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APPROXIMATING FIXED POINTS OF MEAN NONEXPANSIVE MAPPING IN BANACH SPACES

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Abstract: Let X be a uniformly convex Banach space, and T be a mean nonexpansive mapping. It is show that the sequence of Ishikawa iteration associated with T converges to the fixed point of T.

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1. Introduction

Let X be a real Banach space and T is a mapping from X to X. In [2], the mean nonexpansive mapping is defined by

$$\|Tx - Ty\| \leq a\|x - y\| + b\{\|x - Tx\| + \|y - Ty\|\} + c\{\|x - Ty\| + \|y - Tx\|\} \quad (1)$$

for all $x, y \in X, a, b, c \in [0,1]$ with $a + 2b + 2c \leq 1$. The Ishikawa iteration sequence $\{x_n\}$ of T was defined in [7] by

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$$y_n = (1-\beta_n)x_n + \beta_n T x_n\,, \tag{2}$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n\,, \tag{3}$$

where $x_0 \in X, \alpha_n, \beta_n \in [0,1]$.

In [2], Chang Shihsen defined the new mapping which means nonexpansive mapping, and proved the existence of the fixed point in Banach spaces. Zhao Hanbin extended the results and proved the convergence of Picar iteration and Mann iteration for mean nonexpansive mapping in [16]. Since the Ishikawa iteration was defined, numerous papers have been published on the iterative approximation of fixed points for certain classes of operators, using the Mann and Ishikawa iteration methods (see [1], [3], [4], [14]).

A Banach space X is called uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$, where the modulus $\delta(\varepsilon)$ of convexity of X is defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\}$$

for every ε with $0 \le \varepsilon \le 2$. It is easy to see that Banach space X is uniformly convex if and only if for any $x_n, y_n \in B_X = \{x|\ \|x\| \le 1\}, \|x_n+y_n\| \to 2$ implies $\|x_n - y_n\| \to 0$.

Our objective in this paper is to consider an iterative process, which converges to a fixed point of mean nonexpansive mapping. Our result presented in this paper improve and extend the properties of mean nonexpansive mapping.

2. Main Result

In this section, we prove a strong convergence theorem for mean nonexpansive mappings in a uniformly convex Banach space.

Theorem 2.1. *Let X be uniformly convex Banach space, $T : X \to X$ is a mean nonexpansive with some fixed points, $b > 0, 0 < \alpha \le \alpha_n \le \frac{1}{2}, 0 \le \beta_n \le \beta < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the fixed point of T.*

Proof. Firstly, we show that $\|x_n - Tx_n\| \to 0$. Since

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\le \|x_n - Ty_n\| + a\|x_n - y_n\| + b\{\|x_n - Tx_n\| + \|y_n - Ty_n\|\} \\ &+ c\{\|x_n - Ty_n\| + \|y_n - Tx_n\|\} = (1+c)\|x_n - Ty_n\| + a\|x_n - y_n\| + b\|x_n - Tx_n\| \\ &+ b\|y_n - Ty_n\| + c\|y_n - Tx_n\| = (1+c)\|x_n - Ty_n\| + a\|(1-\beta_n)x_n + \beta_n Tx_n - x_n\| \\ &+ b\|x_n - Tx_n\| + b\|(1-\beta_n)x_n + \beta_n Tx_n - Ty_n\| + c\|(1-\beta_n)x_n + \beta_n Tx_n - Tx_n\|\end{aligned}$$

$$
\begin{aligned}
&\le (1+c)\|x_n - Ty_n\| + a\beta_n\|x_n - Tx_n\| \\
&+ b\|x_n - Tx_n\| + b\beta_n\|x_n - Tx_n\| + b\|x_n - Ty_n\| \\
&+ c(1-\beta_n)\|x_n - Tx_n\| = (1+b+c)\|x_n - Ty_n\| \\
&\qquad + (a\beta_n + b + b\beta_n + c(1-\beta_n))\|x_n - Tx_n\|\,,
\end{aligned}
$$

we have

$$(1 - a\beta_n - b - b\beta_n - c(1-\beta_n))\|x_n - Tx_n\| \le (1+b+c)\|x_n - Ty_n\|\,.$$

Let $M = 1 - a\beta_n - b - b\beta_n - c(1-\beta_n)$, then

$$
\begin{aligned}
&M = 1 - a\beta_n - b - b\beta_n - c + c\beta_n = 1 - b - c - (a+b-c)\beta_n \\
&\ge a+b+c-(a+b-c)\beta_n = (a+b)(1-\beta_n)+c(1+\beta_n) \ge (a+b)(1-\beta)+c > 0\,,
\end{aligned}
$$

so

$$\|x_n - Tx_n\| \le \frac{1+b+c}{M}\|x_n - Ty_n\|\,. \tag{4}$$

For the fixed point $p = Tp$, observe that

$$
\begin{aligned}
\|Tx - p\| = \|Tx - Tp\| &\le a\|x-p\| + b\{\|x - Tx\| + \|p - Tp\|\} \\
&\quad + c\{\|x - Tp\| + \|p - Tx\|\} \\
&\le a\|x-p\| + b\{\|x-p\| + \|p - Tx\|\} + c\{\|x - Tp\| + \|p - Tx\|\}\,.
\end{aligned}
$$

Set $L = \frac{a+b+c}{1-b-c}$, by $a + 2b + 2c \le 1$, it is easy to see that $a+b+c \le 1-b-c$, thus $0 \le L \le 1$, and $\|Tx - p\| \le L\|x-p\| \le \|x-p\|$, so that

$$
\begin{aligned}
\|Ty_n - p\| &\le \|y_n - p\| \\
&\le \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\
&= \|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\| \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
&\le \|x_n - p\|\,.
\end{aligned}
$$

Next, we will prove that $\|x_n - p\|$ is bounded.

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n Ty_n - p\| \\
&= \|(1-\alpha_n)(x_n - p) + \alpha_n(Ty_n - p)\|
\end{aligned}
$$

$$\begin{aligned}
&\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|Ty_n-p\| \\
&\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\| \\
&= (1-\alpha_n)\|x_n-p\|+\alpha_n\|(1-\beta_n)x_n+\beta_n Tx_n-p\| \\
&= (1-\alpha_n)\|x_n-p\|+\alpha_n\|(1-\beta_n)(x_n-p)+\beta_n(Tx_n-p)\| \\
&\leq (1-\alpha_n)\|x_n-p\|+\alpha_n(1-\beta_n)\|x_n-p\| \\
&\quad +\alpha_n\beta_n\|Tx_n-p\| \\
&\leq (1-\alpha_n)\|x_n-p\|+\alpha_n(1-\beta_n)\|x_n-p\| \\
&\quad +\alpha_n\beta_n\|x_n-p\| \\
&= ((1-\alpha_n)+\alpha_n(1-\beta_n)+\alpha_n\beta_n)\|x_n-p\| \\
&= \|x_n-p\|\,.
\end{aligned}$$

So

$$\|x_{n+1}-p\|\leq\|x_n-p\|\leq\|x_{n-1}-p\|\leq ...\leq\|x_0-p\|\,. \tag{5}$$

Thus there exists $M_1>0$, such that $\|x_n-p\|\leq M_1$, thus $\|Ty_n-p\|\leq\|x_n-p\|\leq M_1$.

Assume that $\lim_{n\to\infty}\|x_n-Ty_n\|\neq 0$, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a real number $\varepsilon_0>0$, such that

$$\|x_{n_k}-Ty_{n_k}\|\geq\varepsilon_0,\quad k=1,2,3\cdots\,.$$

So we have

$$\begin{aligned}
\|x_{n_k}-Ty_{n_k}\| &\leq \|x_{n_k}-p\|+\|Ty_{n_k}-p\| \\
&\leq \|x_{n_k}-p\|+L\|y_{n_k}-p\| \\
&= \|x_{n_k}-p\|+L\|((1-\beta_{n_k})x_{n_k}+\beta_{n_k}Tx_{n_k})-p\| \\
&= \|x_{n_k}-p\|+L\|(1-\beta_{n_k})(x_{n_k}-p)+\beta_{n_k}(Tx_{n_k}-p)\| \\
&\leq \|x_{n_k}-p\|+(1-\beta_{n_k})L\|x_{n_k}-p\|+\beta_{n_k}L\|Tx_{n_k}-p\| \\
&\leq (1+(1-\beta_{n_k})L+\beta_{n_k}L^2)\|x_{n_k}-p\| \\
&\leq (1+L)\|x_{n_k}-p\| \\
&\leq 2\|x_{n_k}-p\|\,.
\end{aligned}$$

Thus

$$\|x_{n_k}-p\|\geq\frac{1}{2}\|x_{n_k}-Ty_{n_k}\|\geq\frac{\varepsilon_0}{2}=\varepsilon_1>0\,.$$

Furthermore, we have

$$\|\frac{x_{n_k}-p}{\|x_{n_k}-p\|}-\frac{Ty_{n_k}-p}{\|x_{n_k}-p\|}\|=\frac{\|x_{n_k}-Ty_{n_k}\|}{\|x_{n_k}-p\|}\geq\frac{\varepsilon_0}{M_1}>0\,.$$

It follows from

$$\|\frac{x_{n_k}-p}{\|x_{n_k}-p\|}\|=1 \;\text{ and }\; \|\frac{Ty_{n_k}-p}{\|x_{n_k}-p\|}\|\leq L\leq 1$$

and the fact that X is uniformly convex Banach space that there exists $\delta > 0$, such that

$$\|\frac{x_{n_k}-p}{\|x_{n_k}-p\|}+\frac{Ty_{n_k}-p}{\|x_{n_k}-p\|}\|\leq 2-\delta\,.$$

Thus

$$\begin{aligned}
\|x_{n_k+1}-p\| &= \|(1-\alpha_{n_k})x_{n_k}+\alpha_{n_k}Ty_{n_k}-p\|\\
&\leq (1-2\alpha_{n_k})\|x_{n_k}-p\|+\|\alpha_{n_k}(x_{n_k}-p)+\alpha_{n_k}(Ty_{n_k}-p)\|\\
&\leq (1-2\alpha_{n_k})\|x_{n_k}-p\|\\
&+ \alpha_{n_k}\|x_{n_k}-p\|\|\frac{x_{n_k}-p}{\|x_{n_k}-p\|}+\frac{Ty_{n_k}-p}{\|x_{n_k}-p\|}\|\\
&\leq (1-2\alpha_{n_k})\|x_{n_k}-p\|+(2-\delta)\alpha_{n_k}\|x_{n_k}-p\|\\
&= (1-\delta\alpha_{n_k})\|x_{n_k}-p\|\\
&= \|x_{n_k}-p\|-\delta\alpha_{n_k}\|x_{n_k}-p\|\\
&\leq \|x_{n_k}-p\|-\delta\alpha\varepsilon_1\,.
\end{aligned}$$

Using (5), we obtain that

$$\begin{aligned}
\|x_{n_k+1}-p\| &\leq \|x_{n_k}-p\|-\delta\alpha\varepsilon_1\\
&\leq \|x_{n_k-1}-p\|-\delta\alpha\varepsilon_1\\
&\leq \|x_{n_k-2}-p\|-\delta\alpha\varepsilon_1\\
&\leq \cdots\\
&\leq \|x_{n_{k-1}+1}-p\|-\delta\alpha\varepsilon_1\\
&\leq \|x_{n_{k-1}}-p\|-2\delta\alpha\varepsilon_1\,.
\end{aligned}$$

Thus

$$\|x_{n_k}-p\|\leq\|x_{n_{k-1}}-p\|-\delta\alpha\varepsilon_1\,.$$

So

$$\begin{aligned}
\|x_{n_k}-p\| &\leq \|x_{n_{k-1}}-p\|-\delta\alpha\varepsilon_1\\
&\leq \|x_{n_{k-2}}-p\|-2\delta\alpha\varepsilon_1\\
&\leq \cdots\\
&\leq \|x_{n_1}-p\|-(k-1)\delta\alpha\varepsilon_1\,.
\end{aligned}$$

Let $k \to \infty$, we have $\|x_{n_k} - p\| < 0$. It is a contradiction, hence $\lim_{n\to\infty} \|x_n - Ty_n\| = 0$. Using by (4), we conclude that

$$\|x_n - Tx_n\| \to 0\,. \tag{6}$$

Secondly, we will show that $\{Tx_n\}$ is a Cauchy sequence. For any $m, n \in N$.

$$\begin{aligned}
& \|Tx_n - Tx_{n+m}\| \\
\leq\ & a\|x_n - x_{n+m}\| + b\{\|x_n - Tx_n\| + \|x_{n+m} - Tx_{n+m}\|\} \\
+\ & c\{\|x_n - Tx_{n+m}\| + \|x_{n+m} - Tx_n\|\} \\
\leq\ & a\{\|x_n - Tx_n\| + \|Tx_n - Tx_{n+m}\| + \|Tx_{n+m} - x_{n+m}\|\} \\
+\ & b\{\|x_n - Tx_n\| + \|x_{n+m} - Tx_{n+m}\|\} \\
+\ & c\{\|x_n - Tx_n\| + \|Tx_n - Tx_{n+m}\| \\
+\ & \|x_{n+m} - Tx_{n+m}\| + \|Tx_{n+m} - Tx_n\|\}\,.
\end{aligned}$$

Thus

$$\|Tx_n - Tx_{n+m}\| \leq A\|x_n - Tx_n\| + B\|x_{n+m} - Tx_{n+m}\|\,,$$

where $A > 0$ and $B > 0$ are constants.

Following (7), we have $\|Tx_n - Tx_{n+m}\| \to 0$, thus $\{Tx_n\}$ is a Cauchy sequence. □

Finally, since X is a Banach space and $\{Tx_n\}$ is a Cauchy sequence, we know that $\{Tx_n\}$ is a convergent sequence. Let $q = \lim_{n\to\infty} Tx_n$, from (6), we obtain $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, thus $q = \lim_{n\to\infty} x_n$. Using (1), we have

$$\begin{aligned}
& \|Tq - q\| \leq \|q - Tx_n\| + \|Tx_n - Tq\| \\
\leq\ & \|q - Tx_n\| + a\|x_n - q\| + b\{\|x_n - Tx_n\| + \|q - Tq\|\} \\
+\ & c\{\|x_n - Tq\| + \|q - Tx_n\|\}\,.
\end{aligned}$$

Let $n \to \infty$, we get that $\|q - Tq\| \leq b\|q - Tq\| + c\|q - Tq\|$, so $(1 - b - c)\|q - Tq\| \leq 0$. Since $1 - b - c \geq a + b + c \geq b > 0$, we obtain $\|q - Tq\| = 0$. Hence q is the fixed point of T.

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