 4 (1999), 907-930.

[2] M. Defrise, R. Clack, A cone-beam reconstruction algorithm using shift-variant filtering and cone-beam backprojection, *IEEE Trans. Med Imag.*, **13** (1994), 186-195.

[3] L.A. Feldkamp, L.C. Davis, J.W. Kress, Practical cone beam algorithm, *J. Opt. Soc. Am.*, **1**, No. 6 (1984), 612-619.

[4] M. Hanke, P.C. Hansen, Regularization methods for large-scale problems, *Surveys on Mathematics for Industry*, **3** (1993), 253-315.

[5] K. Mueller, R. Yagel, J.J. Wheller, Fast implementation of algebraic methods for 3D reconstruction from cone-beam data, *IEEE Trans. on Med. Imag* (September 1999).

[6] F. Natterer, F. Wübbeling, *Mathematical Methods in Image Reconstruction*, Monographs in Mathematical Modeling and Computation, SIAM (2001).

[7] Elena Loli Piccolomini, Fabiana Zama, The conjugate gradient regularization method, In: *Computed Tomography Problems*, *Applied Mathematics and Computation*, **102**, No. 1 (1999), 87-99.

[8] Elena Loli Piccolomini, Fabiana Zama, An experiment in parallel spect reconstruction, *Parallel Algorithms and Applications* (2002), To Appear.