EXISTENCE AND UNIQUENESS RESULTS FOR
A CLASS OF NONLINEARDIFFERENTIAL SYSTEMS

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Abstract: We study the existence and uniqueness of the strong and weak solutions to a nonlinear differential system with second-order differences, subject to some extreme conditions and initial data.

AMS Subject Classification: 34G20, 35L55, 47H05, 39A12
Key Words: differential system, extreme conditions, maximal monotone operator, Cauchy problem, strong solution, weak solution

1. Introduction

Let $H$ be a real Hilbert space with the scalar product $\langle \cdot , \cdot \rangle$ and the associated norm $\| \cdot \|$. We consider the nonlinear differential system with second-order differences

$$
\begin{align*}
    u_j'(t) + \frac{v_{j+1}(t) - 2v_j(t) + v_{j-1}(t)}{h_j^2} + c_j A(u_j(t)) & \ni f_j(t), \\
v_j'(t) - \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h_j^2} + d_j B(v_j(t)) & \ni g_j(t),
\end{align*}
$$

(S)

0 < t < T, j = 1, N, in $H$, with the extreme conditions

$$
\begin{align*}
    \left( \begin{array}{c}
    u_1(t) - u_0(t) \\
v_1(t) + v_0(t)
\end{array} \right) & \in \alpha \left( \begin{array}{c}
v_1(t) \\
u_1(t)
\end{array} \right), \\
\left( \begin{array}{c}
-u_{N+1}(t) + u_N(t) \\
v_{N+1}(t) - v_N(t)
\end{array} \right) & \in \beta \left( \begin{array}{c}
v_N(t) \\
u_N(t)
\end{array} \right),
\end{align*}
$$

(EC)

Received: August 20, 2007 © 2007, Academic Publications Ltd.
for $0 < t < T$ and the initial data

$$u_j(0) = u_{j0}, \quad v_j(0) = v_{j0}, \quad j = 1, N; \quad (ID)$$

where $N \in \mathbb{N}, N \geq 2, T > 0, c_j, d_j, h_j, h_{ij} > 0$, for all $j = 1, N, \alpha, \beta$ and $A, B$ are operators in $H^2$, respectively $H$, which satisfy some assumptions.

The above problem is a discrete version with respect to $x$ (with $H = \mathbb{R}$) of the nonlinear system

$$\begin{cases}
\frac{\partial u}{\partial t}(t, x) + \frac{\partial^2 v}{\partial x^2}(t, x) + c(x)A(u(t, x)) \ni f(t, x), \\
\frac{\partial v}{\partial t}(t, x) - \frac{\partial^2 u}{\partial x^2}(t, x) + d(x)B(v(t, x)) \ni g(t, x),
\end{cases} \quad (S)_0$$

subject to boundary conditions

$$\begin{pmatrix}
\frac{\partial u}{\partial x}(t, 0) \\
\frac{\partial v}{\partial x}(t, 0)
\end{pmatrix} \in \alpha \begin{pmatrix}
u(t, 0) \\
u(t, 0)
\end{pmatrix}, \quad \begin{pmatrix}
\frac{\partial u}{\partial x}(t, 1) \\
\frac{\partial v}{\partial x}(t, 1)
\end{pmatrix} \in \beta \begin{pmatrix}
u(t, 1) \\
u(t, 1)
\end{pmatrix}, \quad (BC)_0$$

for $0 < t < T$ and the initial data

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad 0 < x < 1. \quad (IC)_0$$

This problem and some generalizations of it (with higher-order partial derivatives, time dependent coefficients in $(S)_0$ or extra functions in $(BC)_0$) have been studied in Luca [6], Luca-Tudorache [9], Moroşanu et al [10]. The conditions $(BC)_0$ are general ones. By making suitable choices of $\alpha$ and $\beta$ we deduce many classical boundary conditions.

In this paper we investigate the existence and uniqueness of the strong and weak solutions for the problem $(S)+(EC)+(ID)$. In our proofs we use some results related to maximal monotone operators and nonlinear evolution equations in Hilbert spaces (see the monographs Barbu [2], Brezis [3], Ladde et al [4], Lakshmikantham et al [5]). For other differential and difference equations in abstract spaces we mention the papers Agarwal et al [1], Luca [7], Luca [8], Rousseau et al [11].

We present the assumptions that we use in the sequel:

(H1) The operators $A : D(A) \subset H \rightarrow H, B : D(B) \subset H \rightarrow H$ are maximal monotone, possibly multivalued.

(H2) The operators $\alpha : D(\alpha) \subset H^2 \rightarrow H^2, \beta : D(\beta) \subset H^2 \rightarrow H^2$ are maximal monotone, possibly multivalued.
(H3) i) The operators $\alpha$ and $\beta$ are bounded on bounded sets.
   
   ii) $(\text{int}D(\alpha)) \cap (D(B) \times D(A)) \neq \emptyset$ and $(\text{int}D(\beta)) \cap (D(B) \times D(A)) \neq \emptyset$.

(H4) The constants $h_j > 0$, $\overline{h}_j > 0$, for all $j = 1, N$.

(H5) The constants $c_j > 0$, $d_j > 0$, for all $j = 1, N$.

2. Existence and Uniqueness Results for Solutions

We express our problem $(S) + (EC) + (ID)$ as a Cauchy problem in a certain Hilbert space, using some maximal monotone operators. We consider the Hilbert space $X = H^{2N} = \{(u_1, u_2, \ldots, u_N, v_1, v_2, \ldots, v_N)^T; \ u_j, v_j \in H, \ j = 1, N\}$ with the scalar product $((u_1, \ldots, u_N, v_1, \ldots, v_N)^T, (\overline{u}_1, \ldots, \overline{u}_N, \overline{v}_1, \ldots, \overline{v}_N)^T)_X = \sum_{j=1}^{N} h_j^2 \langle u_j, \overline{u}_j \rangle + \sum_{j=1}^{N} h_j^2 \langle v_j, \overline{v}_j \rangle$ and the corresponding norm $\| \cdot \|_X$.

We introduce the operator $A_1: D(A_1) = X \to X,$

\[
A_1((u_1, u_2, \ldots, u_N, v_1, v_2, \ldots, v_N)^T) = \left( \frac{v_2 - 2v_1}{h_1^2}, \frac{v_3 - 2v_2 + v_1}{h_2^2}, \ldots, \frac{v_N - 2v_{N-1} + v_{N-2}}{h_{N-1}^2}, \frac{-2v_N + v_{N-1}}{h_N^2}, \frac{-u_N - 2u_{N-1} + u_{N-2}}{h_{N-1}^2}, \frac{-2u_N + u_{N-1}}{h_N^2} \right)^T,
\]

and the operator $A_2: D(A_2) \subset X \to X,$ $D(A_2) = \{(u_1, \ldots, u_N, v_1, \ldots, v_N)^T, (v_1, u_1)^T \in D(\alpha), (v_N, u_N)^T \in D(\beta)\},$

\[
A_2((u_1, \ldots, u_N, v_1, \ldots, v_N)^T) = \left\{ \left( \frac{v_0}{h_1^2}, 0, \ldots, 0, \frac{v_{N+1}}{h_N^2}, -\frac{u_0}{h_1^2}, 0, \ldots, 0, -\frac{u_{N+1}}{h_N^2} \right)^T, \right.
\]

\[
(u_1 - u_0, -v_1 + v_0)^T
\]

$\in \alpha((v_1, u_1)^T), (-u_{N+1} + u_N, v_{N+1} - v_N)^T \in \beta((v_N, u_N)^T) \bigg\}.$

**Lemma 1.** If the assumption (H4) hold, then the operator $A_1$ is maximal monotone in $X$.

**Lemma 2.** If the assumptions (H2) and (H4) hold, then the operator $A_2$ is maximal monotone in $X$. 
We now define the operator \( A : D(A) = D(A_2) \subset X \rightarrow X, A(U) = A_1(U) + A_2(U). \)

**Lemma 3.** If the assumptions (H2) and (H4) hold, then the operator \( A \) is maximal monotone in \( X \).

Next, we define the operator \( B : D(B) \subset X \rightarrow X, D(B) = D(A)^N \times D(B)^N, \)

\[
B((u_1, u_2, \ldots, u_N, v_1, v_2, \ldots, v_N)^T) = \{(c_1\gamma_1, c_2\gamma_2, \ldots, c_N\gamma_N, d_1\delta_1, d_2\delta_2, \ldots, d_N\delta_N)^T, \gamma_i \in A(u_i), \delta_i \in B(v_i), i = 1, N\}.
\]

**Lemma 4.** If the assumptions (H1), (H4) and (H5) hold, then the operator \( B \) is maximal monotone in \( X \).

**Theorem 1.** If the assumptions (H1), (H2), ((H3)i) or (H3)ii), (H4) and (H5) hold, then the operator \( A + B \) is maximal monotone.

Using the operators \( A \) and \( B \) our problem can be written as the following Cauchy problem in the space \( X \)

\[
\begin{aligned}
\{ & \frac{dU}{dt}(t) + A(U(t)) + B(U(t)) \ni F(t), \\
& U(0) = U_0,
\end{aligned}
\]

where \( U = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T, U_0 = (u_{01}, \ldots, u_{0N}, v_{01}, \ldots, v_{0N})^T, F = (f_1, \ldots, f_N, g_1, \ldots, g_N)^T. \)

**Theorem 2.** Assume that the assumptions (H1), (H2), ((H3)i) or (H3)ii), (H4) and (H5) hold. If \((v_{10}, u_{01})^T \in D(\alpha) \cap (D(B) \times D(A)), u_{j0} \in D(A), v_{j0} \in D(B),\) for all \( j = 2, N - 1, (v_{01}, u_{00})^T \in D(\beta) \cap (D(B) \times D(A)) \) (that is \( U_0 \in D(A) \cap D(B) \)), \( f_j, g_j \in W^{1,1}(0, T; H), j = 1, N \), then there exist unique functions \( u_j, v_j \in W^{1,\infty}(0, T; H), j = 1, N \), \( (v_1(t), u_1(t))^T \in D(\alpha) \cap (D(B) \times D(A)), (v_N(t), u_N(t))^T \in D(\beta) \cap (D(B) \times D(A)), u_j(t) \in D(A), v_j(t) \in D(B),\) for all \( j = 2, N - 1,\) for all \( t \in [0, T]\), that verify the system (S) and the extreme conditions (EC) for all \( t \in [0, T] \) and the initial data (ID). Besides, \( u_j, v_j, j = 1, N \) are everywhere differentiable from right in the topology of \( H \) and

\[
\begin{aligned}
\frac{d^+u_j}{dt}(t) &= \left( f_j(t) - c_jA(u_j(t)) - \frac{v_{j+1}(t) - 2v_j(t) + v_{j-1}(t)}{h_j^2} \right)^0, \\
& j = 2, N - 1, \\
\frac{d^+v_j}{dt}(t) &= \left( g_j(t) - d_jB(v_j(t)) + \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h_j^2} \right)^0,
\end{aligned}
\]
for all $t \in [0, T)$, with $(u_1(t) - u_0(t), -v_1(t) + v_0(t))^T \in \alpha((v_1(t), u_1(t))^T),$

$(-u_{N+1}(t) + u_N(t), v_{N+1}(t) - v_N(t))^T \in \beta((v_N(t), u_N(t))^T)$, for all $t \in [0, T)$.

Proof. By Theorem 1, the operator $A + B$ is maximal monotone in $X$. Using Barbu [2], Theorem 2.2, Corollary 2.1, Chapter III, we deduce that for $U_0 \in D(A) \cap D(B)$ and $F \in W^{1,1}(0, T; X)$, the problem $(P) \equiv (S) + (EC) + (ID)$ has a unique strong solution $U = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T \in W^{1,\infty}(0, T; X)$, $U(t) \in D(A) \cap D(B)$, for all $t \in [0, T)$. We can conclude that $U(T) \in D(A) \cap D(B)$, by extending correspondingly the functions $f_j$, $g_j$, $j = 1, N$ and by considering the equation $(P)_j$ in the interval $[0, T + \varepsilon]$, with $\varepsilon > 0$. The solution $U$ is everywhere differentiable from right and $\frac{d^+U}{dt}(0) = (F(t) - A(U(t)) - B(U(t)))^0$, for all $t \in [0, T)$, that is we have the relations from the conclusion of the theorem. Moreover we have

$$
\left\| \frac{d^+U}{dt}(t) \right\|_X \leq \| (F(0) - A(U_0) - B(U_0))^0 \|_X + \int_0^t \left\| \frac{dF}{ds}(s) \right\|_X \, ds, \forall t \in [0, T).
$$

The proof is completed.  \(\square\)

Remark. Under the assumptions of Theorem 2, if $U_0 \in D(A) \cap D(B)$ and $F \in L^1(0, T; X)$, then by Barbu [2], Corollary 2.2, Chapter III, we deduce that the problem $(P)$ has a unique weak solution $U \in C([0, T]; X)$. 

$$j = 2, N - 1,$$

$$\left( \begin{array}{c}
\frac{d^+u_1}{dt}(t) \\
\frac{d^+v_1}{dt}(t)
\end{array} \right) = \left( \begin{array}{c}
f_1(t) - c_1A(u_1(t)) - \frac{v_2(t) - 2v_1(t) + v_0(t)}{h_1^2} \\
g_1(t) - d_1B(v_1(t)) + \frac{u_2(t) - 2u_1(t) + u_0(t)}{h_1^2}
\end{array} \right)^0
$$

$$\left( \begin{array}{c}
\frac{d^+u_N}{dt}(t) \frac{d^+v_N}{dt}(t)
\end{array} \right) = \left( \begin{array}{c}
f_N(t) - c_NA(u_N(t)) - \frac{v_{N+1}(t) - 2v_N(t) + v_{N-1}(t)}{h_N^2} \\
g_N(t) - d_NB(v_N(t)) + \frac{u_{N+1}(t) - 2u_N(t) + u_{N-1}(t)}{h_N^2}
\end{array} \right)^0,
$$
References


