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DYNAMICS OF TRAJECTORIES AND THE FINITE-DIMENSIONAL REDUCTION OF DISSIPATIVE EVOLUTION EQUATIONS

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Abstract: We consider a general class of evolution equations with nonlinear dissipation. Under minimal regularity assumptions, we show that the large-time dynamics can be uniquely described by a system of ODEs with infinite, but exponentially decaying memory. The existence of a finite-dimensional attractor is another corollary.

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1. Introduction

We study an abstract evolutionary dissipative PDE

$$d_t u + A(u) + B(u) = 0\,, \tag{1}$$

$u(t) : [0,\infty) \to H$, where H is a Hilbert space. Here $A : V \to V'$ is a nonlinear elliptic operator, V is a Banach space with $V \hookrightarrow\hookrightarrow H$, and V' its dual. We assume that

$$\begin{gathered} A(0) = 0\,, \\ \|A(u) - A(v)\|_{V'} \le c_1 \|u - v\|_V\,, \\ \langle A(u) - A(v), u - v\rangle \ge a\|u - v\|_V^2\,, \end{gathered} \tag{2}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality between V' and V. Concerning $B : V \to V'$

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– the lower-order nonlinearity – we require that

$$\begin{aligned} \|B(u)-B(v)\|_{V'} &\le b\|u-v\|_H^{\alpha}\|u-v\|_V^{1-\alpha}\,, \qquad \alpha\in(0,1]\,,\\ \langle B(u),u\rangle &\ge -\varepsilon\|u\|_V^2-K_\varepsilon\,, \qquad \varepsilon>0\,. \end{aligned} \tag{3}$$

There are a number of physically relevant examples of PDEs that either fit directly into this scheme, or can be treated along similar lines with appropriate modifications. Here we concentrate on this simple model, because the main ideas are thus not obscured by technical details.

The common topic in studying the evolution of dissipative PDEs is the ultimate finite-dimensionality of the large-time dynamics (Robinson [11], Temam [12]). The existence of global (or exponential) attractor with finite fractal dimension is one of possible results, and has been achieved for a number of equations (see also Eden et al [2]), where an abstract problem similar to our equation is treated.

On the other hand, one would like to be more explicit in showing that the long-time dynamics is governed by a finite number of degrees of freedom. Ideally, this means to rewrite our PDE as a system of ODEs. This corresponds to construction of the so-called inertial manifold, which, unfortunately, is known to exist only for a rather restricted class of equations.

There are, however, a number of results employing the fact that the global attractor has finite fractal dimension, and hence can be parametrized by a finite number of values (see e.g. Eden et al [2]). Alternatively, the dynamics can be uniquely determined by values (i.e. measurements) taken at one point at finite number of different times (the so-called Taken's embedding theorem, see Kukavica et al [6]).

Here we follow a different approach: we show that there is a finite-dimensional projection P such any $u(0)\in\mathcal{A}$ is uniquely determined by $\{Pu(t)\}_{t\le 0}$. The fact that we need complete backward history of the projection is the price we pay for the Lipschitz continuity of the inverse of this projection. This has good consequences for the projected system (e.g. the uniqueness.) Note that the projections guaranteed by the general theorems connected with the finite fractal dimension of $\mathcal{A}$ are only Hölder continuous in general (see Hunt et al [5]).

Similar construction, called "inertial manifolds with delay" was given already in Debussche et al [1], cf. also Hale et al [4]. The advantage of our approach is, in our opinion, a simpler proof, based directly on the smoothing property of the flow, see Pražák [9] and [10]. The particular novelty of the present paper is that the smoothing property is proved in the space of trajectories (cf. Málek et al [7, 8]), which enables us to cover a very general class of problems with nonlinear dissipation, under minimal regularity assumptions.

The paper is organized as follows. The present section is concluded, for the sake of completeness, with proofs of the existence and uniqueness of weak solutions to (1). The long-time dynamics is studied in Section 2. Among others, we prove the finite-dimensionality of the attractor. We also write down the equations governing the evolution of trajectories. The projection to the system of ODEs with delay is constructed in Section 3.

Theorem 1. *Let $u_0 \in H$ and $T > 0$ be given. Then there exists a weak solution*

$$\begin{aligned} u &\in L^\infty(0,T;H) \cap L^2(0,T;V)\,, \\ d_t u &\in L^2(0,T;V')\,, \end{aligned} \tag{4}$$

where $u(0) = u_0$ in the sense of a representative $u \in C([0,T];H)$.

Proof. We will show the compactness of solutions: assuming that u^n solve (1) with $u^n(0) \to u_0$, we prove that $u^n \to u$ (modulo a subsequence) and u is again a solution. This can be turned into a rigorous proof using a suitable approximation scheme. The compactness argument is also an important step in the construction of the global attractor via the method of trajectories.

Testing by u^n, we have – cf. (2), (3) –

$$\begin{aligned} \frac{1}{2} d_t \|u^n\|_H^2 + a\|u^n\|_V^2 &\le \frac{a}{2}\|u^n\|_V^2 + K_a\,, \\ d_t \|u^n\|_H^2 + a\|u^n\|_V^2 &\le 2K_a\,. \end{aligned}$$

Hence u^n satisfy the estimates in $(4)_1$. From the equation one further has $d_t u^n = -A(u^n) - B(u^n)$, hence

$$\int_0^T \|d_t u^n\|_{V'}^2 \le c\big(1 + \int_0^T \|u^n\|_V^2 + \|u^n\|_V^{2(1-\alpha)} \|u^n\|_H^{2\alpha}\big)\,.$$

The gives the second estimate in (4). The key compactness argument of the theory is the Aubin-Lions' Lemma:

$$\big\{u \in L^2(0,T;V);\ d_t u \in L^2(0,T;V')\big\} \hookrightarrow\hookrightarrow L^2(0,T;H)\,, \tag{5}$$

in effect of which we have (extracting a subsequence if necessary)

$$\begin{aligned} u^n &\to u \qquad \text{weakly in } L^2(0,T;V)\,, \\ u^n &\to u \qquad \text{strongly in } L^q(0,T;H), \quad \forall q < \infty\,, \\ d_t u^n &\to d_t u \qquad \text{weakly in } L^2(0,T;V')\,. \end{aligned} \tag{6}$$

By Arzelo-Ascoli type argument, one also gets that

$$u^n(t) \to u(t) \qquad \text{weakly in } H \text{ for } \forall t \in [0,T]\,. \tag{7}$$

To see that u indeed solves (1), observe first that

$$\|B(u^n) - B(u)\|_{V'} \le b\|u^n - u\|_H^{\alpha}\|u^n - u\|_V^{1-\alpha}\,,$$

whence

$$B(u^n) \to B(u) \qquad \text{strongly in } L^q(0,T;V'), \quad \forall q < \infty\,.$$

We also have

$$A(u^n) \to \overline{A(u)} \qquad \text{weakly in } L^2(0,T;V')\,.$$

It remains to show that $\overline{A(u)} = A(u)$. Clearly, one can let $n \to \infty$ to obtain

$$d_t u + \overline{A(u)} + B(u) = 0\,. \tag{8}$$

Let $w \in L^2(0,T;V)$ be arbitrary. We have

$$\begin{aligned} 0 &\le \int_0^T \langle A(u^n) - A(w), u^n - w\rangle \\ &= \int_0^T \langle A(u^n), u^n\rangle - \int_0^T \langle A(u^n), w\rangle - \int_0^T \langle A(w), u^n - w\rangle\,. \end{aligned}$$

Further, testing the equation for u^n by u^n gives

$$\int_0^T \langle A(u^n), u^n\rangle = -\int_0^T \langle B(u^n), u^n\rangle + \frac{1}{2}\|u^n(0)\|_H^2 - \frac{1}{2}\|u^n(T)\|_H^2\,.$$

Altogether we have

$$\begin{aligned} \frac{1}{2}\|u^n(T)\|_H^2 \le \frac{1}{2}\|u^n(0)\|_H^2 &- \int_0^T \langle B(u^n), u^n\rangle + \int_0^T \langle A(u^n), w\rangle \\ &+ \int_0^T \langle A(w), u^n - w\rangle\,. \end{aligned}$$

We can take the limit in all the terms except the left-hand side – however, here the weak convergence preserves the inequality:

$$\begin{aligned} \frac{1}{2}\|u(T)\|_H^2 \le \frac{1}{2}\|u(0)\|_H^2 &- \int_0^T \langle B(u), u\rangle + \int_0^T \langle \overline{A(u)}, w\rangle \\ &+ \int_0^T \langle A(w), u - w\rangle\,. \end{aligned}$$

Finally, testing (8) with u gives

$$\frac{1}{2}\|u(T)\|_H^2 + \int_0^T \langle\overline{A(u)}, u\rangle + \int_0^T \langle B(u), u\rangle = \frac{1}{2}\|u(0)\|_H^2 .$$

Combining the last two formulas implies

$$0 \le \int_0^T \langle\overline{A(u)} - A(w), u - w\rangle ,$$

whence, as w is arbitrary, one concludes by standard Minty's trick that $\overline{A(u)} = A(u)$. □

Theorem 2. *The weak solution is unique.*

Proof. Let $w := u - v$ be the difference of weak solutions. One has

$$\begin{aligned}\frac{1}{2}d_t\|w\|_H^2 + a\|w\|_V^2 &\le b\|w\|_H^\alpha \|w\|_V^{2-\alpha} \\ &\le \frac{a}{2}\|w\|_V^2 + c\|w\|_H^2\end{aligned}$$

by Young's inequality. Hence

$$d_t\|w\|_H^2 + a\|w\|_V^2 \le 2c\|w\|_H^2 . \tag{9}$$

From Gronwall's Lemma we have

$$\|u(t) - v(t)\|_H^2 \le \exp(2cT)\|u(s) - v(s)\|_H^2 , \tag{10}$$

for all $0 \le t - s \le T$. In particular, the uniqueness follows. □

2. Dynamics of Trajectories

We study the large-time dynamics, and thus we can restrict our attention to sufficiently well-behaved absorbing set.

Lemma 3. *For $R > 0$ large enough, the ball $\mathcal{B}_0 = \{u \in H;\ \|u\|_H^2 \le R\}$ is uniformly absorbing and positively invariant.*

Proof. As in the proof of Theorem 1, we see that every solution satisfies

$$d_t\|u\|_H^2 + a\|u\|_V^2 \le 2K_a .$$

Hence, it is enough to take $R > \sqrt{2c^2K_a/a}$, where c is the embedding constant for $V \hookrightarrow H$. □

Instead of working with the usual solution semigroup $S(t) : H \to H$, we prefer the trajectory description of the dynamics. Set

$$\mathcal{B}_\ell := \{\chi : [-\ell, 0] \to H;\ \chi \text{ is weak solution}, \chi(-\ell) \in \mathcal{B}_0\}$$

and define operators $L(t) : \mathcal{B}_\ell \to \mathcal{B}_\ell$ by $L(t)\chi = \psi$, where $\psi(s) = u(t+s)$, $s \in [-\ell, 0]$, with u being the (unique) solution such that $u(0) = \chi(0)$.

It is easy to see that $L(t)$ form a semigroup. The dynamical systems $(L(t), \mathcal{B}_\ell)$ and $(S(t), \mathcal{B}_0)$ are equivalent description of the large time dynamics of (1). However, the advantage of the trajectory approach stand out already in view of the compactness, given in the next lemma. For the future reference, we write

$$H_\ell := L^2(-\ell, 0; H)$$

– this is the topology we implicitly consider in $\mathcal{B}_\ell$.

Lemma 4. *The set $\mathcal{B}_\ell$ is compact in H_ℓ.*

Proof. Let $\{\chi_n\} \subset \mathcal{B}_\ell$. By the arguments of Theorem 1, $\chi_n \to \chi$ (modulo a subsequence), and χ is again weak solution on $[-\ell, 0]$. In particular, $\chi_n(-\ell) \to \chi(-\ell)$ weakly, and $\mathcal{B}_0$ is weakly closed. Hence $\chi \in \mathcal{B}_\ell$. □

The continuity of $L(t)$ is treated next. The key information here is the smoothing property (part (ii)), i.e. the Lipschitz continuity into the the space W_ℓ which is compactly embedded into H_ℓ.

Lemma 5. *(i) $L(t) : H_\ell \to H_\ell$ is Lipschitz continuous on $\mathcal{B}_\ell$.*

(ii) $L(\ell) : H_\ell \to W_\ell$ is Lipschitz continuous on $\mathcal{B}_\ell$, where

$$\|\chi\|^2_{W_\ell} = \int_{-\ell}^0 \|\chi\|^2_V + \|d_t\chi\|^2_{V'} .$$

Proof. Given χ, $\psi \in \mathcal{B}_\ell$, let u, v be the solutions on $[-\ell, T]$ such that $u(0) = \chi(0)$, $v(0) = \psi(0)$ and $w = u - v$.

From (10) we have

$$\|w(t+s)\|^2_H \le c\|w(s)\|^2_H ;$$

integrating over $s \in (-\ell, 0)$ gives (i).

Further, integrating (9) over (s, ℓ), where $s \in (-\ell, 0)$ is fixed, gives

$$a \int_s^\ell \|w\|^2_V \le \|w(s)\|^2_H + c_3 \int_s^\ell \|w\|^2_H .$$

Since $\|w(\tau)\|_H^2 \le c\|w(s)\|_H^2$ for $\tau \ge s$, one comes to

$$a\int_0^\ell \|w\|_V^2 \le c_4\|w(s)\|_H^2 \,.$$

Integrating over $s \in (-\ell, 0)$ gives

$$\int_0^\ell \|w\|_V^2 \le c_5 \int_{-\ell}^0 \|w\|_H^2 \,.$$

Finally, it is straightforward to deduce from (1)-(3) that

$$\int_0^\ell \|d_t w\|_{V'}^2 \le c_6 \int_0^\ell \|w\|_V^2 \,.$$

Combining the last two inequalities gives (ii). □

The existence of global attractor $\mathcal{A}_\ell$ for the semigroup $(L(t), \mathcal{B}_\ell)$ now follows by standard arguments, see e.g. Robinson [11, Theorem 10.5]. The smoothing property (Lemma 5, (ii)) ensures that $\mathcal{A}_\ell$ is finite-dimensional, in fact, an exponential attractor can be constructed (Efendiev et al [3], Málek et al [8]).

Certainly, this is a great advantage of the trajectory description. It would be considerably more difficult to prove the above compactness or the smoothing property for the semigroup $S(t)$; other regularity results for (1) would be clearly needed.

We can proceed further and write down the equation which governs the evolution of the trajectories. From one perspective, it is just a formal exercise, but it serves as an intermediate step to the system of delayed ODEs we will introduce in the last section.

Writing $\chi^t = L(t)\chi^0$, where $\chi^0 \in \mathcal{B}_\ell$ is fixed, one has

$$d_t\chi^t + \mathcal{F}(\chi^t) = 0\,; \tag{11}$$

where $\mathcal{F} : L^2(-\ell, 0; V) \to L^2(-\ell, 0; V')$ is given by

$$\big[\mathcal{F}(\chi)\big](s) = A(\chi(s)) + B(\chi(s)) \qquad s \in (-\ell, 0)\,.$$

Formally, (11) means that

$$d_t\langle\chi^t, \phi\rangle_{-\ell,0} + \langle\mathcal{F}(\chi^t), \phi\rangle_{-\ell,0} = 0 \tag{12}$$

for any $\phi \in L^2(-\ell, 0; V)$. Here $\langle\cdot,\cdot\rangle_{-\ell,0}$ is the duality

$$\langle\chi, \phi\rangle_{-\ell,0} = \int_{-\ell}^0 \langle\chi(s), \phi(s)\rangle\, ds$$

between $L^2(-\ell, 0; V')$ and $L^2(-\ell, 0; V)$. It is straightforward to deduce (12) by integrating the weak formulation of (1) over $(-\ell, 0)$.

One might even try to develop the theory of existence and uniqueness for (11). Will will not pursue this here; in fact, the existence follows directly from Theorem 1, while uniqueness is a consequence of results of the next section. It could be also interesting to consider (11) instead of (1) already on the level of modeling; this is precisely what one does in studying the problems with delay.

3. The ODE-Like Reduction

We will show that the large-time dynamics of (1) is uniquely described by system of ODEs with delay. The delay is infinite, but it fades out exponentially. The two key tools are the smoothing property, and the extension property of Lipschitz functions (cf. Wells et al [13]).

In view of the compact embedding $W_\ell \hookrightarrow\hookrightarrow H_\ell$, for any given $\varepsilon > 0$ there exists a finite-dimensional projection $P : H_\ell \to \mathbb{R}^N$ such that

$$\|\chi\|_{H_\ell} \leq \varepsilon \|\chi\|_{W_\ell} + \|P\chi\|_{\mathbb{R}^N}\,; \qquad \forall \chi \in W_\ell\,. \tag{13}$$

Recall that $\mathcal{A}_\ell$ (the attractor of trajectories) is fully invariant, i.e. the solutions of (11) can be continued for $t < 0$, though the extension is not necessarily unique. More precisely, we set

$$\mathcal{A}^{\leq 0} := \{\chi^t : (-\infty, 0] \to \mathcal{B}_\ell;\ \chi^t \text{ solves (11) and } \chi^0 \in \mathcal{A}_\ell\ \}\,.$$

Further, we define

$$\mathcal{T} := \{p(t) : (-\infty, 0] :\to \mathbb{R}^N;\ \exists \chi^t \in \mathcal{A}^{\leq 0} \text{ such that } p(t) = P\chi^t\}\,.$$

It turns out that $\mathcal{T}$ contains enough information to recover uniquely the final trajectory. This is the following key lemma.

Lemma 6. *For any $p \in \mathcal{T}$, there exists unique $\chi^0 \in \mathcal{A}^{\leq 0}$ such that $p(t) = P\chi^t$, $t \leq 0$. The mapping $E : p \mapsto \chi^0$ is Lipschitz from X to W_ℓ, where X is given by*

$$\|p\|_X = \sup_{s \leq 0} \mathrm{e}^{\gamma s} \|p(s)\|_{\mathbb{R}^N}$$

with suitable $\gamma > 0$.

Proof. Let λ be Lipschitz constant of $L(\ell)$, related to Lemma 5 (ii). Let P be such that (13) holds with $\varepsilon = 1/2\lambda$, and let χ^t, $\psi^t \in \mathcal{A}^{\leq 0}$ be given. Write $p(t) = P\chi^t$, $q(t) = P\psi^t$.

One has

$$\begin{aligned}\|\chi^0-\psi^0\|_{W_\ell} &\le \lambda\|\chi^{-\ell}-\psi^{-\ell}\|_{H_\ell}\\ &\le \frac{1}{2}\|\chi^{-\ell}-\psi{-\ell}\|_{W_\ell}+\lambda\|p(-\ell)-q(-\ell)\|_{\mathbb{R}^N}\\ &\le 2^{-n}\|\chi^{-n\ell}-\psi^{-n\ell}\|_{W_\ell}+\sum_{k=1}^{n}\lambda 2^{1-k}\|p(-k\ell)-q(-k\ell)\|_{\mathbb{R}^N}\\ &=: D_1(n)+D_2(n)\,.\end{aligned}$$

Obviously, $D_1(n)\le c2^{-n}\to 0$ for $n\to\infty$, while

$$D_2(n)\le c\sum_{k=1}^{n}2^{-k}\mathrm{e}^{\gamma k\ell}\|p-q\|_X\le\tilde{c}\|p-q\|_X$$

if γ is small enough. □

This enables us to reduce the dynamics to $\mathbb{R}^N$. We apply the projection P to (11); more precisely, we use (12) with $\phi=\phi_1,\ldots,\phi_N$ which form the basis of PH_ℓ. We can assume that $\phi_i\in L^2(-\ell,0;V)$.

Thus we obtain

$$d_tp(t)+P\mathcal{F}(E(p^{\le t}))=0\,;\qquad t<0$$

for any $p\in\mathcal{T}$. Here $p^{\le t}$ is understood as an element of X, describing the history up to time t:

$$[p^{\le t}](s)=p(t+s)\,,\quad s\le 0\,.$$

Because $P\circ\mathcal{F}\circ E: X\to\mathbb{R}^N$ is Lipschitz mapping, we can use Shane's Lemma to extend it to the Lipschitz map $F: X\to\mathbb{R}^N$ (see Pražák [9] for details.) Hence the dynamics of $p\in\mathcal{T}$ is captured by the delayed ODE system

$$d_tp(t)+F(p^{\le t})=0\,. \tag{14}$$

Remark that decaying weight of the norm X means that the memory fades out exponentially. Such problem was studied in general in Pražák [10]. It is important to note that F is Lipschitz continuous, hence (14) is uniquely solvable.

It would be, of course, more desirable to reduce the dynamics of (1) to a (uniquely solvable) system of ODEs without any delay. This is already equivalent to the existence of inertial manifold, which, unfortunately, is known to exist only for a rather limited class of problems. In view of the generality of (1), our result seems reasonable. Although the dynamics of (14) is still infinitely-dimensional, the decaying weight shows that it is in some sense close to the ODE dynamics. The exponential attractor for (14) (cf. Pražák [10, Theorem 4.5]) can be seen as another evidence.

Let us summarize our considerations in the final theorem. It shows how the dynamics on $\mathcal{A}_\ell$ is uniquely determined by the evolution of (14).

Theorem 7. *Let $\chi^0 \in \mathcal{A}_\ell$ be given trajectory, let $\chi^t \in \mathcal{A}^{\leq 0}$ be its backward extension, and let $p \in \mathcal{T}$ be such that $p(t) = P\chi^t$.*

Extend $p(t)$ to $t > 0$ by solving (14). Then $p^{\leq t} \in \mathcal{T}$, and $E(p^{\leq t}) = L(t)\chi^0$ for all $t > 0$.

Proof. We emphasize that neither the backward extension χ^t, nor the corresponding $p \in \mathcal{T}$ are uniquely determined by χ^0.

Nevertheless, by the construction, the projection is always a solution of (14). On the other hand, (14) is uniquely solvable. Hence the conclusion. □

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References

[1] A. Debussche, R. Temam, Some new generalizations of inertial manifolds, *Discrete Contin. Dynam. Systems*, **2**, No. 4 (1996), 543-558.

[2] A. Eden, C. Foias, B. Nicolaenko, R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Masson, Paris (1994).

[3] M. Efendiev, A. Miranville, S. Zelik, Exponential attractors for a nonlinear reaction-diffusion system in $\mathbf{R}^3$, *C. R. Acad. Sci. Paris Sér. I Math.*, **330**, No. 8 (2000), 713-718.

[4] J.K. Hale, G. Raugel, Regularity, determining modes, Galerkin methods, *J. Math. Pures Appl.*, **82**, No. 9 (2003), 1075-1136.

[5] B.R. Hunt, V.Yu. Kaloshin, Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces, *Nonlinearity*, **12**, No. 5 (1999), 1263-1275.

[6] I. Kukavica, J.C. Robinson, Distinguishing smooth functions by a finite number of point values, and a version of the Takens Embedding Theorem, *Phys. D*, **196** (1-2) (2004), 45-66.

[7] J. Málek, J. Nečas, A finite-dimensional attractor for three-dimensional flow of incompressible fluids, *J. Differential Equations*, **127**, No. 2 (1996), 498-518.

[8] J. Málek, D. Pražák. Large time behavior via the method of l-trajectories, *J. Differential Equations*, **181**, No. 2 (2002), 243-279.

[9] D. Pražák. On reducing the 2d Navier-Stokes equations to a system of delayed ODEs, *Progr. Nonlinear Differential Equations Appl.*, **64** (2005), 403-411.

[10] D. Pražák. On the dynamics of equations with infinite delay, *Cent. Eur. J. Math.*, **4**, No. 4 (2006), 635-647.

[11] J.C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, Cambridge (2001).

[12] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York (1997).

[13] J.H. Wells, L.R. Williams, *Embeddings and Extensions in Analysis*, Springer Verlag, New York (1975).