THE POISSON SUBALGEBRAS OF COMPACT SEMISIMPLE LIE BIALGEBRAS

Steven Benzel
Department of Mathematics
Berry College
Mount Berry, GA 30149-5036, USA
e-mail: sbenzel@berry.edu

Abstract: We begin by reviewing the definition of a Lie bialgebra and recall some standard results. The Lie bialgebra structures on a compact semisimple Lie bialgebra are then determined using Soibelman’s classification of such structures for a compact simple Lie bialgebra. We then determine the Poisson subalgebras for these structures. We conclude with some straightforward remarks on the notion of Poisson homogeneous spaces.

AMS Subject Classification: 17B62
Key Words: Lie bialgebra, Poisson Lie group

1. Introduction

A Lie bialgebra is a Lie algebra structure on a vectorspace together with a Lie algebra structure on the dual space. The two structures cannot be arbitrary but must satisfy a relation which can be formulated in various ways. This relation was first introduced by Drinfel’d [4] and studied by Semenov-Tian-Shansky [12], and Lu-Weinstein [11] to name a few. The classification of such structures when one of them is complex simple was provided in [1] and the classification for compact simple was given by Soibelman in [14]. Lie bialgebras form a category and an arrow injective at the vectorspace level gives rise to the notion of a sub Lie bialgebra.
At the group level, the corresponding notion is that of a Poisson Lie group. This is a Lie group equipped with a Poisson structure such that the multiplication map is a Poisson map. Here too we have sub-objects given by injective arrows. It turns out [12] that the quotient of a Poisson Lie group $G$ by a closed Poisson Lie subgroup $H$ is naturally a Poisson manifold and the action map

$$G \times G/H \to G/H$$

is a Poisson map. As shown in [5], [10], and [7] however, not all homogeneous Poisson Lie spaces arrive as such quotients. In this letter we will extend Soibelman’s classification to cover all compact semisimple Lie bialgebras non-degenerate in a certain sense, and determine all their sub-bialgebras. We will conclude with a simple type of duality between homogeneous Poisson submanifolds of homogeneous Poisson Lie manifolds and compute the homogeneous Poisson structures for some simple examples.

2. Definitions and Basic Theory

We begin with notation. Let $u$ be a finite dimensional real vectorspace with dual space $u^*$. We will use Latin letters for elements of $u$, Greek letters for elements of $u^*$, and will denote the pairing by $\langle \alpha, x \rangle$ for $\alpha \in u^*$ and $x \in u$. We define

$$\bigwedge^{jk} = \bigwedge^j u^* \otimes \bigwedge^k u$$

and identify $\bigwedge^{11}$ with $gl(u)$. $\bigwedge^{jk}$ is dual to $\bigwedge^{kj}$ and we normalize $<>$ by extension via

$$\langle \alpha \wedge \beta, x \wedge y \rangle = \langle \alpha, x \rangle <\beta, y \rangle - \langle \alpha, y \rangle <\beta, y \rangle,$$

where $\alpha, \beta \in u^*$ and $x, y \in u$. We can consider any $r \in \bigwedge^{02}$ as a map $r : u^* \to u^*$ by $\langle \beta, r(\alpha) \rangle = \langle \alpha \wedge \beta, r \rangle$.

Now suppose that $p \in \bigwedge^{21}$ is a Lie algebra structure for $u$ and $\pi \in \bigwedge^{12}$ is a Lie algebra structure for $u^*$. For easy of notation we will forgo the tradition Lie bracket notation in favor of the dot convention, which is that all natural Lie actions are denoted by a dot, with the specific action determined by the elements involved and all actions are left actions. Thus we see that $x \cdot y = p(x \wedge y)$ is the adjoint action of $x$ on $y$, $x \cdot \alpha$ is the coadjoint action of $x$ on $\alpha$, and $\alpha \cdot x$ is the coadjoint action of $\alpha$ on $x$. If we identify

$$\bigwedge^{nk} = \text{Hom}(\bigwedge^n u, \bigwedge^k u),$$
as the space of \( n \) cochains on the Lie algebra \((u, p)\) with values in the \( u \)-space \( \wedge^k u \), we can define a coboundary operator
\[
d : \wedge^n u \to \wedge^{n+1} u.
\]
Similarly if we identify
\[
\wedge^n u = \text{Hom}(\wedge^k u^*, \wedge^n u^*),
\]
as the space of \( k \) cochains on the Lie algebra \((u^*, \pi)\) with values in the \( u^* \)-space \( \wedge^n u^* \), we can define a coboundary operator
\[
\delta : \wedge^n u \to \wedge^{n,k+1}.
\]
Finally, let \( \mathfrak{d} = u \oplus u^* \) and define a bracket on \( \mathfrak{d} \) by
\[
[x + \alpha, y + \beta] = x \cdot y + \alpha \cdot \beta + \alpha \cdot y + x \cdot \beta - y \cdot \alpha - \beta \cdot x,
\]
\( \forall x, y \in u \) and \( \forall \alpha, \beta \in u^* \).

**Theorem 1.** (see [8] [9]) The following are equivalent:

---

- \( x \cdot (\alpha \cdot \beta) = (x \cdot \alpha) \cdot \beta \) \( \beta \) \( (x \cdot \beta) \cdot \alpha \) \( (\beta \cdot x) \cdot \alpha \)
- \( \forall x \in u \) and \( \forall \alpha, \beta \in u^* \).
---

- The bracket on \( \mathfrak{d} \) defined above satisfies the Jacobi identity.

(\( \mathfrak{d} \) is a Lie algebra).

- \( d \pi = 0 \).
- \( \delta \pi = 0 \).
- \( d \delta + \delta d = 0 \) i.e. \( d \) and \( \delta \) form a bicomplex.

The proof is a straightforward computation with the last item resulting from the observation that
\[
\delta + d : \wedge^* \to \wedge^*
\]
can be identified with the Lie algebra cohomology operator of \( \mathfrak{d} \) with values in the trivial representation. The first condition is known as the Drinfel'd identity.

An ordered pair \((u, u^*)\) satisfying the above is known as a Lie bialgebra.

The collection of finite dimensional Lie bialgebras forms a category with an arrow \( \phi : (u_1, u_1^*) \to (u_2, u_2^*) \) given by a Lie map \( \phi : u_1 \to u_2 \) such that the dual map, also denoted by \( \phi \), is also a Lie map. For \( x \in u_1 \) such a \( \phi \) will give \( \phi(\pi_1(x)) = \pi_2(\phi(x)) \) and for \( \alpha \in u_2^* \), \( \phi(\phi(x) \cdot \alpha) = x \cdot \phi(\alpha) \). If \( \pi = dr \) for some \( r \in \wedge^{02} \) then the Lie bialgebra is called exact and \( r \) is known as the (classical) \( r \)-matrix. In this case the Poisson structure on the group is given by
\[
\Pi_g = R^*_g r - L^*_g r,
\]
with $R_g$ and $L_g$ the right and left translations with respect to $g \in G$. The inner automorphism Lie algebra is defined as the subalgebra of $gl(u)$ generated by the adjoint action of $\ker(\pi) \subset u$ and the coadjoint action of $\ker(p) \subset u^*$.

The defining theorem for sub-objects is

**Theorem 2.** (see [13]) For $\mathfrak{h}$ a subalgebra of $u$ the following are equivalent:

- $\mathfrak{h}^\perp$ is an ideal of $u^*$.
- $u^* \cdot \mathfrak{h} \subset \mathfrak{h}$.
- $\mathfrak{h}^\perp \cdot \mathfrak{h} = 0$.
- $\pi(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$.

When these hold, the pair $(\mathfrak{h}, u^*/\mathfrak{h}^\perp)$ forms a Lie bialgebra.

Clearly the injection $\mathfrak{h} \to u$ is a bialgebra map and we call $\mathfrak{h}$ a Poisson subalgebra or simply a sub-bialgebra of $u$. Interestingly, it can happen that $(u, u^*)$ is exact while $(\mathfrak{h}, u^*/\mathfrak{h}^\perp)$ is not and we define a Lie bialgebra to be totally exact if every sub-bialgebra is exact. It is straightforward to see that if every sub-bialgebra $\mathfrak{h}$ of $u$ normalizes a complement $V$

$$u = \mathfrak{h} \oplus V \quad \mathfrak{h} \cdot V \subset V,$$

then $u$ is totally exact.

The defining theorem for direct sums is

**Theorem 3.** Let $\phi : (u_1^*, u_1) \to (u_2, u_2^*)$ be a Lie bialgebra arrow. Then for $\alpha_1 \in u_1^*$ and $\alpha_2 \in u_2^*$, the action

$$\alpha_1 \cdot \alpha_2 = \phi(\alpha_1) \cdot \alpha_2 - \phi(\alpha_2) \cdot \alpha_1,$$

together with the given Lie algebra structures on $u_1^*$ and $u_2^*$ give $u_1^* \bowtie u_2^*$, a twisted Lie algebra structure on the sum $u_1^* \oplus u_2^*$ such that $(u_1 \oplus u_2, u_1^* \bowtie u_2^*)$ is a Lie bialgebra. On the other hand, let $u_1$ and $u_2$ be two Lie algebras with trivial centers and let $(u_1 \oplus u_2, u_1^* \bowtie u_2^*)$ be an exact Lie bialgebra with $r$-matrix

$$r = r_1 + r_2 + r_{12} \in u_1 \wedge u_1 \oplus u_2 \wedge u_2 \oplus u_1 \wedge u_2,$$

then each $r_i$ is an $r$-matrix and

$$r_{12} : (u_1^*, u_1) \to (u_2, u_2^*)$$

is a Lie bialgebra map.

**Proof.** Let $\phi : (u_1^*, u_1) \to (u_2, u_2^*)$ be an arrow and let $\alpha \in u_1^*$, $\beta_1, \beta_2 \in u_2^*$. Then

$$\alpha \cdot (\beta_1 \cdot \beta_2) = \phi(\alpha) \cdot (\beta_1 \cdot \beta_2) - \phi(\beta_1 \cdot \beta_2) \cdot \alpha.$$
\[= (\phi(\alpha) \cdot \beta_1) \cdot \beta_2 - (\beta_1 \cdot \phi(\alpha)) \cdot \beta_2 - (\phi(\alpha) \cdot \beta_2) \cdot \beta_1 + (\beta_2 \cdot \phi(\alpha)) \cdot \beta_1 - \phi(\beta_1) \cdot (\phi(\beta_2) \cdot \alpha) + \phi(\beta_2) \cdot (\phi(\beta_2) \cdot \alpha)\]

\[= (\phi(\alpha) \cdot \beta_1) \cdot \beta_2 - \phi(\phi(\beta_1) \cdot \alpha) \cdot \beta_2 + \phi(\beta_2) \cdot (\phi(\beta_1) \cdot \alpha)\]

\[= (\phi(\alpha) \cdot \beta_1) \cdot \beta_2 - (\phi(\beta_1) \cdot \alpha) \cdot \beta_2 - (\phi(\alpha) \cdot \beta_2) \cdot \beta_1 + (\phi(\beta_2) \cdot \alpha) \cdot \beta_1\]

\[= (\alpha \cdot \beta_1) \cdot \beta_2 - (\alpha \cdot \beta_2) \cdot \beta_1.\]

A similar equality holds for \(\alpha \in u_1^*\) and \(\beta_1, \beta_2 \in u_2^*\). This shows that \(u_1^* \bowtie u_2^*\) is indeed a Lie algebra. To verify that we do have a Lie bialgebra, it suffices to check that the Drinfel’d identity holds. This is a straightforward computation once we note that for \(x \in u_1\) and \(\alpha \in u_2^*\), \(\alpha \cdot x = \phi(\alpha) \cdot x\). For the converse, it is clear that \((u_1, u_1^*)\) and \((u_2, u_2^*)\) are Lie bialgebras, and a computation similar to the above shows that the map

\[Ad \circ r_{12} : u_1^* \rightarrow u_2 \rightarrow gl(u_2)\]

is a Lie algebra map. Thus if \(u_2\) is centerless \(r_{12} : u_1^* \rightarrow u_2\) is a Lie algebra map. Similarly for \(r_{12} : u_2^* \rightarrow u_1\). This gives the theorem. \(\Box\)

3. Compact Semisimple Lie Bialgebras

Now let \(u\) be a compact simple Lie algebra with complexification \(g\), \(\mathfrak{h}\) a Cartan subalgebra of \(g\), \(\Delta^+\) a fixed set of positive roots, and \(e_\alpha \in g_a\) a choice of standard elements, see [6]. For \(v\) an element of an arbitrary \(u\) module we define the isotropy of \(v\) to be the set \(\{x \in u : x \cdot v = 0\}\). We denote by \(x \mapsto \tilde{x}\) the Killing map for \(u\) i.e.

\[< \tilde{x}, y > = -\text{trace}(ad_x ad_y).\]

Let \(\mathfrak{z} = u \cap \mathfrak{h}\), \(x_\alpha = e_\alpha - e_{-\alpha}\), \(y_\alpha = i(e_\alpha + e_{-\alpha})\), \(h_\alpha\) the vector with \(2h_\alpha = x_\alpha \cdot y_\alpha\), and \(u_\alpha = \text{span}\{x_\alpha, y_\alpha\}\). We have chosen this nonstandard definition for \(\Delta^+\) so that \(\Delta^+ \subset \mathfrak{z}^*\). We will assume that \(u\) is given as

\[u = \mathfrak{z} \oplus \sum_{\alpha \in \Delta^+} u_\alpha.\]
Since this decomposition is orthogonal with respect to the Killing form, we can write $u^*$ as

$$u^* = \mathfrak{z}^* \oplus \sum_{\alpha \in \Delta^+} u^*_\alpha,$$

where $\mathfrak{z}^* = \tilde{\mathfrak{z}}$ and $u^*_\alpha = \tilde{u}_\alpha$. As shown in [14], [3], and [2] any $r$-matrix with isotropy $\mathfrak{z}$ has the form $r = r_1 + r_0$, where $r_1$ is an element of $\mathfrak{z} \wedge \mathfrak{z}$ and $r_0$ has the form

$$r_0 = c \sum_{\alpha \in \Delta^+} x_\alpha \wedge y_\alpha$$

and we will call $r$ nondegenerate if $c \neq 0$ in which case we will assume that $c = 1$. The adjoint and coadjoint actions of $u^*$ are given as

$$\xi \cdot \eta = r(\xi) \cdot \eta - r(\eta) \cdot \xi,$$

$$\xi \cdot x = -(x \cdot r)(\xi) = r(\xi) \cdot x + r(x \cdot \xi).$$

The fact that $\mathfrak{z}$ and $\mathfrak{z}^*$ act as derivations on $u^*$ allows us to observe that $u^*_\alpha \cdot u^*_\beta \subset u^*_{\alpha + \beta}$ and $u^*_\alpha \cdot u^*_\beta \subset u^*_{\beta - \alpha}$.

Let $\Phi = \{\alpha_1, ..., \alpha_n\}$ be the simple roots for $\Delta^+$. Then any positive root $\alpha$ can be written uniquely as a sum

$$\alpha = \sum_i n_i \alpha_i$$

with the $n_i$ nonnegative integers. Let $\Psi \subset \Phi$, and define $\Delta^+_0 \subset \Delta^+$ to be the set of all roots which have $n_i = 0$ for every $i$ with $\alpha_i \in \Psi$. Let $t$ be a subspace of $\mathfrak{z}$ containing $h_\alpha$ for every $\alpha \in \Delta^+_0$ and with $r_1(t^+) \perp \text{span}(\Delta^+_0)$. We define $\mathfrak{k}$ a subalgebra of $u$ by

$$\mathfrak{k} = t \oplus \sum_{\alpha \in \Delta^+_0} u^*_\alpha$$

and note that the perpendicular space

$$\mathfrak{k}^\perp = (\mathfrak{z}^* \cap t^+) \oplus \sum_{\alpha \notin \Delta^+_0} u^*_\alpha$$

is an ideal of $u^*$. Thus $\mathfrak{k}$ is a Poisson subalgebra of $u$ and we claim that every Poisson subalgebra is of this form. To see this, we will need the following lemma.
Lemma 4. Let $V = \bigoplus_i V_i$ be a direct sum of vector spaces with $\mathcal{A} \subset \text{End}(V)$ a subalgebra such that for all $a \in \mathcal{A}$, $aV_i \subset V_i$ and $a|_{V_i}$ is either 0 or invertible. If $\mathcal{A}$ satisfies the property that for any $i \neq j$ there exists an $a \in \mathcal{A}$ such that $aV_i = 0$ and $aV_j = V_j$ then $\mathcal{A}$ contains the projections $\pi_i : V \to V_i$.

Proof. Fix an $i$ and for each $j \neq i$ pick $a_j \in \mathcal{A}$ so that $a_jV_j = 0$ and $a_jV_i = V_i$. Let $a = \Pi_{j \neq i}a_j$. Then $\ker(a) = \bigoplus_{j \neq i} V_j$ and $\hat{a} \in \text{End}(V_i)$, the restricted of $a$ to $V_i$ is invertible. We can find a polynomial $q(t)$ such that $q(\hat{a})$ is the inverse of $\hat{a}$ which will give $\pi_i = ag(a) \in \mathcal{A}$ as desired. \hfill \Box

Let $\mathcal{A}$ be the subalgebra of $\text{End}(u^*)$ generated by the identity and $ad_{\xi}$ for $\xi \in \mathfrak{j}^*$. Then $\mathcal{A}$ preserves every ideal of $u^*$ and we claim that $\mathcal{A}$ contains the projections onto the factors of the decomposition

$$u^* = \mathfrak{j}^* \oplus \sum_{\alpha \in \Delta^+} u_{\alpha}^*.$$  

To see this we verify the hypothesis of our lemma. For $\xi \in \mathfrak{j}^*$, $\xi \cdot \mathfrak{j}^* = 0$, $\xi \cdot u_{\alpha}^* \subset u_{\alpha}^*$ and in the basis $\{\tilde{x}_\alpha, \tilde{y}_\alpha\}$, $ad_{\xi}$ has the form

$$ad_{\xi}|_{u_{\alpha}^*} = \begin{pmatrix} < \xi, h_\alpha > & < \xi, r(\alpha) > \\ - < \xi, r(\alpha) > & < \xi, h_\alpha > \end{pmatrix},$$

with determinant $< \xi, h_\alpha >^2 + < \xi, r(\alpha) >^2$. We see that $\mathcal{A}$ restricted to $u_{\alpha}^*$ is isomorphic to $\mathbb{R}$ if $r(\alpha) = 0$ or $\mathbb{C}$ if $r(\alpha) \neq 0$. Now fix $\alpha \neq \beta$. We wish to find an $a \in \mathcal{A}$ with $a(u_{\alpha}^*) = u_{\alpha}^*$ and $a(u_{\beta}^*) = 0$. If $\alpha$ and $\beta$ have the same length then $< \beta, h_\beta > < \alpha, h_\beta >$. If not, we can assume that $\beta$ is the longer root and we still get $< \beta, h_\beta > < \alpha, h_\beta >$. We can take a to be $a = < \beta, h_\beta > I - ad_{\beta}$. Applying the lemma we see that $\mathcal{A}$ contains the desired projections. Let $I$ be an ideal of $u^*$. If we let $I_\alpha = I \cap u_{\alpha}^*$ and $I_0 = I \cap \mathfrak{j}^*$ then

$$I = I_0 \oplus \sum I_\alpha$$

and if $I_\alpha \neq 0$ then $I_{\alpha + \beta} = u_{\alpha + \beta}^*$ for all $\beta \in \Delta^+$ with $\alpha + \beta \in \Delta^+$. Moreover, if $\xi \in I_0$ and $< \xi, h_\alpha > \neq 0$ then $I_\alpha = u_{\alpha}^*$. It is possible to have some $I_\alpha \neq 0$ and $I_\alpha \neq u_{\alpha}^*$. However, if $I^\perp$ is a subalgebra of $u$ this cannot happen. For if $I_\alpha \neq u_{\alpha}^*$ then $h_\alpha \in \mathfrak{j} \cap I^\perp \subset I^\perp$ and $I^\perp$ contains the subalgebra generated by $h_\alpha$ and $u_\alpha \cap I^\perp$. But this subalgebra contains $u_\alpha$ so that $I_\alpha = 0$. Thus, for any ideal $I$ of $u^*$ such that $I^\perp$ is a subalgebra of $u$, there is an $\Omega \subset \Delta^+$ such that

$$I = I_0 + \sum_{\alpha \in \Omega} u_{\alpha}^*,$$
and \( \alpha + \beta \in \Omega \) whenever \( \alpha \in \Omega \) and \( \alpha + \beta \in \Delta^+ \). Now suppose \( \alpha = \sum n_i \alpha_i \in \Omega \) is minimal, i.e. \( \alpha \neq \alpha' + \beta \) for any \( \alpha' \in \Omega \) and suppose \( j \) is such that \( \alpha' = \alpha - \alpha_j \in \Delta^+ \). Then \( u_{\alpha'}, u_{\alpha_j} \subset I^\perp \) and

\[
x_{\alpha'} \cdot x_{\alpha_j} = N_1 x_{\alpha} + N_2 x_{\alpha' - \alpha_j},
\]

where \( N_1 \) is a nonzero constant and \( N_2 \) is nonzero if \( \alpha' - \alpha_j \in \Delta^+ \), see [6]. This implies that \( x_{\alpha'} \cdot x_{\alpha_j} \notin I^\perp \) contradicting the fact that \( I^\perp \) is a subalgebra. Thus we conclude such a \( j \) does not exist and hence, that \( \alpha \) must be simple.

To determine \( I_0 \) we note that \( I \cdot I^\perp = 0 \). For \( \xi \in I_0 \) and \( \alpha \in \Delta^+_0 \) we have

\[
\xi \cdot x_{\alpha} = r(\xi) \cdot x_{\alpha} + r(x_{\alpha} \cdot \xi) = 0
\]

and in fact both terms are 0 since they are perpendicular in the Killing norm. This gives \( r(\xi) \perp \text{span}(\Delta^+_0) \). Since we have \( \xi \in I_0 \) we already have \( \langle \xi, h_{\alpha} \rangle = 0 \) for all \( \alpha \in \Delta^+_0 \). We restate our result.

**Theorem 5.** Every Poisson subalgebra of \( u \) nondegenerate has the form

\[
\mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta^+_0} u_{\alpha},
\]

where \( u, \mathfrak{t} \) and \( \Delta^+_0 \) are given as above.

We now assume that \( u \) is compact semisimple. Then \( u \) can be written as a direct sum of compact simple

\[
u = u_1 \oplus ... \oplus u_l
\]

and the dual space is given as

\[
u^* = u_1^* \oplus ... \oplus u_l^*.
\]

Since \((u, u^*)\) is exact, we know from Theorem 3 that each \((u_i, u_i^*)\) is an exact Lie bialgebra and we call \( u \) nondegenerate if each \( u_i \) is nondegenerate. The mixing matrices are given by

**Proposition 6.** Let \((u_1, u_1^*), (u_2, u_2^*)\) be two Lie bialgebras with \( u_i \) compact simple and \( c_i \neq 0 \). Then the set of Lie bialgebra maps

\[
\phi : (u_1^*, u_1) \rightarrow (u_2, u_2^*)
\]

is the space \( \mathfrak{z}_1 \otimes \mathfrak{z}_2 \).
To prove this, we note that $u_1^*$ is solvable which gives $\phi(u_1^*)$ solvable and hence abelian since $u_2$ is compact. Thus $\mathfrak{z}_1^\perp$, the derived algebra of $u_1^*$, is contained in the kernel of $\phi$. This gives $\phi(u_2^* \mathfrak{u}_2^* \subset \mathfrak{z}_1^\perp$. Similarly $\phi(u_1^* \mathfrak{u}_1^* \subset \mathfrak{z}_2^\perp$. It is also now clear that any element of $\mathfrak{z}_1 \otimes \mathfrak{z}_2$ gives a bialgebra map $\phi$.

We can now find the sub-bialgebras for $u$ nondegenerate by noting that the $r$-matrix is given as

$$r = \sum_i r_i + \sum_{i \neq j} r_{ij}$$

and the isotropy of $r$ as

$$\mathfrak{z} = \sum_i \mathfrak{z}_i,$$

with $r_{ij} \in \mathfrak{z}_i \wedge \mathfrak{z}_j$. As before we let $\mathcal{A}$ be the subalgebra generated by $ad_{\xi}$ for $\xi \in \mathfrak{z}^*$ and it is clear that $\mathcal{A}$ will again contain the projections onto each $u_\alpha^*$. This implies that any ideal $I$ of $u^*$ will be a direct sum $I = I_1 + \ldots + I_l$ where each $I_i$ is an ideal of $u_i$.

**Theorem 7.** A Poisson subalgebra $\mathfrak{k}$ of $u = \bigoplus u_i$ nondegenerate is a direct sum of Poisson subalgebras $\mathfrak{k}_i$ of $u_i$ such that $r(\mathfrak{k}^\perp)$ is perpendicular to the roots of $\mathfrak{k}$.

### 4. Homogeneous Spaces

Now let $H$ and $K$ be closed connected Poisson Lie subgroups of $U$, and $g \in U$. Suppose that $KgH$ is a Poisson submanifold of $U$. Letting $\Pi_m$ denote the Poisson tensor at a point, we see that for all $k \in K$ and $h \in H$

$$\Pi_{kgh} \in \wedge^2 T_{kgh}KgH.$$ 

Left translating to the identity gives

$$(kgh)^{-1} \cdot r - r \in \wedge^2 ((kgh)^{-1} \cdot \mathfrak{k} + \mathfrak{h}), \quad \forall k \in K, \quad \forall h \in H.$$ 

Since $K$ and $H$ are Poisson Lie subgroups we have

$$k \cdot r - r \in \mathfrak{k} \wedge \mathfrak{k}, \quad h \cdot r - r \in \mathfrak{h} \wedge \mathfrak{h}.$$ 

So we see that $KgH$ is a Poisson submanifold of $U$ if and only if

$$r - g \cdot r \in \wedge^2 (\mathfrak{k} + g \cdot \mathfrak{h}).$$

Thus we only need to check the Poisson tensor at the one point $g$. Since $M \subset U$ is a Poisson submanifold if and only if $M^{-1}$ is a Poisson submanifold we also see that
Proposition 8. \( K(gH) \) is a Poisson submanifold of \( U/H \) if and only if \( H(g^{-1}K) \) is a Poisson submanifold of \( U/K \).

Now let \( \Pi^0 \) denote the Poisson bracket on \( U \), and suppose that \( \Pi \) is a homogeneous Poisson structure on \( U/H \), where \( H \) is a Poisson Lie subgroup. Letting \( \sigma : U \to U/H \) be the projection, we have a nonhomogeneous equation for \( \Pi \)

\[
\Pi_{gH} = \sigma^*(\Pi^0_g) + L^*_g(\Pi_{eH}).
\]

We can find an \( \mathfrak{h} \) invariant complement \( V \) for \( \mathfrak{h} \) in \( u \) and note that \( s = \Pi_{eH} \in V \wedge V \) is \( \mathfrak{h} \) invariant. We define a bivectorfield \( \Pi \) on \( U \) by \( \Pi_g = \Pi^0_g + L^*_g(s) \). Then we have \( \sigma^*(\Pi,\Pi) = 0 \), where \([\Pi,\Pi]\) is the Schouten bracket of \( \Pi \). Recalling that left invariant vector fields commute with right invariant vector fields, we can pull the resulting equations back to the identity to obtain

Proposition 9. The Poisson homogeneous structures on \( U/H \) are given by \( H \) invariant elements \( s \in V \wedge V \) such that

\[
2r \cdot s + s \cdot s \in \mathfrak{h} \wedge u \wedge u,
\]

where the action is the Schouten bracket on \( \wedge^* u \).

If \( H \) is maximal then \( U/H \) will be a hermitian symmetric space. The \( H \) fixed subspace of \( V \wedge V \) will be 1 dimensional and \( V \cdot V \subset \mathfrak{h} \) so we see that the space of homogeneous Poisson structures on \( U/H \) is 1-dimensional.

At the other extreme, for \( H = 0 \), we see that the Poisson homogeneous structures on \( U \) itself are given by \( s \in u \wedge u \) so that

\[
2r \cdot s - s \cdot s = 0.
\]

Noting that \( s + r \) will define another \( r \)-matrix on \( U \) allows us to identify the homogeneous Poisson structures on \( U \) with the space \( U/Z \times \wedge^2 u \).

References


