International Journal of Pure and Applied Mathematics
Volume 40 No. 1 2007, 77-88

UNIFIED FORMULAS FOR INTEGER AND FRACTIONAL ORDER SYMBOLIC DERIVATIVES AND INTEGRALS OF THE POWER-INVERSE TRIGONOMETRIC CLASS I

Mhenni M. Benghorbal
Department of Mathematics and Statistics
Concordia University
Montreal, QC, H3G 1M8, CANADA
e-mails: mbenghorbal@gmail.com, mhenni@mathstat.concordia.ca

Abstract: A complete solution to the problem of symbolic differentiation and integration of any order (integer, fractional, or real) of the *power-inverse trigonometric classes* has been given. In this work, we tackle the *power-inverse tangential class*

$$\left\{ f(x) : f(x) = \sum_{j=0}^{\ell} p_j(x^{\alpha_j}) \arctan\left(\beta_j x^{\gamma_j}\right) , \quad \alpha_i \in \mathbb{C} \,, \beta_i \in \mathbb{C}\backslash\{0\} \,, \gamma_i \in \mathbb{R}\backslash\{0\} \right\}, \quad (1)$$

where p_j's are polynomials of certain degrees. We give a unified formula for symbolic derivatives and integrals of any order. The approach does not depend on integration techniques. The formula, in general, is in terms of the H-function, but in many cases can be simplified for less general functions. Arbitrary (integer, fractional or real) order of differentiation is found according to the Riemann-Liouville definition, whereas we adopt the generalized Cauchy n-fold integral definition for arbitrary order of integration.

AMS Subject Classification: 26A33
Key Words: fractional derivatives, fractional integrals, H-function, G-function

1. Introduction

This is a continuation of a series of papers on finding unified formulas for integer

Received: August 7, 2007

and arbitrary order symbolic derivatives and integrals for classes of functions. The motivation of this work stems from the area of symbolic computations. The first paper covered the class of rational polynomials [5]. The second paper tackled the class of meromorphic functions with an infinite number of poles [3]. In the third paper, a unified formula for the *power-exponential class* has been given [4].

In this paper, we introduce a theorem that gives a unified formula for integer and arbitrary order symbolic derivatives and integrals of the *power-inverse tangential class*. This unified formula solves the whole problem of arbitrary differentiation and integration for this class of functions. This means no integration techniques are needed. These formulas enhance the power of integration and differentiation in computer algebra systems (CAS). In fact, it is a new way of finding derivative and anti-derivatives of elementary and special functions.

2. Riemann-Liouville Fractional Derivative Definition

The most widely known definition of the fractional derivative is the Riemann-Liouville definition (R-L), see [8], [11], [10]. It appears as a result of unification of the notions of integer-order integration and differentiation. The definition is given by

$$\mathcal{R}^q f = \frac{1}{\Gamma(k-q)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-q-1} f(t)\, dt \quad (k-1 < q < k)\,, \tag{2}$$

where $k = \lceil q \rceil$ and $f(x)$ is a function with a weak singularity over the interval of integration. Note that if $f(x)$ is continuous over the interval $[a, x]$, then by letting $q \to k$, one gets $f^{(k)}(x)$.

2.1. R-L Fractional Derivative of $(x-a)^m$

We are interested in the fractional derivative of the function

$$f(x) = (x-a)^m\,, \tag{3}$$

because of its use later. Substituting (3) in (2) yields

$$\mathcal{R}^q\,(x-a)^m = \frac{1}{\Gamma(\alpha)} \frac{d^k}{dx^k} \int_a^x (x-y)^{\alpha-1} (y-a)^m\, dy, \quad m > -1\,, \tag{4}$$

where $k = \lceil q \rceil$, $\alpha = k - q \in (0,1)$, $k-1 < q \leq k$, and $x - a > 0$.

Using the substitution $y = xz$ in the integral in the above equation simplifies it in terms of the Beta function [1] to

$$\mathcal{R}^q\,(x-a)^m = \frac{1}{\Gamma(\alpha)}\frac{d^k}{dx^k}[\,\beta(\alpha,m+1)(x-a)^{\alpha+m}\,] \tag{5}$$

$$= \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)}(\alpha+m)^{\underline{k}}\,(x-a)^{m-q}\,, \tag{6}$$

where

$$(\alpha+m)^{\underline{k}} = (\alpha+m)\,(\alpha+m-1)...(\alpha+m-(k-1))\,, \tag{7}$$

and

$$\beta(u,v) = \int_{x=0}^{\infty} x^{u-1}(1-x)^{v-1}dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\,. \tag{8}$$

Therefore, based on equation (6), we observe two things: the fractional derivative of $(x-a)^m$ is zero for $m=-\alpha,-\alpha+1,...,-\alpha+k-1$, and the condition $m>-1$ in equation (4) can be replaced by a weaker condition, $m\neq -1,-2,...$. The extension of m is justified by means of analytic continuation of the Γ function. So equation (6) takes the form

$$\mathcal{R}^q\,(x-a)^m = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m-q+1)}(x-a)^{m-q}, & m\neq -1,-2,\dots\,, \\ 0, & m=-\alpha,-\alpha+1,...,-\alpha+k-1\,. \end{cases} \tag{9}$$

In the above if q replaced with $-q$ it results in getting integrals of any order of the function $(x-a)^m$ and it is given by

$$\frac{\Gamma(m+1)}{\Gamma(m+q+1)}(x-a)^{m+q}\,. \tag{10}$$

The last formula can be proved by using the following definition of arbitrary integration [10]

$$f^{(-q)}(x) = \frac{1}{\Gamma(q)}\int_0^x (x-t)^{q-1}f(t)dt\,, \quad q>0\,. \tag{11}$$

3. Mellin-Barnes Integrals

Mellin-Barnes integrals [9] have the form

$$\frac{1}{2\pi i}\int_C g(s)z^s ds\,, \tag{12}$$

where the contour C is a suitable contour, $i=\sqrt{-1}$, $z \neq 0$, and

$$z^s = e^{(s\mathrm{Log}|z| + is\arg(z))}, \tag{13}$$

in which $\mathrm{Log}\,|z|$ represents the natural logarithm of $|z|$ and $\arg(z)$ is not necessarily the principal value. The integrand is assumed to have the form

$$g(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)}, \quad 0 \le m \le q\,, 0 \le n \le p\,. \tag{14}$$

For the above, we require that:

(i) A_j and B_j are positive numbers.

(ii) a_j and b_j are complex numbers such that

$$A_j(b_k+\nu) \neq B_k(a_j - \lambda - 1) \tag{15}$$

for

$$\nu, \lambda = 0,1,2\ldots\,; j = 1,\ldots,n\,; k = 1,\ldots,m\,.$$

That means the poles of $\Gamma(b_k - B_k s)$ for $k = 1,\ldots,m$ and $\Gamma(1-a_j+A_j s)$ for $j = 1,\ldots,n$ do not coincide.

(iii) The contour C separates the poles resulting from $\Gamma(b_k - B_k s)$, $1 \le k \le m$, from those of $\Gamma(1-a_j+A_j s)$, $1 \le j \le n$.

3.1. Existence Conditions for Mellin-Barnes Integrals

For the existence conditions, we need to construct the following two equations

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \tag{16}$$

and

$$\beta = \prod_{j=1}^{p} {A_j}^{A_j} \prod_{j=1}^{q} {B_j}^{-B_j}\,. \tag{17}$$

Three main paths are used to find existence conditions for the integral (12):

Path 1. C_1 goes from $c - i\infty$ to $c + i\infty$ and has the property of separating the poles of $\Gamma(1-a_k+A_k s)$, $k = 1,\ldots,n$ and $\Gamma(b_j - B_j s)$, $j = 1,\ldots,m$.

Path 2. C_2 is a loop beginning and ending at ∞ and encircling all the poles of $\Gamma(b_j - B_j s), j = 1,\ldots,m$ once in the negative direction.

Path 3. C_3 is a loop beginning and ending at $-\infty$ and encircling all the poles of $\Gamma(1-a_k+A_k s), k=1,\ldots,m$ once in the positive direction.

The existence conditions are:

(1) The Mellin-Barnes integral (12) taken along Path 1 converges in the sector defined by

$$|\arg(z)| < \frac{\pi}{2}\left(\sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \sum_{j=n+1}^{p} A_j - \sum_{j=m+1}^{q} B_j\right). \tag{18}$$

(2) With Path 2, the integral converges:

(i) for all $z \neq 0$ with $\mu > 0$,

(ii) for $0 < |z| < \beta^{-1}$ with $\mu = 0$.

(3) Path 3 is similar to Path 2.

4. Meijer *G*-Function

The G-function is very general in nature. A large number of special functions are special cases of this function. In this section we give some definitions of the function without any proofs. For a detailed discussion of the G-function, we refer the reader to [7].

The G-function has been implemented in *Maple*, together with representation of elementary and some special functions in terms of the G-function.

Notation.

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| z\right) \equiv G_{p,q}^{m,n}\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| z\right) \equiv G_{p,q}^{m,n}(z) \equiv G(z). \tag{19}$$

These are the standard notations used in the literature. In the following definition an empty product is interpreted as unity and $0 \le m \le q$, $0 \le n \le p$. The Meijer G -function with the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ is defined as a Mellin-Barnes type integral as follows [7].

Definition 1.

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| z\right) = \frac{1}{2\pi i}\int_L g(s) z^s ds, \tag{20}$$

$$g(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}.$$

It is clear that the Meijer G-function is a special case of the Mellin-Barnes integral and it is derived from the latter by letting A_j's and B_j's equal one.

5. The *H*-Function

The H-function is a very general function that encompasses the G-function; see [6].

Notation.

$$H_{p,q}^{m,n}\left(\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B1),\dots,(b_q,B_q)\end{matrix}\middle|\, z\right) \equiv H_{p,q}^{m,n}\left(\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\middle|\, z\right) \equiv H_{p,q}^{m,n}(z) \equiv H(z)\,. \tag{21}$$

Definition 2. The H-function is defined by the Mellin-Barnes integral (12)

$$H(z) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)} z^s ds\,. \tag{22}$$

In fact, it is the Mellin-Barnes integral (12) itself.

5.1. Existence Conditions for the *H*-Function

They are the same as for the Mellin-Barnes integral (12).

5.2. Properties of *H*-Function

The following are some properties of the H-function that are useful for us (see [6]).

Property 1.

$$z^{\alpha}\, H_{p,q}^{m,n}\left(\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\middle|\, z\right) = H_{p,q}^{m,n}\left(\begin{matrix}(a_p+\alpha A_p,A_p)\\(b_q+\alpha B_q,B_q)\end{matrix}\middle|\, z\right)\,. \tag{23}$$

Property 2.

$$H_{p,q}^{m,n}\left(\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\middle|\, z^{\alpha}\right) = \frac{1}{\alpha}\, H_{p,q}^{m,n}\left(\begin{matrix}(a_p,\frac{A_p}{\alpha})\\(b_q,\frac{B_q}{\alpha})\end{matrix}\middle|\, z\right) \qquad \alpha > 0\,. \tag{24}$$

Property 3.

$$H^{m,n}_{p,q}\left(\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\middle| z\right) = H^{n,m}_{q,p}\left(\begin{matrix}(1-b_q,B_q)\\(1-a_p,A_p)\end{matrix}\middle| \frac{1}{z}\right). \tag{25}$$

5.3. Arbitrary Order Symbolic Derivative and Integral of the *H*-Function

One can express the integer or arbitrary order derivatives and integrals of the H-function in terms of a new H-function. That means the class of H-function is closed under the operations of arbitrary differentiations and arbitrary integrations. This is a very nice property of the H-function, which is not possessed by other classes of functions. The following lemma illustrates this idea [2].

Lemma 1. *The formula [2]*

$$(H^{m,n}_{p,q}(z))^{(r)}(z) = H^{m,n+1}_{p+1,q+1}\left(\begin{matrix}(a_p - rA_p, A_p), (-r,1)\\(b_q - rB_q, B_q), (0,1)\end{matrix}\middle| z\right) \tag{26}$$

gives:

(i) derivatives of arbitrary order if $r > 0$,

(ii) integrals of arbitrary order if $r < 0$, *of the* H*-function*

$$H^{m,n}_{p,q}\left(\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\middle| z\right). \tag{27}$$

Proof. We give a proof for part (i); that for part (ii) is similar. We recall the definition of the H-function (22)

$$H(z) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n \Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j+B_j s)\prod_{j=n+1}^p \Gamma(a_j - A_j s)} z^s ds\,. \tag{28}$$

One can differentiate both sides of the above equation, provided the above integral converges uniformly for some z. Formula (9)

$$\frac{d^r}{dx^r}x^s = \frac{\Gamma(s+1)}{\Gamma(s-r+1)}x^{s-r}\,, \tag{29}$$

gives

$$(H^{m,n}_{p,q}(z))^{(r)} = \frac{1}{2\pi i}\int_C h(s) z^{s-r} ds\,, \tag{30}$$

where

$$h(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n \Gamma(1-a_j+A_j s)\Gamma(s+1)}{\prod_{j=m+1}^q \Gamma(1-b_j-B_j s)\prod_{j=n+1}^p \Gamma(a_j - A_j s)\Gamma(s-r+1)}\,. \tag{31}$$

Using the notation of the H-function, the above can be written as

$$(H^{m,n}_{p,q}(z))^{(r)}(z) = x^{-r} H^{m,n+1}_{p+1,q+1}\left(\begin{matrix} (a_p, A_p), (0,1) \\ (b_q, B_q), (r,1) \end{matrix} \middle| z \right) . \tag{32}$$

Property 1 of the H-function simplifies the last equation to a more compact form

$$(H^{m,n}_{p,q}(z))^{(r)}(z) = H^{m,n+1}_{p+1,q+1}\left(\begin{matrix} (a_p - rA_p, A_p), (-r,1) \\ (b_q - rB_q, B_q), (0,1) \end{matrix} \middle| z \right) . \tag{33}$$

If C is taken as Path 1, the existence condition for the above H-function is

$$|\arg(z)| < \frac{\pi}{2}\left(\sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \sum_{j=n+1}^{p} A_j - \sum_{j=m+1}^{q} B_j \right) . \tag{34}$$

For Part ii, one needs formula (10)

$$\frac{\Gamma(s+1)}{\Gamma(s+r+1)} x^{s+r} , \tag{35}$$

which is derived from the generalized Cauchy formula for the n-fold integral. The rest of the proof is exactly similar to that for part (i). □

Formula (26) is very important for finding integer and arbitrary order symbolic derivatives and integrals of both elementary and special functions as long as they are representable in terms of the H-function.

6. A Unified Formula for Arbitrary Order Symbolic Derivatives and Integrals of The *Power-Inverse Tangential Class*

Definition 3. A function $f(x)$ is said to be of *power-inverse tangential* type if it belongs to the following class of functions

$$\left\{ f(x) : f(x) = \sum_{j=0}^{\ell} p_j(x^{\alpha_j}) \arctan \beta_j x^{\gamma_j} , \quad \alpha_j \in \mathbb{C} , \beta_j \in \mathbb{C}\backslash\{0\} , \gamma_j \in \mathbb{R}\backslash\{0\} \right\} , \tag{36}$$

where p_j's are polynomials of certain degrees.

The following is a theorem which gives the main result of this paper.

Theorem 1. *Let*

$$f(x) = \sum_{j=0}^{\ell} p_j(x^{\alpha_j}) \arctan \beta_j x^{\gamma_j} \,, \tag{37}$$

where p_j's are polynomials of certain degrees, $m \in \mathbb{N}$, $\alpha_i, \beta_i \in \mathbb{C}$, and $\gamma_i \in \mathbb{R}$, then the formula

$$f^{(r)}(x) = \sum_{j=1}^{\ell} \sum_{i_j=0}^{m_j} \frac{a_{i_j}\beta_j}{2\gamma_j (\mathrm{i}\beta_j)^{1+\frac{i_j-r}{\gamma_j}}} \times H_{3,3}^{1,3}\left(\left. \begin{matrix} (\frac{i_j\alpha_j-r}{\gamma_j}+1, \frac{1}{\gamma_j}), (\frac{i_j\alpha_j-r}{2\gamma_j}+\frac{1}{2}, \frac{1}{2\gamma_j}), (-r,1) \\ (\frac{i_j\alpha_j-r}{2\gamma_j}+\frac{1}{2}, \frac{1}{2\gamma_j}), (\frac{i_j\alpha_j-r}{\gamma_j}, \frac{1}{\gamma_j}), (0,1) \end{matrix} \right| (\mathrm{i}\beta_j)^{1/\gamma_j} x \right)$$

$$\left|\arg((\mathrm{i}\beta_j)^{1/\gamma_j} x)\right| < \min\left\{ \frac{\pi}{2\gamma_j}, j = 1 \ldots \ell \right\} , \tag{38}$$

yields:

(a) Derivatives of any order if $r > 0$.

(b) Integrals of any order if $r < 0$.

Proof.

$$f(x) = \sum_{j=1}^{\ell} p_j(x^{\alpha_j}) \arctan \beta_j x^{\gamma_j} = \sum_{j=1}^{\ell} \sum_{i_j=0}^{m_j} a_{i_j} x^{i_j\alpha_j} \arctan \beta_j x^{\gamma_j}$$

$$= \sum_{j=1}^{\ell} \sum_{i_j=0}^{m_j} a_{i_j} x^{i_j\alpha_j} \sum_{k=0}^{\infty} \frac{(-1)^k (\beta_j x^{\gamma_j})^{2k+1}}{(2k+1)} \,. \tag{39}$$

The last infinite sum admits the Mellin-Barnes integral form

$$f(x) = \sum_{j=1}^{\ell} \sum_{i_j=0}^{m_j} a_{i_j} \beta_j x^{i_j\alpha_j+\gamma_j} \times \frac{1}{2\pi i} \int_C \frac{\Gamma(-s)\Gamma(2s+1)\Gamma(s+1)}{\Gamma(2s+2)} (-{\beta_j}^2 x^{2\gamma_j})^s ds \,, \tag{40}$$

where C is a suitable contour (see Section 3). Using the notation of the H-function, we have

$$f(x) = \sum_{j=1}^{\ell} \sum_{i_j=0}^{m_j} a_{i_j} \beta_j \, x^{i_j\alpha_j+\gamma_j} H_{2,2}^{1,2}\left(\left. \begin{matrix} (0,1),(0,2) \\ (0,1),(-1,2) \end{matrix} \right| -{\beta_j}^2 x^{2\gamma_j} \right) . \tag{41}$$

Applying property (2) of the H-function, the above can be simplified to

$$f(x)=\sum_{j=1}^{\ell}\sum_{i_j=0}^{m_j}\frac{a_{i_j}\beta_j}{2\gamma_j}\,x^{i_j\alpha_j+\gamma_j}\,H^{1,2}_{2,2}\left(\begin{matrix}(0,\frac{1}{\gamma_j}),(0,\frac{1}{2\gamma_j})\\(0,\frac{1}{2\gamma_j}),(-1,\frac{1}{\gamma_j})\end{matrix}\middle|(i\beta_j)^{1/\gamma_j}x\right),\qquad i=\sqrt{-1}\,. \tag{42}$$

Exploiting Property 1, a further simplification to the above equation follows, and we find

$$f(x)=\sum_{j=1}^{\ell}\sum_{i_j=0}^{m_j}\frac{a_{i_j}\beta_j}{2\gamma_j(i\beta_j)^{1+\frac{i_j}{\gamma_j}}}\times H^{1,2}_{2,2}\left(\begin{matrix}(\frac{i_j\alpha_j}{\gamma_j}+1,\frac{1}{\gamma_j}),(\frac{i_j\alpha_j}{2\gamma_j}+\frac{1}{2},\frac{1}{2\gamma_j})\\(\frac{i_j\alpha_j}{2\gamma_j}+\frac{1}{2},\frac{1}{2\gamma_j}),(\frac{i_j\alpha_j}{\gamma_j},\frac{1}{\gamma_j})\end{matrix}\middle|(i\beta_j)^{1/\gamma_j}x\right). \tag{43}$$

Applying formula (26), given in Lemma 1, to the above H-functions in the last equation, a unified formula for integer and arbitrary order symbolic derivatives and integrals follows

$$f^{(r)}(x)=\sum_{j=1}^{\ell}\sum_{i_j=0}^{m_j}\frac{a_{i_j}\beta_j}{2\gamma_j(i\beta_j)^{1+\frac{i_j-r}{\gamma_j}}}\times H^{1,3}_{3,3}\left(\begin{matrix}(\frac{i_j\alpha_j-r}{\gamma_j}+1,\frac{1}{\gamma_j}),(\frac{i_j\alpha_j-r}{2\gamma_j}+\frac{1}{2},\frac{1}{2\gamma_j}),(-r,1)\\(\frac{i_j\alpha_j-r}{2\gamma_j}+\frac{1}{2},\frac{1}{2\gamma_j}),(\frac{i_j\alpha_j-r}{\gamma_j},\frac{1}{\gamma_j}),(0,1)\end{matrix}\middle|(i\beta_j)^{1/\gamma_j}x\right). \tag{44}$$

Using Path 1; see Section 3, the existence condition becomes

$$\left|\arg((i\beta_j)^{1/\gamma_j}x)\right|<\min\left\{\frac{\pi}{2\gamma_j},j=1\ldots\ell\right\}. \tag{45}$$

Example 1. The integral,

$$\int_0^x t^{\alpha}\,\arctan(\beta\,t^{\gamma})\,dt\,, \tag{46}$$

can be evaluated by applying the unified formula (38) and the answer can be simplified in terms of the Meijer G-function

$$\frac{\beta}{4\gamma}\,x^{\gamma+\alpha+1}\,G^{1,3}_{3,3}\left(\begin{matrix}0,\frac{1}{2},\frac{\gamma-\alpha-1}{2\gamma}\\0,-\frac{1}{2},-\frac{\gamma+\alpha+1}{2\gamma}\end{matrix}\middle|\beta^2x^{2\gamma}\right). \tag{47}$$

The above G-function can be simplified to a less general function, namely to the hypergeometric function

$$\frac{\beta}{\gamma+\alpha+1}\, x^{\gamma+\alpha+1}\, F\left(\begin{matrix} \frac{1}{2}, 1, \frac{\gamma+\alpha+1}{2\gamma} \\ \frac{3}{2}, \frac{3\gamma+\alpha+1}{2\gamma} \end{matrix} \,\middle|\, -\beta^2 x^{2\gamma}\right). \tag{48}$$

7. Conclusion

In this work, we have given a complete solution of the problem of symbolic differentiation and integration of any order of the *power-inverse tangential class* of functions. We emphasis that the approach does not depend on integration techniques. The formula makes complicated integrals easy to evaluate, since It is just a plug-in formula. That shows the important rule played by special functions in evaluating elementary and non-elementary integrals. Several unified formulas for different classes of functions can be found in the author's previous work. A more general approach that covers this class and other classes of functions is being worked on by the author.

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover (1970).

[2] Mhenni M. Benghorbal, *Power Series Solutions of Fractional Differential Equations and Symbolic Derivatives and Integrals*, Ph.D. Thesis, London (2004).

[3] Mhenni M. Benghorbal, The Mittag-Leffer Theorem and integer order derivative formulas of meromorphic functions, *Int. J. of Comp. and Num. Anal. and App.* (2007), To Appear.

[4] Mhenni M. Benghorbal, A uniffed formula for arbitrary order symbolic derivatives and integrals of the power-exponential class, *International Journal of Pure and Applied Mathematics*, **37**, No. 3 (2007), 349-365.

[5] Mhenni M. Benghorbal, Robert M. Corless, A uniffed formula for arbitrary order symbolic derivatives and integrals of a rational polynomial, *International Journal of Pure and Applied Mathematics*, **16**, No. 2 (2004).

[6] A.M. Mathai, *The H-Function with Applications in Statistics and Other Disciplines*, John Wiley and Sons (1978).

[7] A.M. Mathai, *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Oxford Science Publications (1993).

[8] Keith B. Oldham, Jerome Spanier, *The Fractional Calculus*, Acadamic Press (1974).

[9] R.B. Paris, D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, Cambridge (2001).

[10] Igor Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, California (1999).

[11] S.E. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach Science Publishers (1996).