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WEAKLY CONTRA-CONTINUOUS FUNCTIONS

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Abstract: A new form of contra-continuity, called weak contra-continuity, is introduced and used to investigate S-closed spaces and strongly S-closed spaces. It is shown that weak contra-continuity is strictly between contra-continuity and slight continuity. Also the basic properties of weakly contra-continuous functions are developed.

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1. Introduction

Dontchev [5] introduced the notion of a contra-continuous function in 1996 and used this concept to investigate S-closed and strongly S-closed spaces. In 1998 a weak form of contra-continuity, subcontra-continuity, was developed by Baker [2] and recently the concept of firm contra-continuity was introduced by Baker, Caldas, and Jafari [3]. Both of these concepts have been used to study S-closed spaces and strongly S-closed spaces. In this note we introduce a form of contra-continuity, which we call weak contra-continuity, that is weaker than both contra-continuity and firm contra-continuity but yet is still related to S-closed spaces and strongly S-closed spaces. Conditions are established under which weak contra-continuous images of S-closed spaces and strongly S-closed spaces are compact. Also we show that weak contra-continuity is related to several other classes of functions, including slightly continuous functions, g-continuous functions, and a-continuous functions. Finally, several closed graph type properties and other basic properties of these functions are developed.

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2. Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$. A set A is semi-open [9] provided that $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. A subset A of a space X is g-closed [10] if $\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X. A set A is regular closed (respectively, semi-closed, g-open) if its complement is regular open (respectively, semi-open, g-closed).

Definition 1. A space X is said to be S-closed [14] if every semi-open cover of X has a finite subfamily, the closures of whose members cover X.

From [7] a space X is S-closed if and only if every cover of X by regular closed sets has a finite subcover.

Definition 2. A space X is said to be strongly S-closed [5] if every closed cover of X has a finite subcover.

Definition 3. A function $f : X \to Y$ is said to be contra-continuous [5] if $f^{-1}(V)$ is closed for every open subset V of Y.

Definition 4. A function $f : X \to Y$ is said to be subcontra-continuous [2] provided there is an open base $\mathcal{B}$ for Y such that $f^{-1}(V)$ is closed for every $V \in \mathcal{B}$.

Definition 5. A function $f : X \to Y$ is said to be firmly contra-continuous [3] if for every open cover Λ of Y there exists a finite closed cover $\mathcal{F}$ of X such that for every $F \in \mathcal{F}$ there exists $V \in \Lambda$ such that $f(F) \subseteq V$.

Definition 6. A function $f : X \to Y$ is said to be slightly continuous [13] provided that, for every $x \in X$ and for every clopen subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $f(U) \subseteq V$.

Obviously a function is slightly continuous if and only if inverse images of clopen sets are closed.

Definition 7. A function $f : X \to Y$ is said to be weakly continuous [8] if for every $x \in X$ and for every open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $f(U) \subseteq \mathrm{Cl}(V)$.

The following characterization of weak continuity is due to Noiri [11] and Rose [12].

Theorem 2.1. (see [11, 12]) *A function $f : X \to Y$ is weakly continuous if and only if $Cl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every open subset V of Y.*

Definition 8. A function $f : X \to Y$ is said to be g-continuous (see [4]) if $f^{-1}(A)$ is g-closed for every closed subset A of Y.

3. Weakly Contra-Continuous Functions

We define a function $f : X \to Y$ to be weakly contra-continuous provided that for every open subset V of Y and every closed subset A of Y such that $A \subseteq V$, $\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$.

Theorem 3.1. *If $f : X \to Y$ is contra-continuous, then f is weakly contra-continuous.*

Proof. Let V be an open subset of Y and let A be a closed subset of Y such that $A \subseteq V$. Since $f^{-1}(V)$ is closed and $f^{-1}(A) \subseteq f^{-1}(V)$, we have $\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$, which proves that f is weakly contra-continuous. □

Recall that a space is zero dimensional provided that it has a clopen base.

Theorem 3.2. *If $f : X \to Y$ is weakly contra-continuous and Y is zero dimensional, then f is subcontra-continuous.*

Proof. Let $\mathcal{B}$ be a clopen base for Y and let $B \in \mathcal{B}$. Then, since f is weakly contra-continuous, $\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(B)$, which proves that $f^{-1}(B)$ is closed. Hence f is subcontra-continuous with respect to the base $\mathcal{B}$. □

Corollary 3.3. (see [3], Corollary 4.3) *If $f : X \to Y$ is firmly contra-continuous and Y is zero dimensional, then f is subcontra-continuous.*

Theorem 3.4. *If $f : X \to Y$ is continuous, then f is weakly contra-continuous.*

Proof. Let V be an open subset of Y and let A be a closed subset of Y such that $A \subseteq V$. Since $f^{-1}(A)$ is closed, $\mathrm{Cl}(f^{-1}(A)) = f^{-1}(A) \subseteq f^{-1}(V)$, which proves that f is weakly contra-continuous. □

It is an immediate consequence of the following result from [3] that firm contra-continuity implies weak contra-continuity.

Theorem 3.5. (see [3], Theorem 4.1) *If $f : X \to Y$ is firmly contra-continuous, then for every open subset V of Y and every closed subset A of Y such that $A \subseteq V$, $Cl(f^{-1}(A)) \subseteq f^{-1}(V)$.*

Since continuity and contra-continuity are independent [5], weak contra-continuity does not imply either continuity or contra-continuity. The following example shows that weak contra-continuity does not imply firm contra-continuity.

Example 3.6. (see [3]) Let $X = \{a, b, c\}$ and assume the topology on X is given by $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. The identity mapping $f : X \to X$ is obviously continuous and therefore weakly contra-continuous. However, since any closed cover of X must contain X, f is not firmly contra-continuous.

Theorem 3.7. *If $f : X \to Y$ is weakly contra-continuous, then f is slightly continuous.*

Proof. Let V be a clopen subset of Y. If we take $A = V$ in the definition of weak contra-continuity, we have $\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(V)$, which proves that $f^{-1}(V)$ is closed. Therefore f is slightly continuous. □

The following example shows that slight continuity is strictly weaker than weak contra-continuity.

Example 3.8. Let X denote the real numbers with the indiscrete topology and let Y be the real numbers with the usual topology. Since Y is connected, the identity mapping $f : X \to Y$ is slightly continuous. However, f is not weakly contra-continuous. Note that $\{0\} \subseteq (-1,1) \subseteq Y$ and that $\{0\}$ is closed, but in X $\mathrm{Cl}(f^{-1}(0)) \not\subseteq f^{-1}((-1,1))$.

Therefore we have the following implications, none of which is reversible.

firm contra-continuity

$\Downarrow$

continuity $\Rightarrow$ weakly contra-continuous $\Rightarrow$ slightly continuous

$\Uparrow$

contra-continuous

Recall that a space X is extremally disconnected (or briefly an ED space) provided that closures of open sets are open.

Theorem 3.9. *If $f : X \to Y$ is weakly contra-continuous and X is an ED space, then for every open set V of Y, $f^{-1}(Cl(V))$ is closed.*

Proof. Since $\mathrm{Cl}(V) \subseteq \mathrm{Cl}(V)$ and $\mathrm{Cl}(V)$ is clopen, we have $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))) \subseteq f^{-1}(\mathrm{Cl}(V))$. □

Corollary 3.10. *If $f : X \to Y$ is weakly contra-continuous and X is an ED space, then f is weakly continuous.*

Proof. Let V be an open subset of Y. Since $f^{-1}(\mathrm{Cl}(V))$ is closed, we have $\mathrm{Cl}(f^{-1}(V)) \subseteq \mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))) = f^{-1}(\mathrm{Cl}(V))$, which proves that f is weakly continuous. □

Definition 9. A function $f : X \to Y$ is said to be approximately continuous (or briefly a-continuous), see [1], if $\mathrm{Cl}(A) \subseteq f^{-1}(V)$ whenever V is an open subset of Y, A is a g-closed subset of X, and $A \subseteq f^{-1}(V)$.

Theorem 3.11. *If $f : X \to Y$ is g-continuous and a-continuous, then f is weakly contra-continuous.*

Proof. Assume V is an open subset of Y, A is a closed subset of Y, and $A \subseteq V$. Since f is g-continuous, $f^{-1}(A)$ is g-closed. Then, since $f^{-1}(A) \subseteq f^{-1}(V)$ and f is a-continuous, $\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$, which proves that f is weakly contra-continuous. □

Theorem 3.12. *If $f : X \to Y$ is weakly contra-continuous and images of g-closed sets are closed, then f is a-continuous.*

Proof. Let V be an open subset of Y and let A be a g-closed subset of X such that $A \subseteq f^{-1}(V)$. Then $f(A)$ is closed and $f(A) \subseteq V$ and, since

f is weakly contra-continuous, $\mathrm{Cl}(f^{-1}(f(A))) \subseteq f^{-1}(V)$. Therefore $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(f^{-1}(f(A))) \subseteq f^{-1}(V)$, which proves that f is a-continuous. □

4. Properties of Weakly Contra-Continuous Functions

We establish conditions under which weakly contra-continuous images of S-closed spaces and strongly S-closed spaces are compact.

In [15] a space is called a P_Σ-space provided that every open set is the union of regular closed sets. We define a slightly weaker property by replacing regular closed sets with closed sets.

Definition 10. A space X is said to be a weakly P_Σ-space provided that, for every open set U of X and every $x \in U$, there exists a closed set A such that $x \in A \subseteq U$.

Definition 11. A function $f : X \to Y$ is said to be contra-semicontinuous (see [6]) if, for every open set V of Y, $f^{-1}(V)$ is semi-closed.

Theorem 4.1. *Assume $f : X \to Y$ is weakly contra-continuous and Y is weakly P_Σ. If X is strongly S-closed, then $f(X)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(X)$ by open subsets of X and let $y \in f(X)$. Then let $V_y \in \mathcal{C}$ such that $y \in V_y$. Since Y is weakly P_Σ, there exists a closed set A_y such that $y \in A_y \subseteq V_y$. Since f is weakly contra-continuous, $\mathrm{Cl}(f^{-1}(A_y)) \subseteq f^{-1}(V_y)$. Therefore the family $\{\mathrm{Cl}(f^{-1}(A_y)) : y \in f(X)\}$ is a closed cover of X and, since X is strongly S-closed, there exists a finite subcover $\{\mathrm{Cl}(f^{-1}(A_{y_i})) : i = 1, \ldots, n\}$. Then we see that $\{V_{y_i} : i = 1, \ldots, n\}$ is a finite subcover of $\mathcal{C}$, which proves that $f(X)$ is compact. □

Theorem 4.2. *Let $f : X \to Y$ be weakly contra-continuous and contra-semicontinuous and assume that Y is weakly P_Σ. If X is S-closed, then $f(X)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(X)$ by open subsets of X and let $y \in f(X)$. Then let $V_y \in \mathcal{C}$ such that $y \in V_y$. Since Y is weakly P_Σ, there exists a closed set A_y such that $y \in A_y \subseteq V_y$. Since f is contra-semicontinuous, $f^{-1}(A_y)$ is semi-open. Therefore the family $\{f^{-1}(A_y) : y \in f(X)\}$ is a cover of X by semi-open sets. Since Y is S-closed, there is a finite subfamily $\{f^{-1}(A_{y_i}) : i = 1, \ldots, n\}$ such that $X = \cup_{i=1}^{n} \mathrm{Cl}(f^{-1}(A_{y_i}))$. Because f is weakly contra-continuous, $\mathrm{Cl}(f^{-1}(A_{y_i})) \subseteq f^{-1}(V_{y_i})$ for every i. It follows that $\{V_{y_i} : i = 1, \ldots, n\}$ is a finite subcover of $\mathcal{C}$ and therefore that $f(X)$ is compact. □

Recall that the graph of a function $f : X \to Y$, denoted by $G(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

Definition 12. A function $f : X \to Y$ is said to have a (regular) contra-closed graph if, for every $(x, y) \in X \times Y - G(f)$, there exist a closed subset A of X containing x and a (regular) closed subset B of Y containing y such that $(A \times B) \cap G(f) = \emptyset$.

Theorem 4.3. *If $f : X \to Y$ is weakly contra-continuous and Y is T_1, then $G(f)$ is contra-closed.*

Proof. Suppose $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$ and, since Y is T_1, there exists an open set V of Y such that $f(x) \in V$ and $y \notin V$. Also, since $\{f(x)\}$ is closed and f is weakly contra-continuous, $\mathrm{Cl}(f^{-1}(f(x))) \subseteq f^{-1}(V)$. Hence we have $(x, y) \in \mathrm{Cl}(f^{-1}(f(x))) \times (Y - V) \subseteq X \times Y - G(f)$, which shows that $G(f)$ is contra-closed. □

Theorem 4.4. *If $f :\to Y$ is weakly contra-continuous and Y is Urysohn, then $G(f)$ is regular contra-closed.*

Proof. Suppose $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$ and, since Y is Urysohn, there exist open subsets U and V of Y containing y and $f(x)$, respectively, such that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. Therefore $\mathrm{Cl}(V) \subseteq Y - \mathrm{Cl}(U)$ and, since f is weakly contra-continuous, $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))) \subseteq f^{-1}(Y - \mathrm{Cl}(U))$. Thus $(x, y) \in \mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))) \times \mathrm{Cl}(U) \subseteq X \times Y - G(f)$. Hence $G(f)$ is regular contra-closed. □

Recall that the graph function of $f : X \to Y$ is the function $g : X \to X \times Y$ given by $g(x) = (x, f(x))$ for every $x \in X$.

Theorem 4.5. *If the graph unction of $f : X \to Y$, $g : X \to X \times Y$, is weakly contra-continuous, then f is weakly contra-continuous.*

Proof. Assume $A \subseteq V \subseteq Y$, where A is closed and V is open. Then $X \times A \subseteq X \times V \subseteq X \times Y$, $X \times A$ is closed in $X \times Y$, and $X \times V$ is open in $X \times Y$. Then, using the fact that g is weakly contra-continuous, we obtain $\mathrm{Cl}(f^{-1}(A)) = \mathrm{Cl}(g^{-1}(X \times A)) \subseteq g^{-1}(X \times V) = f^{-1}(V)$. Therefore f is weakly contra-continuous. □

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