International Journal of Pure and Applied Mathematics
Volume 40 No. 2 2007, 233-240

IMPROVEMENT OF THE CONVERGENCE RATE OF THE DISCOUNTED LIMIT THEOREM

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Abstract: Let $\xi_v = \sum\limits_{k=0}^{\infty} v^k \xi_k$, where $0 \leq v \leq 1$ and $\xi_0,\ \xi_1,\ \xi_2,\ \ldots$ be a sequence of independent and identically distributed random variables with the mean $E\xi_0 = \mu < \infty$ and variance $\sigma^2 = E(\xi_0 - \mu)^2 < \infty$. The author considers the approximation of the distribution funtion of the normed sum of the random variable $Z_v = \sqrt{1-v}\sigma^{-1}(\xi_v - \mu(1-v)^{-1})$. Its estimate was obtained through pseudomoments.

AMS Subject Classification: 60G30, 60G40, 60G50, 60G20
Key Words: distributions, characteristic functions, discounted limit theorems, normal distribution function, Berry-Esseen Theorem

1. Introduction

Let $\xi_0,\ \xi_1,\ \xi_2, \ldots$ be a sequence of independent random variables (i.r.v.) with the common distribution $F(x)$ and characteristic function

$$f(t) = E\exp(itx)dF(x).$$

Let v be a discount factor ($0 < v < 1$). Then we define

$$\xi_v = \sum_{k=0}^{\infty} v^k \xi_k, \tag{1}$$

which may be interpreted as the present value of the sum of certain periodic and identically distributed payments ξ_k. We assume that three moments of ξ_k are finite:

Received: July 21, 2007

$$\mu = \int_{-\infty}^{\infty} x dF(x) < \infty, \quad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 dF(x) < \infty,$$
$$\rho_3 = \int_{-\infty}^{\infty} (x-\mu)^3 dF(x) < \infty\,. \tag{2}$$

Then it is easy to see that the mean and variance of the r.v. ξ_v are

$$\mathbf{E}\xi_v = \mu(1-v)^{-1} \text{ and } \mathbf{D}\xi_v = \sigma^2(1-v^2)^{-1}.$$

We form a normed (normalized) sum

$$Z_v = B_v(\xi_v - A_v) = \frac{\sqrt{1-v}}{\sigma}\left(\xi_v - \frac{\mu}{1-v}\right), \tag{3}$$

where $A_v = \frac{\mu}{1-v}$ and $B_v = \frac{\sqrt{1-v}}{\sigma} > 0$ are sequences of properly selected numbers.

It has been shown that the normalized r.v. Z_v with the mean $\mathbf{E}Z_v = 0$ and variance $\mathbf{D}Z_v = (1+v)^{-1}$ is asimptotically normal for $v \to 1$. We denote the distribution function of the r.v. Z_v as $\mathcal{F}_v$ and that of the normal distribution with zero mean and variance $(1+v)^{-1}$ by

$$\mathcal{N}_v(x) = \sqrt{\frac{1+v}{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(1+v)t^2}{2}\right) dt. \tag{4}$$

Hans U. Gerber in [2] has proved the discounted version of the Berry-Esseen Theorem.

Theorem 1.1. (Hans U. Gerber) *If (2) holds, then for all* x

$$|\mathcal{F}_v(x) - \mathcal{N}_v(x)| \le 5.4\,(\rho^3/\sigma^3)\,\sqrt{1-v}. \tag{5}$$

L. Saulis et al [5] considered a nonuniform estimate for the difference $F_v(x) - N_v(x)$ employing the cumulant method when the centered moments $\mathbf{E}(X_0-\mu)^s$ of the r.v. X_0 satisfy the condition: there exist quantities $\gamma \ge 0$ and $K > 0$ such that

$$|\mathbf{E}(X_0-\mu)^s| \le (s!)^{1+\gamma} K^{s-2}\sigma^2, \quad s = 3, 4, \ldots, l.$$

2. The Main Result

Without loss of generality, suppose that $\mu = E\xi_k = 0,\ A_v = 0$.

Let $\eta_0,\ \eta_1,\ \eta_2, \ldots$ be other sequence of independent normal random variables. Let $E\eta_k = 0,\ D\eta_k = \sigma^2$ with the comon distribution $\Phi_\sigma(x)$ and characteristic function $\phi_\sigma(t) = E\exp(itx)d\Phi_\sigma(x) = \exp(-\frac{\sigma^2 t^2}{2})$.

In [7], V.M. Zolotarev paid attention to the fact that estimates of type (6) disregard the degree of proximity of $F(x)$ and $\Phi_\sigma(x)$ though such a proximity is of essential importance. It seems that pseudomoments

$$\begin{aligned} \mu(m) &= \int_{-\infty}^{\infty} x^m d(F(x) - \Phi_\sigma(x)), \quad m - \text{integer number}, \\ \nu(r) &= \int_{-\infty}^{\infty} |x|^r |d(F(x) - \Phi_\sigma(x))|, \end{aligned} \tag{6}$$

can serve as a natural measure of proximity of random variables ξ_k and η_k.

Analogously as (1) we define the sum $\eta_v = \sum\limits_{k=0}^{\infty} v^k \eta_k$ for the normal r.v. η_k and normalized sum

$$N_v = B_v \eta_v,$$

where $B_v = \frac{\sqrt{1-v}}{\sigma} > 0$ as in (3).

Then it is easy to see that the distribution function

$$P(\eta_v \le x) = \mathcal{N}_v(x) = \sqrt{\frac{1+v}{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(1+v)t^2}{2}\right) dt$$

as in (4) and the characteristic function $f_{N_v}(t) = \exp\left(-\frac{t^2}{2(1+v)}\right)$.

The main rezult of the work is reduced to the following

Theorem 2.1. *If* $\mu(1) = \mu(2) = 0,\ \ \nu(3) < \infty$, *then*

$$|\mathcal{F}_v(x) - \mathcal{N}_v(x)| \le 4.42 \max(\gamma(3), \gamma(3)^{\frac{1}{4}})\sqrt{1-v}, \tag{7}$$

where

$$\gamma(3) = \frac{\nu(3)}{\sigma^3} = \frac{\int_{-\infty}^{\infty} |x|^3 |d(F(x) - \Phi_\sigma(x))|}{\sigma^3}.$$

3. Auxiliary Lemmas

We obtain the expression of the characteristic function of $f_{Z_v}(t)$

$$f_{Z_v}(t) = \prod_{k=0}^{\infty} f\left(B_v v^k t\right) \tag{8}$$

and analogously

$$f_{N_v}(t) = \prod_{k=0}^{\infty} \phi_\sigma\left(B_v v^k t\right)$$
$$= \prod_{k=0}^{\infty} \exp\left(-\frac{(1-v)\, v^{2k} t^2}{2}\right) = \exp\left(-\frac{t^2}{2(1+v)}\right). \tag{9}$$

Lemma 3.1. (Esseen's Inequality) *For all x and T*

$$|\mathcal{F}_v(x) - \mathcal{N}_v(x)| \leq \frac{2}{\pi}\int_0^T \frac{|f_{Z_v}(t) - f_{N_v}(t)|}{|t|} dt + \frac{24q}{\pi T} \tag{10}$$

with $q = \max \mathcal{N}_v'(x) = \mathcal{N}_v'(0) = \sqrt{\left(\frac{1+v}{2\pi}\right)}$.

See the proof in [5, 6].

Lemma 3.2. *Let conditions of Theorem 2.1 be fulfilled. If $\gamma(3) \leq 1$, then for all $|t| \leq T = \frac{c}{\gamma_3\sqrt{1-v}}$, where $\gamma(3) = \frac{\nu(3)}{\sigma^3}$*

$$|f_{Z_v}(t) - f_{N_v}(t)| \leq \frac{c_1 t^3 \gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left\{-\frac{t^2 A(c,v)}{2)}\right\}. \tag{11}$$

If $\gamma(3) < 1$, then for all $|t| \leq \hat{T} = \frac{c}{\gamma_3^{1/4}\sqrt{1-v}}$

$$|f_{Z_v}(t) - f_{N_v}(t)| \leq \frac{c_1 t^3 \gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left\{-\frac{t^2 \sqrt{\gamma(3)} A(c,v)}{2)}\right\}, \tag{12}$$

where $c_1 = \frac{\exp(c^2/2)}{6}$ and $A(c,v) = \frac{1}{1+v} - \frac{cc_1}{3(1+v+v^2)} > 0$, if $0 < c < 1.4$ and $0 \leq v \leq 1$.

Proof. To prove the lemma, we make use of source ideas of works [5, 6]. First let us consider the case $\gamma(3) \geq 1$.

Employing the expressions of the characteristic functions (8) and (9), it is easy to obtain the following expression:

$$\begin{aligned}\delta_v(t) = |f_{Z_v}(t) - f_{N_v}(t)| &= |\prod_{k=0}^{\infty}\alpha_k - \prod_{k=0}^{\infty}\beta_k| \cdot \exp\left(-\frac{t^2}{2(1+v)}\right)\\ &= |\sum_{k=0}^{\infty}(\alpha_k-\beta_k)\prod_{j=0}^{k-1}\alpha_j \prod_{j=k+1}^{\infty}\beta_k| \cdot \exp\left(-\frac{t^2}{2(1+v)}\right),\end{aligned} \tag{13}$$

where

$$\alpha_k = \frac{f\left(B_v\, v^k t\right)}{\phi_\sigma\left(B_v v^k t\right)}, \quad \beta_k \equiv 1, \quad k = 1, 2, \ldots .$$

Using the estimate

$$|e^{iz} - 1 - iz + \frac{(iz)^2}{2!}| \le \frac{|z|^2}{2!} \cdot \min\left(2,\ \frac{|z|}{3}\right) \le \frac{|z|^3}{6},$$

that is valid for all real z, we easily obtain the estimate

$$\begin{aligned}|\alpha_k - \beta_k| &= |f\left(B_v v^k t\right) - \phi_\sigma\left(B_v v^k t\right)| \cdot \phi_\sigma\left(B_v v^k t\right)^{-1}\\ &= |E\exp(iB_v v^k t x)|d(F(x) - \Phi_\sigma(x))| \cdot \exp\left(\frac{(1-v)\, v^{2k}t^2}{2}\right)\\ &\le \frac{(1-v)^{3/2} v^{3k} t^3 \gamma(3)}{6} \cdot \exp\left(\frac{(1-v)\, v^{2k}t^2}{2}\right).\end{aligned}$$

However $|t| \le T = \frac{c}{\gamma_3\sqrt{1-v}}$, therefore

$$\exp\left(\frac{(1-v)\, v^{2k}t^2}{2}\right) \le \exp\left(\frac{c^2}{2}\right)$$

and

$$|\alpha_k - \beta_k| = c_1 \cdot (1-v)^{3/2} v^{3k} t^3 \gamma(3), \text{ where } c_1 = \frac{\exp(c^2/2)}{6}. \tag{14}$$

Hence, due to the fact that $\beta_k \equiv 1$, for all $|t| \le T$ we have

$$|\alpha_k| \le 1 + |\alpha_k - \beta_k| \le 1 + c_1 \cdot (1-v)^{3/2} v^{3k} t^3 \gamma(3)$$

and

$$\prod_{j=0}^{k-1} |\alpha_j| \le \exp\left(\frac{c_1 t^3 \gamma(3)\sqrt{1-v}}{1+v+v^2}\right). \tag{15}$$

From (8)-(10) we derive

$$|f_{Z_v}(t) - f_{N_v}(t)| \le \frac{c_1 t^3 \gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left(-\frac{t^2}{2(1+v)} + \frac{cc_1 t^2}{1+v+v^2}\right)$$
$$= \frac{c_1 t^3 \gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left\{-\frac{t^2}{2}\left(\frac{1}{1+v} - \frac{2cc_1}{1+v+v^2}\right)\right\}.$$

It is not difficult to calculate that the inequality

$$\frac{1}{1+v} - \frac{2cc_1}{1+v+v^2} > 0 \tag{16}$$

holds then $0 < c \le 1.4$ and $0 \le v \le 1$. Then (10) is proved.

Let us now consider the case $\gamma(3) < 1$. Instead of (13) we take the expansion

$$|f_{Z_v}(t) - f_{N_v}(t)| = |\sum_{k=0}^{\infty} (\hat{\alpha}_k - \hat{\beta}_k) \prod_{j=0}^{k-1} \hat{\alpha}_j \prod_{j=k+1}^{\infty} \hat{\beta}_k| \cdot \exp\left(-\frac{t^2\sqrt{\gamma(3)}}{2(1+v)}\right), \tag{17}$$

where

$$\hat{\alpha}_k = \frac{f\left(B_v\ v^k t\right)}{\phi_\sigma\left(B_v v^k t \gamma(3)^{1/4}\right)} \text{ and } \hat{\beta}_k = \frac{\phi_\sigma\left(B_v v^k t\right)}{\phi_\sigma\left(B_v v^k t \gamma(3)^{1/4}\right)}, k = 1, 2, \dots.$$

Now we get

$$|\hat{\alpha}_k - \hat{\beta}_k| \le \frac{(1-v)^{3/2} v^{3k} t^3 \gamma(3)}{6} \cdot \exp\left(\frac{(1-v)\ v^{2k} t^2 \gamma(3)^{1/2}}{2}\right).$$

For all $|t| \le T = \frac{c}{\gamma(3)^{1/4}\sqrt{1-v}}$, $\exp\left(\frac{(1-v)\ v^{2k}t^2\gamma(3)^{1/2}}{2}\right) \le \exp\left(\frac{c^2}{2}\right)$, therefore for all $|t| \le T$

$$|\hat{\alpha}_k - \hat{\beta}_k| \le c(1-v)^{3/2} v^{3k} t^3 \gamma(3), \quad \text{where } c_1 = \frac{\exp(c^2/2)}{6}. \tag{18}$$

Applying (18) and inequality

$$\hat{\beta}_k = \frac{\phi_\sigma\left(B_v v^k t\right)}{\phi_\sigma\left(B_v v^k t \gamma(3)^{1/4}\right)} = \exp\left(-t^2(1-v)v^{2k}(1-\gamma(3)^{1/2})/2)\right) < 1,$$

we obtain

$$\hat{\alpha}_k \le \hat{\beta}_k + |\hat{\alpha}_k - \hat{\beta}_k| \le 1 + \hat{c}_1 \cdot (1-v)^{3/2} v^{3k} t^3 \gamma(3)$$

and

$$\prod_{j=0}^{k-1} \hat{\alpha}_j \leq \exp\left(\frac{\hat{c}_1 t^3 \gamma(3)\sqrt{1-v}}{1+v+v^2}\right), \tag{19}$$

since $\gamma(3) < 1$.

By gathering inequalities (17)-(19) together, for all $|t| \leq T = \frac{\hat{c}}{\gamma(3)^{1/4}\sqrt{1-v}}$ and $\gamma(3) < 1$, we obtain

$$\begin{aligned}
|f_{Z_v}(t) - f_{N_v}(t)| &\leq \frac{c_1|t|^3\gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left(-\frac{t^2\sqrt{\gamma(3)}}{2(1+v)} + \frac{c_1|t|^3\gamma(3)\sqrt{1-v}}{1+v+v^2}\right) \\
&= \frac{c_1|t|^3\gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left\{-\frac{t^2}{2}\left(\frac{1}{1+v} - \frac{2c_1|t|\gamma(3)\sqrt{1-v}}{1+v+v^2}\right)\right\} \\
&\leq \frac{c_1|t|^3\gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left\{-\frac{t^2}{2}\left(\frac{1}{1+v} - \frac{2cc_1\gamma(3)^{1/4}}{1+v+v^2}\right)\right\} \\
&\leq \frac{c_1|t|^3\gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left\{-\frac{t^2}{2}\left(\frac{1}{1+v} - \frac{2cc_1}{1+v+v^2}\right)\right\}.
\end{aligned} \tag{20}$$

The inequalities (16) and (20) together yield (12). □

4. Proof of Theorem 2.1

Assume that $\gamma(3) \geq 1$. Then, using Lemmas 3.1 and 3.2, we get

$$\begin{aligned}
&|\mathcal{F}_v(x) - \mathcal{N}_v(x)| \\
&\quad\leq \frac{2}{\pi}\int_0^T \frac{c_1 t^2\gamma(3)\sqrt{1-v}}{1+v+v^2} \cdot \exp\left\{-\frac{t^2 A(c,v)}{2)}\right\} + \frac{24T\sqrt{1+v}}{c\pi\sqrt{2\pi}} \\
&\quad\leq C\gamma(3)\sqrt{1-v},
\end{aligned} \tag{21}$$

where

$$C \leq \frac{\exp(c^2/2)}{3\sqrt{2\pi}(1+v+v^2)A(c,v)^{3/2}} + \frac{24\sqrt{1+v}}{c\pi\sqrt{2\pi}}. \tag{22}$$

One may take $C = 4.42$ because for $0.94 \leq v \leq 1$ the above formuls leads to smaller constants whereas for $1 \leq v \leq 0.94$ we find that $4.42 \cdot \sqrt{1-v}$ is greater than one, such that (5) is true anyway. The proof for $\gamma(3) < 1$ is analogical.

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