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INEQUALITIES FOR A CLASS OF POSITIVE SOLUTIONS OF DISCRETE EQUATION $\Delta u(n+k) = -p(n)u(n)$ IN THE CRITICAL CASE

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Abstract: The delayed discrete equation $\Delta u(n+k) = -p(n)u(n)$ with a positive coefficient p is considered. The coefficient p has a special form to reflect the critical case. We prove that there exists a class of positive solutions of the above equation if $n \to \infty$ and give their estimation.

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1. Introduction

Let $N(a) := \{a, a+1, \dots\}$ with $a > 1$ a positive integer. In this paper, the delayed scalar linear discrete equation

$$\Delta u(n+k) = -p(n)u(n) \tag{1}$$

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with fixed $k \in \{1, 2, \dots\}$ and variable $n \in N(a)$ is considered. The function $p\colon N(a) \to \mathbf{R}$ is assumed to be positive and has a special form

$$p(n) := \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + \frac{k}{8n^2}\right) \tag{2}$$

which reflects what is called the critical case. We prove that there exists a class of positive solutions of the above equation if $n \to \infty$ and we give their estimation. The proof uses the method described in [1, 2].

2. Preliminary

Let us consider the scalar discrete equation

$$\Delta u(n + \tilde{k}) = f(n, u(n), u(n+1), \dots, u(n+\tilde{k})), \tag{3}$$

where $f(n, u_0, u_1, \dots, u_{\tilde{k}})$ is defined on $N(a) \times \mathbf{R}^{\tilde{k}+1}$ with values in $\mathbf{R}$, $a \in \mathbf{N}$ and $\tilde{k} \in \mathbf{N}$.

Together with the discrete equation (3) we consider an initial problem. It is defined as follows: for a given $s \in \mathbf{N}$, we are seeking the solution of (3) satisfying the $\tilde{k}+1$ initial conditions

$$u(a+s+m) = u^{s+m} \in \mathbf{R},\ m = 0, 1, \dots, \tilde{k} \tag{4}$$

with prescribed constants u^{s+m}.

Let us recall that the solution of the initial problem (3), (4) is defined as an infinite sequence of numbers

$$\{u(a+s) = u^s, u(a+s+1) = u^{s+1}, \dots, u(a+s+\tilde{n}) = u^{s+\tilde{n}},\\ u(a+s+\tilde{n}+1), u(a+s+\tilde{n}+2), \dots\}$$

such that, for any $n \in N(a+s)$, the equality (3) holds.

The existence and uniqueness of the solution of the initial problem (3), (4) is obvious for every $n \in N(a+s)$. Moreover, the initial problem (3), (4) depends continuously on the initial data.

For every $n \in N(a)$, let us define a set $\omega(n)$ as

$$\omega(n) := \{u \in \mathbf{R} : b(n) < u < c(n)\}, \tag{5}$$

where $b(n)$, $c(n)$, $b(n) < c(n)$ are real functions defined on $N(a)$.

The following theorem is a slightly improved version of Theorem 1 in [1].

Theorem 1. *Let $f(n, u_0, u_1, \dots, u_{\tilde{k}})$ be defined on $N(a) \times \mathbf{R}^{\tilde{k}+1}$ with* *values in* $\mathbf{R}$*. If the inequalities*

$$f(n, u_0, u_1, \dots, u_{\tilde{k}-1}, b(n+\tilde{k})) - b(n+\tilde{k}+1) + b(n+\tilde{k}) < 0 \tag{6}$$

and

$$f(n, u_0, u_1, \dots, u_{\tilde{k}-1}, c(n+\tilde{k})) - c(n+\tilde{k}+1) + c(n+\tilde{k}) > 0 \tag{7}$$

hold for every $n \in N(a)$ *and every* $u_0 \in \omega(n), u_1 \in \omega(n+1), \dots, u_{\tilde{k}-1} \in \omega(n+\tilde{k}-1)$*, then there exists an initial problem*

$$u^*(a+m) = u^*_m \in \mathbf{R},\ m = 0, 1, \dots, \tilde{k}$$

with

$$u^*_0 \in \omega(a), u^*_1 \in \omega(a+1), \dots, u^*_{\tilde{k}} \in \omega(a+\tilde{k})$$

such that the corresponding solution $u = u^*(n)$ *of* (3) *satisfies the inequalities*

$$b(n) < u^*(n) < c(n)$$

for every $n \in N(a)$*.*

3. Existence of a Class of Positive Solutions

In this part we prove the existence of a class of positive solutions of equation (1). In the proof of the corresponding theorem, the following elementary lemmas concerning asymptotic expansions of the involved functions are necessary. Their proofs are omitted since they can be done easily using the binomial formula.

Lemma 1. *For* $n \to \infty$*, and fixed* $\sigma, d \in \mathbf{R}$*, the following asymptotic representation holds:*

$$\left(1 + \frac{d}{n}\right)^{\sigma} = 1 + \frac{\sigma d}{n} + \frac{\sigma(\sigma-1)d^2}{2n^2} + O\left(\frac{1}{n^3}\right). \tag{8}$$

Lemma 2. *For* $n \to \infty$*, and fixed* $\kappa, d \in \mathbf{R}$*, the following asymptotic representation holds:*

$$\ln^{\kappa}(n+d) = (\ln n)^{\kappa}\left[1 + \frac{\kappa d}{n \ln n} - \frac{\kappa d^2}{2n^2 \ln n} + \frac{\kappa(\kappa-1)d^2}{2(n \ln n)^2} + o\left(\frac{1}{n^3}\right)\right]. \tag{9}$$

Theorem 2. *Let* $a \in \mathbf{N}$ *and* $k \in \mathbf{N} \setminus \{0\}$ *be fixed. Let* $\sigma > 0$ *be fixed. Then there exists a positive integer* $a_1 \geq a$ *and a positive solution* $u = u(n)$*,* $n \in N(a_1)$ *of equation* (1) *such that the inequalities*

$$\left(\frac{k}{k+1}\right)^{n} \cdot \sqrt{n \ln^{-\sigma} n} < u(n) < \left(\frac{k}{k+1}\right)^{n} \cdot \sqrt{n \ln^{\sigma} n} \tag{10}$$

hold for every $n \in N(a_1)$*.*

Proof. In the proof, Theorem 1 with $\tilde{k} = k$ is used. We define

$$f(n, u(n), u(n+1), \dots, u(n+k)) := -p(n)u(n)$$

and

$$b(n) := \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln^{-\sigma} n}\,, \qquad c(n) := \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln^{\sigma} n}$$

for every $n \in N(a)$. In this case (see (5))

$$\begin{aligned} \omega(n) &: = \{u \in \mathbf{R} : b(n) < u < c(n)\} \\ &= \left\{u \in \mathbf{R}\colon \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln^{-\sigma} n} < u < \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln^{\sigma} n}\right\}. \end{aligned}$$

Let us verify that the inequality of the type (6) holds. Let $n \in N(a)$ be sufficiently large and $\tilde{k} = k$. Then

$$\begin{aligned} & f(n, u_0, u_1, \dots, u_{k-1}, b(n+k)) - b(n+k+1) + b(n+k) \\ &\quad = -p(n)u_0 - \left(\frac{k}{k+1}\right)^{n+k+1} \cdot \sqrt{(n+k+1)\ln^{-\sigma}(n+k+1)} \\ &\qquad + \left(\frac{k}{k+1}\right)^{n+k} \cdot \sqrt{(n+k)\ln^{-\sigma}(n+k)}. \end{aligned}$$

Since $u_0 \in \omega(n)$, i.e.

$$-u_0 < -\left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln^{-\sigma} n}, \;\; n \in N(a),$$

we get

$$\begin{aligned} & f(n, u_0, u_1, \dots, u_{k-1}, b(n+k)) - b(n+k+1) + b(n+k) < -p(n)\left(\frac{k}{k+1}\right)^n \\ &\quad \cdot \sqrt{n \ln^{-\sigma} n} - \left(\frac{k}{k+1}\right)^{n+k+1} \cdot \sqrt{(n+k+1)\ln^{-\sigma}(n+k+1)} \\ &\qquad + \left(\frac{k}{k+1}\right)^{n+k} \cdot \sqrt{(n+k)\ln^{-\sigma}(n+k)} = \mathcal{B}_1 \end{aligned}$$

with

$$\begin{aligned} \mathcal{B}_1 := \mathcal{V}_1 \cdot \Bigg[& -p(n)\left(\frac{k}{k+1}\right)^{-k} - \frac{k}{k+1} \cdot \sqrt{1 + \frac{k+1}{n}} \cdot \sqrt{\frac{\ln^{\sigma} n}{\ln^{\sigma}(n+k+1)}} \\ & + \sqrt{1 + \frac{k}{n}} \cdot \sqrt{\frac{\ln^{\sigma} n}{\ln^{\sigma}(n+k)}}\Bigg] \end{aligned}$$

and

$$\mathcal{V}_1 := \left(\frac{k}{k+1}\right)^{n+k} \sqrt{n \ln^{-\sigma} n}.$$

Applying formula (8) with $\sigma = 1/2$, $d = k+1$ to the expression

$$\sqrt{1 + \frac{k+1}{n}}$$

and with $\sigma = 1/2$, $d = k$ to the expression

$$\sqrt{1 + \frac{k}{n}}$$

and formula (9) with $\kappa = -\sigma/2$, $d = k+1$ to the expression

$$\sqrt{\frac{\ln^\sigma n}{\ln^\sigma(n+k+1)}}$$

and with $\kappa = -\sigma/2$, $d = k$ to the expression

$$\sqrt{\frac{\ln^\sigma n}{\ln^\sigma(n+k)}}\,,$$

we get

$$\sqrt{1 + \frac{k+1}{n}} = 1 + \frac{k+1}{2n} - \frac{(k+1)^2}{8n^2} + O\left(\frac{1}{n^3}\right),$$

$$\sqrt{1 + \frac{k}{n}} = 1 + \frac{k}{2n} - \frac{k^2}{8n^2} + O\left(\frac{1}{n^3}\right),$$

$$\sqrt{\frac{\ln^\sigma n}{\ln^\sigma(n+k+1)}} = 1 - \frac{\sigma(k+1)}{2n\ln n} + \frac{\sigma(k+1)^2}{4n^2\ln n} + \frac{\sigma(\sigma+2)(k+1)^2}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right)$$

and

$$\sqrt{\frac{\ln^\sigma n}{\ln^\sigma(n+k)}} = 1 - \frac{\sigma k}{2n\ln n} + \frac{\sigma k^2}{4n^2\ln n} + \frac{\sigma(\sigma+2)k^2}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right).$$

Then

$$\begin{aligned}\sqrt{1 + \frac{k+1}{n}} \cdot \sqrt{\frac{\ln^\sigma n}{\ln^\sigma(n+k+1)}} \\ &= \left[1 + \frac{k+1}{2n} - \frac{(k+1)^2}{8n^2} + O\left(\frac{1}{n^3}\right)\right] \\ &\times \left[1 - \frac{\sigma(k+1)}{2n\ln n} + \frac{\sigma(k+1)^2}{4n^2\ln n} + \frac{\sigma(\sigma+2)(k+1)^2}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right)\right]\end{aligned}$$

$$= 1 + \frac{k+1}{2n} - \frac{\sigma(k+1)}{2n\ln n} - \frac{(k+1)^2}{8n^2} + \frac{\sigma(\sigma+2)(k+1)^2}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right)$$

and

$$\sqrt{1+\frac{k}{n}} \cdot \sqrt{\frac{\ln^\sigma n}{\ln^\sigma(n+k)}}$$

$$= \left[1 + \frac{k}{2n} - \frac{k^2}{8n^2} + O\left(\frac{1}{n^3}\right)\right]$$

$$\times \left[1 - \frac{\sigma k}{2n\ln n} + \frac{\sigma k^2}{4n^2\ln n} + \frac{\sigma(\sigma+2)k^2}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right)\right]$$

$$= 1 + \frac{k}{2n} - \frac{\sigma k}{2n\ln n} - \frac{k^2}{8n^2} + \frac{\sigma(\sigma+2)k^2}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right).$$

Now

$$\mathcal{B}_1 = \mathcal{V}_1 \cdot \left[-\left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + \frac{k}{8n^2}\right) \cdot \left(\frac{k}{k+1}\right)^{-k} \right.$$

$$- \frac{k}{k+1} \cdot \left(1 + \frac{k+1}{2n} - \frac{\sigma(k+1)}{2n\ln n} - \frac{(k+1)^2}{8n^2} + \frac{\sigma(\sigma+2)(k+1)^2}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right)\right)$$

$$\left. +1 + \frac{k}{2n} - \frac{\sigma k}{2n\ln n} - \frac{k^2}{8n^2} + \frac{\sigma(\sigma+2)k^2}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right) \right]$$

$$= \mathcal{V}_1 \cdot \left[-\frac{\sigma(\sigma+2)k}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right)\right].$$

It is obvious that there exists an integer $a_1 \geq a$ such that, for $n \geq a_1$, the inequality

$$-\frac{\sigma(\sigma+2)k}{8n^2\ln^2 n} + O\left(\frac{1}{n^3}\right) < 0$$

holds, i.e.

$$\mathcal{B}_1 < 0$$

for every $n \in N(a_1)$. Consequently,

$$f(n, u_0, u_1, \ldots, u_{k-1}, b(k+n)) - b(n+k+1) + n(n+k) < 0,$$

i.e. the inequality (6) holds for every $n \in N(a_1)$.

Let us start the verification of inequality (7). For sufficiently large $k \in N(a)$ and for $\tilde{n} = n$, we get:

$$f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n)$$

$$= -p(k)u_0 - \sqrt{k+n+1} \cdot \left(\frac{n}{n+1}\right)^{k+n+1} + \sqrt{k+n} \cdot \left(\frac{n}{n+1}\right)^{k+n}.$$

Since $u_0 \in \omega(k)$, i.e.

$$-u_0 > -\sqrt{k} \cdot n^k/(n+1)^k, \ \ k \in N(a),$$

we get

$$f(k,u_0,u_1,\dots,u_{n-1},c(k+n))-c(k+n+1)+c(k+n) > -p(k)\sqrt{k}\cdot\left(\frac{n}{n+1}\right)^k$$
$$-\left(\frac{n}{n+1}\right)^k\cdot\left(\frac{n}{n+1}\right)^{n+1}\sqrt{k+n+1}+\left(\frac{n}{n+1}\right)^k\cdot\left(\frac{n}{n+1}\right)^n\sqrt{k+n}=\mathcal{H}_1$$

with

$$\mathcal{H}_1 := \left(\frac{n}{n+1}\right)^k\sqrt{k}$$
$$\cdot\left[-p(k)-\left(\frac{n}{n+1}\right)^{n+1}\cdot\sqrt{1+\frac{n+1}{k}}+\left(\frac{n}{n+1}\right)^n\cdot\sqrt{1+\frac{n}{k}}\,\right].$$

Applying now twice formula (8): with $\sigma = 1/2$, $d = n+1$ to the expression

$$\sqrt{1+\frac{n+1}{k}}$$

and with $\sigma = 1/2$, $d = n$ to the expression

$$\sqrt{1+\frac{n}{k}}\ ,$$

we continue:

$$\mathcal{H}_1 = \left(\frac{n}{n+1}\right)^k\sqrt{k}$$
$$\times\left[-p(k)-\left(\frac{n}{n+1}\right)^{n+1}\cdot\left(1+\frac{n+1}{2k}-\frac{(n+1)^2}{8k^2}+\frac{(n+1)^3}{16k^3}+O\left(\frac{1}{k^4}\right)\right)\right.$$
$$\left.+\left(\frac{n}{n+1}\right)^n\cdot\left(1+\frac{n}{2k}-\frac{n^2}{8k^2}+\frac{n^3}{16k^3}+O\left(\frac{1}{k^4}\right)\right)\right]=$$
$$\left(\frac{n}{n+1}\right)^k\sqrt{k}\cdot\left[-p(k)-\left(\frac{n}{n+1}\right)^{n+1}+\left(\frac{n}{n+1}\right)^n+\frac{1}{k}\left(\frac{-n^{n+1}}{2(n+1)^n}+\frac{n^{n+1}}{2(n+1)^n}\right)\right.$$
$$\left.+\frac{1}{k^2}\left(\frac{n^{n+1}}{8(n+1)^{n-1}}-\frac{n^{n+2}}{8(n+1)^n}\right)+\frac{1}{k^3}\left(\frac{-n^{n+1}}{16(n+1)^{n-2}}+\frac{n^{n+3}}{16(n+1)^n}\right)+O\left(\frac{1}{k^4}\right)\right]$$
$$=\left(\frac{n}{n+1}\right)^k\sqrt{k}\cdot\left[-p(k)+\left(\frac{n}{n+1}\right)^n\frac{-n+n+1}{n+1}+\frac{1}{k^2}\frac{n^{n+1}(n+1)-n^{n+2}}{8(n+1)^n}\right.$$
$$\left.+\frac{1}{k^3}\frac{-n^{n+1}(n+1)^2+n^{n+3}}{16(n+1)^n}+O\left(\frac{1}{k^4}\right)\right]=\mathcal{H}_2$$

with

$$\mathcal{H}_2 := \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \left[-p(k) + \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} + \frac{1}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n \right.$$
$$\left. + \frac{1}{16k^3} \frac{-2n^{n+2} - n^{n+1}}{(n+1)^n} + O\left(\frac{1}{k^4}\right)\right].$$

Due to (2) we obtain

$$\mathcal{H}_2 \geq \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \left[- \left(\frac{n}{n+1}\right)^n \cdot \left(\frac{1}{n+1} + \frac{\theta n}{8k^2}\right)\right.$$
$$\left. + \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} + \frac{1}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n + \frac{1}{16k^3} \frac{-2n^{n+2} - n^{n+1}}{(n+1)^n} + O\left(\frac{1}{k^4}\right)\right]$$
$$= \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \mathcal{H}_3$$

with

$$\mathcal{H}_3 := \frac{1-\theta}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n - \frac{1}{16k^3} \frac{n^{n+1}(1+2n)}{(n+1)^n} + O\left(\frac{1}{k^4}\right).$$

Now it is obvious that there exists an integer $a_1 \geq a$ such that the inequality

$$\mathcal{H}_3 > 0$$

holds for every $k \in N(a_1)$. Consequently,

$$f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) > 0,$$

i.e. the inequality (7) holds for every $k \in N(a_1)$.

This means that all the suppositions of Theorem 1 are met with $a := a_1$, $\tilde{k} = k$ and thus there exists an initial problem

$$u^*(a_1 + m) = u^*_m \in \mathbf{R},\ m = 0, 1, \ldots, k$$

with

$$u^*_0 \in \omega(a_1), u^*_1 \in \omega(a_1+1), \ldots, u^*_k \in \omega(a_1+k)$$

such that the corresponding solution $u = u^*(k)$ of equation (1) satisfies the inequalities

$$b(n) = \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln^{-\sigma} n} < u(n) < c(n) = \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln^{\sigma} n}$$

for every $n \in N(a_1)$, i.e. (10) holds. The theorem is proved. □

Remark 3. Due to the linearity of equation (1), we deduce that inequalities (10) in Theorem 2 suggests the asymptotic behavior of (at least) one parametric family of solutions generated with the indicated solution $u(n)$.

4. Comparisons and Concluding Remarks

The following result is well-known (see [3, p. 192]).

Theorem 4. *Let* $k \in \mathbf{N} \setminus \{0\}$, $p(n) > 0$ *for* $n \geq 0$, *and*

$$p(n) \leq \frac{k^k}{(k+1)^{k+1}}\,. \tag{11}$$

Then the difference equation (1) *(where* $n = 0, 1, 2, \dots$ *) has a positive solution*

$$\{u(0), u(1), u(2), \dots\}.$$

As noted in [3, p. 179], for $p(n) \equiv p = \text{const}$, the inequality (11) is sharp since, in this case, the necessary and sufficient condition for the oscillation of all the solutions of (1) is the inequality

$$p > \frac{k^k}{(k+1)^{k+1}}.$$

Therefore the value

$$p_c = \frac{k^k}{(k+1)^{k+1}}$$

is called a critical value since it separates the case of all the solutions oscillating from the case of there existing a positive solution. The function $p(n)$ we defined by formula (2) can be called a critical function since

$$\lim_{n\to\infty} p(n) = \lim_{n\to\infty} \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + \frac{k}{8n^2}\right) = \frac{k^k}{(k+1)^{k+1}} = p_c.$$

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