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A BOUND ON NORMAL APPROXIMATION OF NUMBER OF VERTICES OF A FIXED DEGREE IN A RANDOM GRAPH

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Abstract: In this paper, we use the Stein-Chen method to give non-uniform bounds on normal approximation of number of vertices of a fixed degree in a random graph.

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1. Introduction and Main Results

A random graph is a graph generated by some random procedure. The study of random graphs has a long history. A systematic study of random graphs began with the influential work of Erdös and Rényi, (see [6], [7], [8]). The random graph theory has developed into one of the mainstays of modern discrete mathematics. There are many applications of a random graph, which can be seen in [10], [11], [4], [9].

Let $\mathbb{G}(n,p)$ be a random graph on n labeled vertices $\{1, 2, \ldots, n\}$, where possible edge $\{i, j\}$ is present randomly and independently with the probability p, $0 < p < 1$.

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Let S_n be the number of vertices of a fixed degree d, where $d \geq 0$, in $\mathbb{G}(n,p)$. Then $S_n = Y_1 + Y_2 + \cdots + Y_n$, where

$$Y_i = \begin{cases} 1, & \text{if vertex } i \text{ has degree } d \text{ in } \mathbb{G}(n,p), \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$.

Note that the expectation of Y_i for $i = 1, 2, \ldots, n$ is

$$\mu = P(Y_i = 1) = \binom{n-1}{d} p^d q^{n-1-d} \leq C n^d p^d \text{ and } ES_n = n\mu\,, \tag{1}$$

where $q = 1 - p$.

In 1989, A.D. Barbour, M. Karoñski, A. Ruciñski [3] proved that the distribution function of $W_n := \frac{S_n - ES_n}{\sqrt{\text{Var}S_n}}$, where $\text{Var}S_n > 0$ can be approximated by the standard normal distribution function, $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$, as follows:

Theorem 1.1. *Let $d \geq 1$. If $ES_n \to \infty$ and $\log n + d \log\log n - np \to \infty$ then $\sup_{z \in \mathbb{R}} |P(W_n \leq z) - \Phi(z)| \to 0$ as $n \to \infty$.*

We observe that, in Theorem 1.1 we know only sufficient conditions in order for the distribution function of W_n to converge to Φ. But, in case of an isolated vertex, i.e., $d = 0$, Theorem 1.2 gives us both necessary and sufficient conditions for the convergence of W_n.

Theorem 1.2. *For $d = 0$, $\sup_{z \in \mathbb{R}} |P(W_n \leq z) - \Phi(z)| \to 0$ as $n \to \infty$ if and only if $n^2 p \to \infty$ and $\log n - np \to \infty$.*

From Theorem 1.1 and Theorem 1.2 we know that there exist conditions that W_n weakly converges to Φ. The important question is how closed between $P(W_n \leq z)$ and $\Phi(z)$, i.e., we need to know a Berry-Esseen bound between W_n and Φ. In Theorem 1.3, A.D. Barbour, M. Karoñski, A. Ruciñski [3] gave Berry-Esseen bound for the normal approximation of W_n in case $d \geq 1$ under a metric d_1 which is defined by, for any random variables X and Y

$$d_1(X,Y) = \sup \Big\{ |Eh(X) - Eh(Y)| : \sup_{x \in \mathbb{R}} |h(x)| + \sup_{x \in \mathbb{R}} |h'(x)| \leq 1$$
$$\text{for all bounded test functions } h \text{ with bounded derivative} \Big\}.$$

Theorem 1.3. *If $d \geq 1$ then $d_1(W_n, \mathcal{N}(0,1)) \leq \frac{C}{\sqrt{\text{Var}S_n}}$, where $\mathcal{N}(0,1)$ is the standard normal random variable.*

A Berry-Esseen bound between distribution function of W_n and Φ in the form $\delta_n := \sup_{z \in \mathbb{R}} |P(W_n \leq z) - \Phi(z)|$ and the metric d_1 are different (see Barbour and Hall [1]). In general, $\delta_n = O(\varepsilon_n^{\frac{1}{2}})$, where ε_n is an upper estimate in metric d_1. Chen [5] proved that $\delta_n \approx \varepsilon_n$ at the cost of much greater effort.

In this work, we give non-uniform bounds between $P(W_n \leq z)$ and $\Phi(z)$. The followings are our main results.

Theorem 1.4. *Let $d \geq 1$. If $p = \frac{1}{n^\delta}$ for a fixed $\delta \in [1, 1+\frac{1}{d})$, then there exists a constant C, independent of z, such that for $0 < \beta < \frac{1}{2} - \frac{d(\delta-1)}{2}$,*

$$|P(W_n \leq z) - \Phi(z)| \leq \frac{C}{(1+|z|)n^\beta}.$$

Theorem 1.5. *Let $d = 0$. If $p = \frac{1}{n^\delta}$ for a fixed $\delta \in [1, 2)$, then there exists a constant C, independent of z, such that for $0 < \beta < \frac{1}{2}$,*

$$|P(W_n \leq z) - \Phi(z)| \leq \frac{C}{(1+|z|)n^\beta}.$$

Note that the condition of δ in Theorem 1.4 and Theorem 1.5 imply that $p = \frac{1}{n^\delta}$ satisfies the sufficient condition of Theorem 1.1 and Theorem 1.2.

Theorem 1.4 and Theorem 1.5 give uniform bound corollaries.

Corollary 1.6. *Let $d \geq 1$ and $p = \frac{1}{n^\delta}$ for a fixed $\delta \in [1, 1+\frac{1}{d})$. Then there exists a constant C, independent of z, such that for $0 < \beta < \frac{1}{2} - \frac{d(\delta-1)}{2}$,*

$$\sup_{z \in \mathbb{R}} |P(W_n \leq z) - \Phi(z)| \leq \frac{C}{n^\beta}.$$

Corollary 1.7. *Let $d = 0$ and $p = \frac{1}{n^\delta}$ for a fixed $\delta \in [1, 2)$. Then there exists a constant C, independent of z, such that for $0 < \beta < \frac{1}{2}$,*

$$\sup_{z \in \mathbb{R}} |P(W_n \leq z) - \Phi(z)| \leq \frac{C}{n^\beta}.$$

We observe that from Proposition 2.1, we have the order of the bound in Theorem 1.3 is $\frac{1}{\sqrt{\mathrm{Var} S_n}} \leq \frac{1}{n^{\frac{1}{2} - \frac{d(\delta-1)}{2}}}$ but the order of bound in Theorem 1.4 and Corollary 1.6 is $\frac{1}{n^\beta}$, where $0 < \beta < \frac{1}{2} - \frac{d(\delta-1)}{2}$.

In case $d = 0$, we know that S_n is a number of isolated vertices in $\mathbb{G}(n, p)$. In 1987, Kordecki [12] gave a non-uniform bound for the normal approximation of W_n, as follows.

Theorem 1.8. *If $\log n - np \to \infty$ and $n^2 p \to \infty$, then there exists a constant $C(z)$ such that*

$$|P(W_n \leq z) - \Phi(z)| \leq \frac{C(z)}{\sqrt{\mathrm{Var} S_n}}. \tag{2}$$

We note that the constant $C(z)$ of Kordecki is a bound of the second derivative of Stein's solution in Stein's equation, $f'(\omega) - \omega f(\omega) = I_{\omega \leq z} - \Phi(z)$, where

I_A is an indicator random variable on A.

In 2006, Punkla and Chaidee[13] show that the constant $C(z)$ is of the form $C|z|$, where $z \geq 1$. In this sense the bound in (2) is not good when $z \to \infty$. Theorem 1.5 improve (2) by giving the constant C which does not depend on z and giving a non-uniform bound of this convergence. This paper is organized as follows. In Section 2, we prove auxiliary results for the proof of main results which are given in Section 3.

Throughout this paper, C stands for an absolute constant with possible different values in different places and does not depend on z.

2. Auxiliary Results

In this section, we give auxiliary results for proving Theorem 1.4 and Theorem 1.5. For each $i \in \{1, 2, \ldots, n\}$, let $X_i = \frac{Y_i - EY_i}{\sqrt{\mathrm{Var} S_n}}$. Then $EX_i = 0$, $W_n = \sum_{i=1}^n X_i$ and $EW_n^2 = 1$. For any $\Lambda \subset \{1, 2, \ldots, n\}$ and $i, j \in \{1, 2, \ldots, n\}$, we define

$$Y_j^{(\Lambda)} = \begin{cases} 1\,, & \text{if vertex } j \text{ has degree } d \text{ in } \mathbb{G}(n,p) - \{\Lambda\}, \\ 0\,, & \text{otherwise,} \end{cases}$$

where a random graph $\mathbb{G}(n,p) - \{\Lambda\}$ obtained from $\mathbb{G}(n,p)$ by dropping the vertex in Λ. For $i, j = 1, 2, \ldots, n$, let

$$Z_{ij} = \begin{cases} \frac{1}{\sigma_n} Y_i\,, & i = j\,, \\ \frac{1}{\sigma_n}(Y_j - Y_j^{(i)})\,, & i \neq j\,, \end{cases} \quad Z_i = \sum_{j=1}^n Z_{ij}\,, \quad W^{(i)} = W_n - Z_i,$$

$$V_{ij} = \begin{cases} 0\,, & i = j\,, \\ \frac{1}{\sigma_n}\left\{ Y_j^{(i)} + \sum_{\substack{l=1 \\ l \neq i,j}}^n (Y_l^{(i)} - Y_l^{(i,j)}) \right\}, & i \neq j, \end{cases}$$

and $W_{ij} = W^{(i)} - V_{ij}$, where $\sigma_n^2 = \mathrm{Var} S_n$.

Proposition 2.1. *Let $d \geq 0$ and $p = \frac{1}{n^\delta}$ for $\delta \geq 1$. Then for large n, there exists a constant $C > 0$ such that:*

1. *$\sigma_n^2 \geq Cn\mu$, and*
2. *$\sigma_n^2 \geq Cn^{1-d(\delta-1)}$.*

Proof. First, we will show that there exist constants C_1 and C_2 such that

$$C_1 n^d p^d \leq \mu \leq C_2 n^d p^d. \tag{3}$$

It is clear from (1) that $\mu \leq Cn^d p^d$ for some a constant $C \in (0, 1)$. Since

$\lim_{n\to\infty} q^n = \lim_{n\to\infty}(1-\frac{1}{n^\delta})^n = 1$, for large n we have

$$\mu = P(Y_i = 1) = \binom{n-1}{d} p^d q^{n-1-d} \geq C n^d p^d.$$

Hence (3) holds. From (3) and the fact that

$$\sigma_n^2 = \frac{n}{n-1}\binom{n-1}{d}^2 (d-(n-1)p)^2 p^{2d-1}(1-p)^{2(n-d)-3} + ES_n - \frac{(ES_n)^2}{n}$$

(see Barbour, Karoñski and Ruciñski [3], 1989), we have

$$\sigma_n^2 = n\binom{n-1}{d} p^d q^{n-1-d}\Big\{\frac{1}{(n-1)}\binom{n-1}{d}(d-(n-1)p)^2 p^{d-1} q^{n-2-d}$$
$$+1-\frac{ES_n}{n}\Big\} = n\mu\Big\{\binom{n-1}{d}p^d q^{n-1-d}\frac{(d-(n-1)p)^2}{(n-1)}\frac{1}{pq}+1-\frac{n\mu}{n}\Big\}$$
$$= n\mu\Big\{\mu\frac{(d-(n-1)p)^2}{(n-1)}\frac{1}{pq}+1-\mu\Big\} \geq n\mu\Big\{1-\mu\Big\} \geq Cn\mu.$$

So from (3) we have $\sigma_n^2 \geq Cn\mu \geq Cn^{d+1}p^d = Cn^{1-d(\delta-1)}$. □

Proposition 2.2. *Let $p = \frac{1}{n^\delta}$ for $\delta \geq 1$. Then for each $r_1, r_2 \in \mathbb{N}$ there exists a constant $C \equiv C(r_1, r_2)$ such that*

$$E|X_i^{r_1} Z_i^{r_2}| \leq \frac{C\mu}{\sigma_n^{r_1+r_2}} \quad \text{for every } i \in \{1,2,\dots,n\}.$$

Proof. Note that

$$|Y_j - Y_j^{(i)}| \leq E_{ij} I[\deg(j) = d \text{ or } d+1] \tag{4}$$

where

$$E_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are joined in } \mathbb{G}(n,p), \\ 0 & \text{otherwise} \end{cases}$$

and $\deg(j)$ is a degree of a vertex j for all $j \in \{1,2,\dots,n\}$.

Hence, for $r_1, r_2, \dots, r_m \in \mathbb{N}$ and for $j_1, j_2, \dots, j_m (\neq i)$ we have

$$E|Y_{j_1} - Y_{j_1}^{(i)}|^{r_1}|Y_{j_2} - Y_{j_2}^{(i)}|^{r_2}\cdots|Y_{j_m} - Y_{j_m}^{(i)}|^{r_m}$$
$$\leq EE_{ij_1}^{r_1}(I[\deg(j_1) = d \text{ or } d+1])^{r_1} E_{ij_2}^{r_2}\cdots E_{ij_m}^{r_m}$$
$$= P(E_{ij_1} = 1, I[\deg(j_1) = d \text{ or } d+1] = 1, E_{ij_2} = 1, \dots, E_{ij_m} = 1)$$
$$= p^{m-1}\Big[\binom{n-2}{d-1}p^d q^{n-1-d} + \binom{n-2}{d}p^{d+1}q^{n-2-d}\Big]$$
$$= p^{m-1}\binom{n-1}{d}p^d q^{n-1-d}\Big[\frac{d}{n-1} + \frac{(n-1-d)p}{(n-1)q}\Big] \leq \frac{p^{m-1}\mu}{(n-1)}\Big[d + \frac{np}{q}\Big]$$

$$\leq \frac{Cp^{m-1}\mu}{n}. \tag{5}$$

From this fact and the fact that $np \leq 1$ we have,

$$\begin{aligned}
&E\Big|\sum_{\substack{j=1\\j\neq i}}^{n}(Y_j - Y_j^{(i)})\Big|^r \\
&\leq \sum_{\substack{j=1\\j\neq i}}^{n} E|Y_j - Y_j^{(i)}|^r + \sum_{\substack{j_1=1\\j_1\neq i}}^{n}\sum_{\substack{j_2=1\\j_2\neq i,j_1}}^{n} E|(Y_{j_1} - Y_{j_1}^{(i)})^{r_1}(Y_{j_2} - Y_{j_2}^{(i)})^{r_2}| \\
&\quad + \sum_{\substack{j_1=1\\j_1\neq i}}^{n}\sum_{\substack{j_2=1\\j_2\neq i,j_2}}^{n}\cdots\sum_{\substack{j_r=1\\j_r\neq i,j_1,\ldots,j_{r-1}}}^{n} E|(Y_{j_1} - Y_{j_1}^{(i)})\cdots(Y_{j_r} - Y_{j_r}^{(i)})| \\
&\leq C\Big(\frac{n\mu}{n} + \frac{n^2p\mu}{n} + \cdots + \frac{n^rp^{r-1}\mu}{n}\Big) \leq C\mu
\end{aligned} \tag{6}$$

for any $r \in \mathbb{N}$. Then

$$\begin{aligned}
E|Z_i^r| &= \frac{1}{\sigma_n^r}E\Big|\Big(Y_i + \sum_{\substack{j=1\\j\neq i}}^{n}(Y_j - Y_j^{(i)})\Big)^r\Big| \\
&\leq \frac{C}{\sigma_n^r}\Big\{E|Y_i^r| + E\Big|\sum_{\substack{j=1\\j\neq i}}^{n}(Y_j - Y_j^{(i)})\Big|^r\Big\} \leq \frac{C\mu}{\sigma_n^r}.
\end{aligned} \tag{7}$$

From (7) and the fact that

$$|X_i| = \Big|\frac{(Y_i - \mu)}{\sigma_n}\Big| \leq \frac{1}{\sigma_n}, \tag{8}$$

we have $E|X_i^{r_1}Z_i^{r_2}| \leq \frac{1}{\sigma_n^{r_1}}E|Z_i^{r_2}| \leq \frac{C\mu}{\sigma_n^{r_1+r_2}}$. □

Proposition 2.3. *Let $p = \frac{1}{n^\delta}$ for $\delta \geq 1$. Then for $r_1, r_2, r_3 \in \mathbb{N}$ there exist constants $C \equiv C(r_1, r_2, r_3)$ such that for every $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$*

1. *$E|X_i^{r_1}Z_i^{r_2}V_{ij}^{r_3}| \leq \frac{C\mu}{\sigma_n^{r_1+r_2+r_3}}$, and*
2. *$E|X_i^{r_1}Z_{ij}^{r_2}V_{ij}^{r_3}| \leq \frac{C\mu}{n\sigma_n^{r_1+r_2+r_3}}$.*

Proof. Similarly to (4), we observe that

$$|Y_l^{(i)} - Y_l^{(i,j)}| \le E_{jl} I^{(i)}[\deg(l) = d \text{ or } d+1]\,, \tag{9}$$

where

$$I^{(i)}[\deg(l) = d \text{ or } d+1] = \begin{cases} 1 & \text{if } \deg(l) = d \text{ or } d+1 \text{ in } \mathbb{G}(n,p) - \{i\}, \\ 0 & \text{otherwise.} \end{cases}$$

1. By the same argument of (6), one can use (9) to show that,

$$E\Big|\sum_{\substack{l=1 \\ l\ne i,j}}^{n} (Y_l^{(i)} - Y_l^{(i,j)})\Big|^r \le C\mu$$

for $r \in \mathbb{N}$. From this fact and the fact that

$$EY_j^{(i)} = \binom{n-2}{d} p^d q^{n-2-d} = \binom{n-1}{d} p^d q^{n-1-d} \Big[\frac{(n-1-d)}{(n-1)q}\Big] \le C\mu,$$

we have,

$$\begin{aligned} E|V_{ij}^r| &= \frac{1}{\sigma_n^r} E\Big|\Big\{Y_j^{(i)} + \sum_{\substack{l=1 \\ l\ne i,j}}^{n} (Y_l^{(i)} - Y_l^{(i,j)})\Big\}^r\Big| \\ &\le \frac{1}{\sigma_n^r}\Big\{E|Y_j^{(i)}|^r + E\Big|\sum_{\substack{l=1 \\ l\ne i,j}}^{n} (Y_l^{(i)} - Y_l^{(i,j)})\Big|^r\Big\} \le \frac{C\mu}{\sigma_n^r} \end{aligned} \tag{10}$$

for any $r \in \mathbb{N}$.

Thus, from (7), (8) and (10), we have

$$\begin{aligned} E|X_i^{r_1} Z_i^{r_2} V_{ij}^{r_3}| &\le \frac{1}{\sigma_n^{r_1}} E|Z_i^{r_2} V_{ij}^{r_3}| \\ &\le \frac{1}{\sigma_n^{r_1}} \Big\{E|Z_i^{2r_2}|\Big\}^{\frac{1}{2}} \Big\{E|V_{ij}^{2r_3}|\Big\}^{\frac{1}{2}} \le \frac{C\mu}{\sigma_n^{r_1+r_2+r_3}}. \end{aligned}$$

2. Note from (4) and (9) that, for any $i, j \in \{1, 2, \ldots, n\}, l_1, \ldots, l_m, \in \{1, 2, \ldots, n\} - \{i, j\}$ and $r_1, r_2, \ldots, r_{m+1} \in \mathbb{N}$,

$$\begin{aligned} E|(Y_j^{(i)})^{r_1}(Y_j - Y_j^{(i)})^{r_2}| &\le E|Y_j^{(i)}|^{r_1} E_{ij}^{r_2} (I[\deg(j) = d \text{ or } d+1])^{r_2} \\ &= P(Y_j^{(i)}{=}E_{ij}{=}1, I[\deg(j) = d \text{ or } d{+}1] = 1) = \binom{n-2}{d} p^{d+1} q^{n-2-d} \\ &= \binom{n-1}{d} p^d q^{n-1-d} \frac{(n-1-d)p}{(n-1)q} \le \frac{C\mu}{n}, \end{aligned} \tag{11}$$

and

$$E|(Y_j - Y_j^{(i)})^{r_1}(Y_{l_1}^{(i)} - Y_{l_1}^{(i,j)})^{r_2}(Y_{l_2}^{(i)} - Y_{l_2}^{(i,j)})^{r_3} \cdots (Y_{l_m}^{(i)} - Y_{l_m}^{(i,j)})^{r_{m+1}}|$$

$$
\begin{aligned}
&\leq EE_{ij}^{r_1}E_{jl_1}^{r_2}(I^{(i)}[\deg(l_1)=d \text{ or } d+1])^{r_2}E_{jl_2}^{r_3}\cdots E_{jl_m}^{r_{m+1}}\\
&= P(E_{ij}=1, E_{jl_1}=1, I^{(i)}[\deg(l_1)=d \text{ or } d+1]=1,\\
&\qquad E_{jl_2}=1,\dots,E_{jl_m}=1)\\
&= pp^{m-1}\Big[\binom{n-3}{d-1}p^d q^{n-2-d}+\binom{n-3}{d}p^{d+1}q^{n-3-d}\Big]\\
&= p^m\binom{n-1}{d}p^d q^{n-1-d}\Big[\frac{(n-1-d)d}{(n-1)(n-2)}+\frac{(n-1-d)(n-2-d)p}{(n-1)(n-2)q^2}\Big]\\
&= \frac{p^m\mu}{(n-1)}\Big[\frac{(n-1-d)d}{(n-2)}+\frac{(n-1-d)(n-2-d)p}{(n-2)q^2}\Big]\\
&\leq \frac{Cp^m\mu}{n}. \qquad (12)
\end{aligned}
$$

From (12) we see that $E\Big|(Y_j-Y_j^{(i)})^{r_2}\Big[\sum_{\substack{l=1\\ l\neq i,j}}^n(Y_l^{(i)}-Y_l^{(i,j)})\Big]^{r_3}\Big|\leq\frac{C\mu}{n}$. From this fact, (8) and (11), we have

$$
\begin{aligned}
E|X_i^{r_1}Z_{ij}^{r_2}V_{ij}^{r_3}| &\leq \frac{1}{\sigma_n^{r_1+r_2+r_3}}E\Big|(Y_j-Y_j^{(i)})^{r_2}\Big\{Y_j^{(i)}+\sum_{\substack{l=1\\ l\neq i,j}}^n(Y_l^{(i)}-Y_l^{(i,j)})\Big\}^{r_3}\Big|\\
&\leq \frac{C}{\sigma^{r_1+r_2+r_3}}\Big\{E|(Y_j-Y_j^{(i)})^{r_2}(Y_j^{(i)})^{r_3}|\\
&\quad + E\Big|(Y_j-Y_j^{(i)})^{r_2}\Big[\sum_{\substack{l=1\\ l\neq i,j}}^n(Y_l^{(i)}-Y_l^{(i,j)})\Big]^{r_3}\Big|\Big\}\leq\frac{C\mu}{n\sigma^{r_1+r_2+r_3}}. \quad \square
\end{aligned}
$$

To prove the main theorem by using Stein's method, we need to bound the solution

$$
f_z(x)=\begin{cases}\sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(x)[1-\Phi(z)]\,, & \text{if } x\leq z\,,\\ \sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(z)[1-\Phi(x)]\,, & \text{if } x> z\,,\end{cases}
$$

of the Stein's equation, $f'(\omega)-\omega f(\omega)=I_{\{\omega\leq z\}}-\Phi(z)$, see [14].

The following proposition gives us the bounds of the second derivative of f_z proved by Punkla and Chaidee in 2006 (see [13]).

Proposition 2.4. *For $z\geq 0$, we have:*

1. $|f_z''(x)|\leq 4.32$ *for $0\leq z<1$,*
2. $|f_z''(x)|\leq \frac{Ce^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$ *for $x\leq 0$ and $z\geq 1$,*
3. $|f_z''(x)|\leq\frac{C}{1+z}$ *for $0<x\leq\frac{z}{2}$ and $z\geq 1$,*

4. $|f_z''(x)| \le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} + \frac{1}{z} + z$ *for* $x > \frac{z}{2}$ *and* $z \ge 1$.

Proof. See [13], p. 3. □

3. Proof of Main Results

3.1. Proof of Theorem 1.4.

It suffices to consider $z \ge 0$ as we can apply the result to $-W$ when $z < 0$. Let $z \ge 0$. Barbour, Karoñski and Ruciñski [3] use the Stein's method and the independence between $W^{(i)}$ and X_i to show that

$$|P(W_n \le z) - \Phi(z)| \le A_1 + A_2 + A_3,$$

where

$$A_1 = \frac{1}{2}\sum_{i=1}^{n} E|X_i Z_i^2 f_z''(W^{(i)} + \theta_i Z_i)|\,, \text{ for some } \theta_i \in [0,1]\,,$$

$$A_2 = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} E|X_i Z_{ij} V_{ij} f_z''(W_{ij} + \theta_{ij} V_{ij})|\,, \text{ for some } \theta_{ij} \in [0,1]\,,$$

$$A_3 = \sum_{i=1}^{n}\sum_{j=1}^{n} E|X_i Z_{ij}|E|(Z_i + V_{ij}) f_z''(W - \theta(Z_i + V_{ij}))|\,, \text{ for some } \theta \in [0,1]\,.$$

If $0 \le z < 1$. From (5), (8) and Proposition 2.1(1), we have

$$E|X_i Z_{ij}| \le \frac{1}{\sigma_n^2} E|Y_j - Y_j^{(i)}| \le \frac{C\mu}{n\sigma_n^2} \le \frac{C}{n^2}. \tag{13}$$

From Propositions 2.1, 2.2, 2.3 and the fact that $E|Z_i + V_{ij}| \le \frac{1}{\sigma_n}$, we have

$$A_1 + A_2 + A_3 \le 4.32\Big\{ \sum_{i=1}^{n} E|X_i Z_i^2| + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} E|X_i Z_{ij} V_{ij}|$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{n} E|X_i Z_{ij}|E|(Z_i + V_{ij})| \le \frac{C}{\sigma_n} \le \frac{C}{n^{\frac{1}{2} - \frac{d(\delta-1)}{2}}} \le \frac{C}{n^{\beta}}.$$

Suppose $z \ge 1$. We will bound A_1, A_2, A_3 by $\frac{C}{n^\beta}$. To do this we divide the proof into 3 steps as follows.

Step 1. To show $A_1 \le \frac{C}{n^\beta}$, it suffices to show that there exists a constant

$C > 0$ such that for each $i \in \{1, 2, \ldots, n\}$

$$E|X_i Z_i^2 f_z''(W_i + \theta_i Z_i)| \le \frac{C}{(1+z)n^{\beta+1}}$$

for $0 < \beta < \frac{1}{2} - \frac{d(\delta-1)}{2}$, $\theta_i \in [0,1]$.

By Propositions 2.1, 2.2 and 2.4 we have

$$\begin{aligned}
&E|X_i Z_i^2 f_z''(W_i+\theta_i Z_i)| \\
&= E|X_i Z_i^2||f_z''(W_i+\theta_i Z_i)|I\{W_i+\theta_i Z_i \le 0\}+E|X_i Z_i^2||f_z''(W_i+\theta_i Z_i)| \\
&\quad I\{0<W_i+\theta_i Z_i \le \frac{z}{2}\}+E|X_i Z_i^2||f_z''(W_i+\theta_i Z_i)|I\{W_i+\theta_i Z_i>\frac{z}{2}\} \\
&\le C\Big(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}+\frac{1}{1+z}\Big)E|X_i Z_i^2|+CzE|X_i Z_i^2|I\{W_i+\theta_i Z_i > \frac{z}{2}\} \\
&\le \frac{C\mu}{(1+z)\sigma_n^3}+CzE|X_i Z_i^2|I\{W_i+\theta_i Z_i > \frac{z}{2}\} \\
&\le \frac{C}{(1+z)n^{\beta+1}}+CzE|X_i Z_i^2|I\{W_i+\theta_i Z_i > \frac{z}{2}\}. \qquad (14)
\end{aligned}$$

To bound $E|X_i Z_i^2|I\{W_i + \theta_i Z_i > \frac{z}{2}\}$, we note from (7) and (8) that

$$\begin{aligned}
P\Big(W-(1-\theta_i)Z_i > \frac{z}{2}\Big) &\le \frac{16}{z^4}E(W-(1-\theta_i)Z_i)^4 \\
&\le \frac{C}{z^4}(EW^4+EZ_i^4) \le \frac{C}{z^4}(E\Big[\sum_{i=1}^{n} X_i\Big]^4+\frac{\mu}{\sigma_n^4}) \le \frac{Cn^4}{z^4\sigma_n^4}, \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
P\Big(W_{ij} > \frac{z}{3}\Big) &\le \frac{9}{z^2}E(W_{ij}^2) \\
&\le \frac{C}{z^2}(EW^2+EZ_i^2+EV_{ij}^2) \le \frac{C}{z^2}\Big[1+\frac{1}{\sigma_n^2}+\frac{1}{\sigma_n^2}\Big] \le \frac{C}{z^2}. \qquad (16)
\end{aligned}$$

Let $r > 0$ be such that $r[\frac{d}{2}+\frac{1}{2}-(\beta+\frac{\delta d}{2})]+\frac{1}{2}-2d(\delta-1) \ge \beta+1$. Then for $0 < \beta < \frac{1}{2} - \frac{d(\delta-1)}{2}$ and $j \ne i$, by using Propositions 2.1, 2.3, (7) and (15), we have

$$\begin{aligned}
&E|X_i Z_i^2|I\{W_i+\theta_i Z_i > \frac{z}{2}, |V_{ij}| > \frac{1}{n^\beta}\} \\
&\qquad \le E|X_i Z_i^2 n^{r\beta} V_{ij}^r|I\{W-(1-\theta_i)Z_i > \frac{z}{2}, |V_{ij}| > \frac{1}{n^\beta}\} \\
&\quad \le n^{r\beta}\Big\{E|X_i Z_i^2 V_{ij}^r|^2\Big\}^{\frac{1}{2}}\Big\{P\Big(W-(1-\theta_i)Z_i > \frac{z}{2}\Big)\Big\}^{\frac{1}{2}} \le \frac{Cn^{r\beta}\mu^{\frac{1}{2}}n^2}{z^2\sigma_n^{r+3}\sigma_n^2}
\end{aligned}$$

$$\leq \frac{Cn^{r\beta+2}\mu^{\frac{1}{2}}}{z^2\sigma_n^{r+5}} \leq \frac{C}{z^2 n^{r[\frac{d}{2}+\frac{1}{2}-(\beta+\frac{\delta d}{2})]+\frac{1}{2}-2d(\delta-1)}} \leq \frac{C}{z^2 n^{\beta+1}}. \quad (17)$$

and, by Propositions 2.1, 2.2, (15), (16) and independence of W_{ij} and (X_i, Z_{ij}) (see [3], p. 137), we have

$$\begin{aligned}
&E|X_iZ_i^2|I\{W_i+\theta_iZ_i > \frac{z}{2}, |V_{ij}| \leq \frac{1}{n^\beta}\} \\
&= E|X_iZ_i^2|I\{W_{ij}+V_{ij}+\theta_iZ_i > \frac{z}{2}\}I\{|Z_i| \leq \frac{1}{n^\beta}, |V_{ij}| \leq \frac{1}{n^\beta}\} \\
&\quad + E|X_iZ_i^2|I\{W_{ij}+V_{ij}+\theta_iZ_i > \frac{z}{2}\}I\{|Z_i| > \frac{1}{n^\beta}, |V_{ij}| \leq \frac{1}{n^\beta}\} \\
&\leq E|X_iZ_i^2|I\{W_{ij} > \frac{z}{3}\} + E|X_iZ_i^2|I\{W-(1-\theta_i)Z_i > \frac{z}{2}\}I\{|Z_i| > \frac{1}{n^\beta}\} \\
&\leq E|X_iZ_i^2|EI\{W_{ij} > \frac{z}{3}\} \\
&\quad + n^{r\beta}E|X_iZ_i^{r+2}|I\{W-(1-\theta_i)Z_i > \frac{z}{2}\}I\{|Z_i| > \frac{1}{n^\beta}\} \\
&\leq E|X_iZ_i^2|P\Big(W_{ij} > \frac{z}{3}\Big) \\
&\quad + n^{r\beta}\Big\{E|X_iZ_i^{r+2}|^2\Big\}^{\frac{1}{2}}\Big\{P\Big(W-(1-\theta_i)Z_i > \frac{z}{2}\Big)\Big\}^{\frac{1}{2}} \\
&\leq \frac{C}{z^2}E|X_iZ_i^2| + \frac{Cn^{r\beta+2}\mu^{\frac{1}{2}}}{z^2\sigma_n^{r+5}} \\
&\leq \frac{C}{z^2n^{\beta+1}} + \frac{C}{z^2 n^{r[\frac{d}{2}+\frac{1}{2}-(\beta+\frac{\delta d}{2})]+\frac{1}{2}-2d(\delta-1)}} \leq \frac{C}{z^2n^{\beta+1}}. \qquad (18)
\end{aligned}$$

Then from (14), (17) and (18), we have

$$E|X_iZ_i^2 f_z''(W_i+\theta_iZ_i)| \leq \frac{C}{(1+z)n^{\beta+1}}.$$

Step 2. To show $A_2 \leq \frac{C}{n^\beta}$, it suffices to show that there exists a constant $C > 0$ such that for each $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$,

$$E|X_iZ_{ij}V_{ij}f_z''(W_{ij}+\theta_{ij}V_{ij})| \leq \frac{C}{(1+z)n^{\beta+2}} \quad (19)$$

for $0 < \beta < \frac{1}{2} - \frac{d(\delta-1)}{2}$, $\theta_{ij} \in [0,1]$.

By Proposition 2.3 and the same argument of (14), we have

$$\begin{aligned}
&E|X_iZ_{ij}V_{ij}f_z''(W_{ij}+\theta_{ij}V_{ij})| \\
&\quad \leq \frac{C}{(1+z)n^{\beta+2}} + zE|X_iZ_{ij}V_{ij}|I\{W_{ij}+\theta_{ij}V_{ij} > \frac{z}{2}\}. \qquad (20)
\end{aligned}$$

Similarly to (15), by using Chebyshev's inequality, we can show that

$$P\Big(W_{ij}+\theta_{ij}V_{ij}>\frac{z}{2}\Big)\le\frac{Cn^4}{z^4\sigma_n^4}. \tag{21}$$

Let $r>0$ be such that $r[\frac{d}{2}+\frac{1}{2}-(\beta+\frac{\delta d}{2})]+1-2d(\delta-1)>\beta+2$. Then for $0<\beta<\frac{1}{2}-\frac{d(\delta-1)}{2}$ we have

$$\begin{aligned}
&E|X_iZ_{ij}V_{ij}|I\{W_{ij}+\theta_{ij}V_{ij}>\frac{z}{2},|V_{ij}|>\frac{1}{n^\beta}\}\\
&\le n^{r\beta}E|X_iZ_{ij}V_{ij}^{r+1}|I\{W_{ij}+\theta_{ij}V_{ij}>\frac{z}{2},|V_{ij}|>\frac{1}{n^\beta}\}\\
&\le n^{r\beta}\Big\{E|X_iZ_{ij}V_{ij}^{r+1}|^2\Big\}^{\frac{1}{2}}\Big\{P(W_{ij}+\theta_{ij}V_{ij}>\frac{z}{2})\Big\}^{\frac{1}{2}}\\
&\le\frac{Cn^{r\beta}}{z^2}\Big\{E|X_iZ_{ij}V_{ij}^{r+1}|^2\Big\}^{\frac{1}{2}}\Big\{\frac{n^4}{\sigma_n^4}\Big\}^{\frac{1}{2}}\le\frac{Cn^{r\beta}n^2}{z^2\sigma_n^2}\Big\{\frac{\mu}{(n-1)\sigma_n^{2r+6}}\Big\}^{\frac{1}{2}}\\
&\le\frac{Cn^{r\beta+2}\mu^{\frac{1}{2}}}{z^2n^{\frac{1}{2}}\sigma_n^{r+5}}\le\frac{C}{z^2n^{r[\frac{d}{2}+\frac{1}{2}-(\beta+\frac{\delta d}{2})]+1-2d(\delta-1)}}\le\frac{C}{z^2n^{\beta+2}}
\end{aligned} \tag{22}$$

and by the fact that W_{ij} is independent of the pair (X_i,Z_{ij}), (13), (16), we have

$$\begin{aligned}
E|X_iZ_{ij}V_{ij}|I\{W_{ij}+\theta_{ij}V_{ij}>\frac{z}{2},|V_{ij}|\le\frac{1}{n^\beta}\}&\le\frac{1}{n^\beta}E|X_iZ_{ij}|I\Big\{W_{ij}>\frac{z}{3}\Big\}\\
=\frac{1}{n^\beta}E|X_iZ_{ij}|EI\Big\{W_{ij}>\frac{z}{3}\Big\}&\le\frac{1}{n^\beta}E|X_iZ_{ij}|P\Big(W_{ij}>\frac{z}{3}\Big)\\
&\le\frac{C}{z^2n^\beta}E|X_iZ_{ij}|\le\frac{C\mu}{z^2n^{\beta+1}\sigma_n^2}\le\frac{C}{z^2n^{\beta+2}}.
\end{aligned} \tag{23}$$

Thus, from (20), (22) and (23), we have (19).

Step 3. To show $A_3\le\frac{C}{n^\beta}$, (13) enables us to show that there exists a constant $C>0$ such that for each $i,j\in\{1,2,\dots,n\}$

$$E|(Z_i+V_{ij})f_z''(W-\theta(Z_i+V_{ij}))|\le\frac{C}{(1+z)n^\beta} \tag{24}$$

for $0<\beta<\frac{1}{2}-\frac{d(\delta-1)}{2}$, $\theta\in[0,1]$ and $z\ge1$.

From (7) and (10), by using the same argument of (14), we have

$$\begin{aligned}
&E|(Z_i+V_{ij})f_z''(W-\theta(Z_i+V_{ij}))|\\
&\quad\le\frac{C}{(1+z)n^\beta}+CzE|Z_i+V_{ij}|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}.
\end{aligned} \tag{25}$$

Similarly (15) and (16), by using Chebyshev's inequality, we can show that

$$P(W-\theta(Z_i+V_{ij})>\frac{z}{2})\le\frac{Cn^4}{z^4\sigma^4}, \tag{26}$$

and

$$P(W-\theta(Z_i+V_{ij})>\frac{z}{2})\le\frac{C}{z^2}. \tag{27}$$

Let $r>0$ be such that

$$r[\frac{d}{2}+\frac{1}{2}-(\beta+\frac{\delta d}{2})]-\frac{1}{2}-d(\delta-1)]>\beta.$$

Then for $0<\beta<\frac{1}{2}-\frac{d(\delta-1)}{2}$ by (7), (26) and (27), we have

$$\begin{aligned}
E|Z_i|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}&=E|Z_i|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}I\{|Z_i|\ge\frac{1}{n^\beta}\}\\
&+E|Z_i|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}I\{|Z_i|<\frac{1}{n^\beta}\}\\
&\le E|n^{r\beta}Z_i^{r+1}|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}\\
+\frac{1}{n^\beta}EI\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}&\le\Big\{E|n^{r\beta}Z_i^{r+1}|^2\Big\}^{\frac{1}{2}}\Big\{P(W-\theta(Z_i+V_{ij})>\frac{z}{2})\Big\}^{\frac{1}{2}}\\
+\frac{1}{n^\beta}P(W-\theta(Z_i+V_{ij})>\frac{z}{2})&\le\frac{Cn^{r\beta}n^2}{z^2\sigma_n^2}\Big\{E|Z_i^{r+1}|^2\Big\}^{\frac{1}{2}}+\frac{C}{n^\beta z^2}\\
&\le\frac{C}{z^2}\Big\{\frac{n^{r\beta}n^2}{\sigma_n^2}\Big\{\frac{\mu}{\sigma_n^{2r+2}}\Big\}^{\frac{1}{2}}+\frac{1}{n^\beta}\Big\}\le\frac{C}{z^2}\Big\{\frac{n^{r\beta}n^2\mu^{\frac{1}{2}}}{\sigma_n^2\sigma_n^{r+1}}+\frac{1}{n^\beta}\Big\}\\
&\le\frac{C}{z^2}\Big\{\frac{n^{r\beta+2}\mu^{\frac{1}{2}}}{\sigma_n^{r+3}}+\frac{1}{n^\beta}\Big\}\le\frac{C}{z^2}\Big\{\frac{C}{n^{r[\frac{d}{2}+\frac{1}{2}-(\beta+\frac{\delta d}{2})]-\frac{1}{2}-d(\delta-1)}}+\frac{1}{n^\beta}\Big\}\\
&\le\frac{C}{z^2n^\beta}.
\end{aligned} \tag{28}$$

From (10), (26) and (27), we have

$$\begin{aligned}
E|V_{ij}|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}&=E|V_{ij}|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}I\{|V_{ij}|\ge\frac{1}{n^\beta}\}\\
&+E|V_{ij}|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}I\{|V_{ij}|<\frac{1}{n^\beta}\}\\
&\le E|n^{r\beta}V_{ij}^{r+1}|I\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}+\frac{1}{n^\beta}EI\{W-\theta(Z_i+V_{ij})>\frac{z}{2}\}\\
&\le\Big\{E|n^{r\beta}V_{ij}^{r+1}|^2\Big\}^{\frac{1}{2}}\Big\{P(W-\theta(Z_i+V_{ij})>\frac{z}{2})\Big\}^{\frac{1}{2}}+\frac{1}{n^\beta}P(W-\theta(Z_i+V_{ij})>\frac{z}{2})\\
&\le\frac{Cn^{r\beta}n^2}{z^2\sigma_n^2}\Big\{E|V_{ij}^{r+1}|^2\Big\}^{\frac{1}{2}}+\frac{C}{n^\beta z^2}\le\frac{C}{z^2}\Big\{\frac{n^{r\beta}n^2}{\sigma_n^2}\Big\{\frac{\mu}{\sigma_n^{2r+2}}\Big\}^{\frac{1}{2}}+\frac{1}{n^\beta}\Big\}\\
&\le\frac{C}{z^2}\Big\{\frac{n^{r\beta+2}\mu^{\frac{1}{2}}}{\sigma_n^{r+3}}+\frac{1}{n^\beta}\Big\}\le\frac{C}{z^2}\Big\{\frac{C}{n^{r[\frac{d}{2}+\frac{1}{2}-(\beta+\frac{\delta d}{2})]-\frac{1}{2}-d(\delta-1)}}+\frac{1}{n^\beta}\Big\}
\end{aligned}$$

$$\leq \frac{C}{z^2 n^\beta}. \quad (29)$$

Therefore from (25), (28) and (29), we have (24). This completes the proof of Theorem 1.4.

3.2. Proof of Theorem 1.5.

We use the same argument of Theorem 1.4 by taking $d = 0$ and let $\beta \in (0, \frac{1}{2})$.

Remark. In general when we refer to the probability approximation, normal approximation and Poisson approximation are also involved. For Poisson approximation of S_n in case $d \geq 1$, A.D. Barbour, L. Holst and S. Janson [2] showed that the distribution of S_n can be approximated by Poisson distribution with parameter $\lambda_n := ES_n$, when $np \to 0$, i.e. $p = \frac{1}{n^\delta}$, $\delta \geq 1$ and in case $d = 0$ the distribution of S_n can be approximated by Poisson distribution with parameter $\lambda_n := ES_n$, when np is large, i.e., $p = \frac{1}{n^\delta}$, $\delta \in (0,1)$, and give uniform bounds. A. Suntadkarn and K. Neammanee [15], improve a bound of A.D. Barbour, L. Holst and S. Janson in case of a non-uniform bounds in 2007. So from Theorem 1.2 ($d = 0$) and this note, the approximation of S_n with $\delta > 2$ remains an open problem.

References

[1] A.D. Barbour, P. Hall, On bounds to the rate of convergence in the central limit theorem, *Bull. London Math. Soc.*, **17** (1985), 151-156.

[2] A.D. Barbour, L. Holst, S. Janson, *Poisson Approximation*, Oxford Studies in Probability, **2**, Clarendon Press, Oxford (1992).

[3] A.D. Barbour, M. Karoñski, A. Ruciñski, A central limit theorem for decomposable random variables with application to random graphs, *J. Comb. Th. Ser. B*, **47** (1989), 125-145.

[4] H.R. Bernard, P. Kilworth, M. Evans, C. McCarty, G. Selley, Studying social relations crossculturally, *Ethnology*, **2** (1998), 155-179.

[5] L.H.Y. Chen, The rate of convergence in a central limit theorem for dependent random variables with arbitary index set, *Ann. Probab.* (1987).

[6] P. Erdös, A. Rény, On random graphs. I., *Publ. Math. Debrecen*, **6** (1959), 290-297.

[7] P. Erdös, A. Rény, On the evolution of random graphs, *Magyar Tud. Akad. Mat. Kutato Int. Kozl.*, **5** (1960), 17-61.

[8] P. Erdös, A. Rény, On the strength of connectedness of a random graph, *Acta Math. Acad. Sci. Hungar.*, **12** (1961), 261-267.

[9] T.J. Fararo, M. Sunshine, *A Study of a Biased Friendship Network*, Syracuse University Press (1964).

[10] C.C. Foster, A. Rapoport, C. Orwant, A study of large sociogram: Elimination of free parameters, *Behav. Sci.*, **8** (1963), 56-65.

[11] D.D. Heckatorn, Respondent-driven sampling: A new approach to the study of hidden population, *Soc. Prob.*, **44** (1997).

[12] W. Kordecki, Normal approximation, isolated vertices in random graphs, In: *Random Graphs' 97, Proceedings, Pozan* (Ed-s: M. Karoñski, J. Jaworski, A. Ruciñski), John Wiley and Sons, Chichester (1987), 131-139.

[13] Y. Punkla, N. Chaidee, Normal approximation of number of isolated vertices in a random graph, *Thai Journal of Mathematics*, **4**, No. 1 (2006), 1-10.

[14] C.M. Stein, A bound for the error in the normal approximation to the distribution of a sum of dependent random variables, *Proc. Sixth Berkeley Sympos. Math. Statist. Probab.*, **3** (1972), 583-602.

[15] A. Suntadkarn, K. Neammanee, Poisson approximation of the number of vertices in a random graph, To Appear.