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COMPUTING CASIMIR OPERATORS FOR THE GENERALIZED UPPER HALF PLANE

Kevin A. Broughan
Department of Mathematics
University of Waikato
3105, Private Bag, Hamilton, NEW ZEALAND
e-mail: kab@waikato.ac.nz

Abstract: A program which computes the Casimir partial differential operators for the generalized upper half plane $\mathfrak{h}^n = GL(n,\mathbb{R})/O(n,\mathbb{R}) \times \mathbb{R}^\times$ is described. The output from this program for small values of the dimension n and order m and eigenvalues for the operators acting on the "power function" I_ν are also given. In dimension 3 the relationship between Bump's forms for the operators and the operators computed here is found to be $\Delta_1 = \Delta_{2,3}/2, \Delta_2 = \Delta_{3,3}/3 - \Delta_{2,3}/2$ provided that a small amendment is made to the exponent of one factor in one term in Bump's operator.

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1. Introduction

The upper half plane $\mathfrak{h}^n$ is a generalization of the complex upper half plane $\mathfrak{h}^2$. Automorphic forms on $\mathfrak{h}^n$ for $SL(n,\mathbb{Z})$ and its subgroups are generalizations of modular forms for $SL(2,\mathbb{Z})$ and its subgroups. The automorphic functions are eigenfunctions of all operators in the center of the real algebra of partial differential operators on $\mathfrak{h}^n$ with real polynomial coefficients. This algebra is generated by the so-called Casimir operators defined below. They are gener-

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alizations of the hyperbolic Laplacian $\Delta = y^2(\partial_{x,x} + \partial_{y,y})$. Thus the Casimir operators have a central role in studies of forms on $\mathfrak{h}^n$.

The natural coordinates on $\mathfrak{h}^n$ come from the Iwasawa form for matrices in $GL(n, \mathbb{R})$, so as the dimension and order increase, "heroic" algebraic manipulation is required and errors are easy to make. There have been no publicly available programs to compute these operators. Because of this a number of programs are provided in the *Mathematica* package GL(n)pack [1] for deriving and applying them. There are examples in the literature of operators which, at least on the surface, differ from those given by the program. For example $\Delta_{3,3}$ the third order operator in dimension 3 as given in [2, p. 34] is different. So the purpose of this paper is (a) to show explicitly how the operators are derived, (b) to list expressions for the operators for small values of the dimension and order, (c) to show how the size of the operators grows with the dimension, (d) to validate the operators and (e) to compute their eigenvalues when acting on the power function as defined below. In addition some observations are made concerning the qualitative properties of the operators and eigenvalues.

First some standard definitions. The real general linear and orthogonal groups are $GL(n, \mathbb{R})$ and $O(n, \mathbb{R})$ respectively. The subgroup of diagonal matrices with non-zero constant value is $\mathbb{R}^\times$. Each $g \in GL(n, \mathbb{R})$ can be expressed uniquely as $g = x.y.o.d$, the so-called Iwasawa form, where x is uppper triangular unipotent, y is positive diagonal with 1 in the bottom entry, o orthogonal and d is in $\mathbb{R}^\times$. Then $O(n, \mathbb{R}) \times \mathbb{R}^\times$ acts on the right on $GL(n, \mathbb{R})$ by matrix multiplication and we set

$$\mathfrak{h}^n = GL(n, \mathbb{R})/O(n, \mathbb{R}) \times \mathbb{R}^\times.$$

From the Iwasawa form we can express each element of $\mathfrak{h}^n$ as the matrix $x.y$. It is convenient to use the coordinates $y_1, \cdots, y_{n-1}$ for the matrix y where the 1,1 entry is $y_1 \cdots y_{n-1}$, the (i,i)-th is $y_1 \cdots y_{n-i}$ so the $(n-1, n-1)$-th is y_1. These $n-1$ variables (y_i), together with the $n(n-1)/2$ variables $(x_{i,j})$ from the above diagonal terms of the unipotent matrix x, constitute the so-called Iwasawa coordinates for $\mathfrak{h}^n$.

If $f : GL(n, \mathbb{R}) \to \mathbb{C}$ is a smooth (i.e. C^∞) function and $\alpha \in \mathfrak{gl}(n, \mathbb{R})$ is a real $n \times n$ matrix let

$$D_\alpha f(g) := \frac{\partial}{\partial t} f(g.\exp(\mathrm{t}\alpha))|_{\mathrm{t}=0}.$$

It follows that if $a_1, \cdots, \alpha_m$ are matrices the composite operator can be written

$$D_{\alpha_1} \circ \cdots \circ D_{\alpha_m} f(g) = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_m} f(g.(id + t_1\alpha_1) \cdots (id + t_m\alpha_m))|_{t_i=0}\,,$$

where $1 \le i \le m$. If $e_{i,j}$ is the $n \times n$ matrix with 1 in the $(i,j)^{\text{th}}$ position and 0 elsewhere write $D_{i,j} := D_{e_{i,j}}$. Then the Casimir operator for dimension n and order m is the summation

$$\Delta_{m,n} := \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} D_{i_1,i_2} \circ \cdots \circ D_{i_m,i_1}.$$

Note that there are $m.n^m$ operators in this sum - e.g. for $n = 4, m = 4$ this is over 1000, already beyond hand calculation.

The complexity in any computation of the Casimir operators comes from the fact that the functions f are defined on $\mathfrak{h}^n$, so the variables are the Iwasawa coordinates, rather than those inherited from $GL(n, \mathbb{R})$. This is, on the face of it, $O(mn^m)$, and is not expected to a able to be improved.

Introductory information regarding the operators is set out in [3, Sections 2.3, 2.4]. For background literature on the construction of the ring of invariant differential operators see [4, 6, 5]. For information about GL(n)pack and its free availability see the web page [2].

2. The Program

The *Mathematica* code set out below is a clarified and simplified form of that included in the GL(n)pack package [1]. The function calls UpperTriangular, IwasawaForm, IwasawaXVariables, IwasawaYVariables are self explanatory. The functions IntegerDigits, Expand, Module and Simplify are from *Mathematica*. There are two steps. First the indices $(i_1, \cdots, i_m)$ are computed. Then the operators, for each index m-tuple, are computed and added to the existing operator.

```
GetCasimirOperator[m_, n_, x_, y_, fun_] :=
    Module[{op = 0, term, is},
       Do[(is = ZeroPad[Reverse[IntegerDigits[i, n]], m] + 1;
           term = OneCasimir[n, is, x, y, fun];
           op = Expand[op + term]), {i, 0, n^m - 1}];
       Return[Simplify[op, Assumptions->Table[y[i]>0, {i,1,n-1}]]]]

OneCasimir[n_, is_, x_, y_, f_] :=
  Module[{X, Y, e = 1, M, m = Length[is], t, Mij, Id, xvars, yvars,
  i, term, A},
    X = UpperTriangular[n, x];
    Y = Table[If[i == j, 1, 0], {i, 1, n}, {j, 1, n}];
    Do[{e = e*y[i], Y[[n - i, n - i]] = e}, {i, 1, n - 1}];
```

```
    Id = Table[If[i == j, 1, 0], {i, 1, n}, {j, 1, n}];
    Mij = Table[0, {i, 1, n}, {j, 1, n}];
    M = Id;
    Do[(Mij = Id;
        Mij[[Part[is, i], Part[is, i + 1]]] =
          t[i] + Mij[[Part[is, i], Part[is, i + 1]]];
        M = M.Mij), {i, 1, m - 1}];
    Mij = Id;
    Mij[[Part[is, m], Part[is, 1]]] = Mij[[Part[is, m], Part[is, 1]]]
                                                      + t[m];
    M = M.Mij;
    A = IwasawaForm[X.Y.M];
    xvars=IwasawaXVariables[A];  yvars=IwasawaYVariables[A];
    fun = Apply[f, Join[xvars,yvars]];
    Do[ fun = D[fun, t[i]], {i, 1, m}];
    term=fun;
    Do[term=term /.t[i]->0, {i,1,m}];
    Return[term]]

ZeroPad[x_, m_] :=
    If[Length[x] >= m, x, Join[x, Table[0, {i, 1, m - Length[x]}]]]
```

3. The Operators

First some notation: For dimension $n = 2$ we use $y := y_1$ and $x := x_{1,2}$ so

$$x.y = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

is the generic element representing $x + \mathrm{iy} \in \mathfrak{h}^2$. For $\Delta_{3,4}$ and $\Delta_{4,4}$ the multi-index symbol ∂ is used defined as

$$\partial^{(n_1,\cdots,n_9)} = \frac{\partial^{n_1+\cdots+n_9}}{\partial x_{1,2}^{n_1}\cdots y_3^{n_9}}$$

reflecting the lexical ordering $(x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}, y_1, y_2, y_3)$ of the Iwasawa variables.

The operators given below are a selection based on the twin criteria size and computation time, given the author's rather standard contemporary workstation and the restrictions on space and time afforded by the interpreted *Mathematica*. However, this sample is regarded as sufficiently extensive to provide exact expressions for operators with small dimension and order and to enable qualitative properties of their form to be induced. Some of these properties are given.

$$\Delta_{2,2} = \Delta_{3,2} = 2y^2\,(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\,,$$

$$\Delta_{4,2} = 2\,y^2\,(3\,\frac{\partial^2}{\partial y^2} + 4\,y\,\frac{\partial^3}{\partial y^3} + y^2\,\frac{\partial^4}{\partial y^4} + 3\,\frac{\partial^2}{\partial x^2} + 4\,y\,\frac{\partial^3}{\partial x^2\partial y} + 2\,y^2\,\frac{\partial^4}{\partial x^2\partial y^2} + y^2\,\frac{\partial^4}{\partial x^4})\,,$$

$$\Delta_{5,2} = 2\,y^2\,(5\,\frac{\partial^2}{\partial y^2} + 8\,y\,\frac{\partial^3}{\partial y^3} + 2\,y^2\,\frac{\partial^4}{\partial y^4} + 5\,\frac{\partial^2}{\partial x^2} + 8\,y\,\frac{\partial^3}{\partial x^2\partial y} + 4\,y^2\,\frac{\partial^4}{\partial x^2\partial y^2} + 2\,y^2\,\frac{\partial^4}{\partial x^4})\,,$$

$$\Delta_{2,3} = 2\,({y_2}^2\,\frac{\partial^2}{\partial y_2^2} - y_1\,y_2\,\frac{\partial^2}{\partial y_1\partial y_2} + {y_1}^2\,\frac{\partial^2}{\partial y_1^2} + {y_1}^2\,\frac{\partial^2}{\partial x_{2,3}^2} + 2\,x_{1,2}\,{y_1}^2\,\frac{\partial^2}{\partial x_{1,3}\partial x_{2,3}} + {x_{1,2}}^2\,{y_1}^2\,\frac{\partial^2}{\partial x_{1,3}^2} + {y_1}^2\,{y_2}^2\,\frac{\partial^2}{\partial x_{1,3}^2} + {y_2}^2\,\frac{\partial^2}{\partial x_{1,2}^2})\,,$$

$$\begin{aligned}\Delta_{3,3} = 3\,y_1\,(&-y_2\,\frac{\partial^2}{\partial y_1\partial y_2} + {y_2}^2\,\frac{\partial^3}{\partial y_1\partial y_2^2} + 2\,y_1\,\frac{\partial^2}{\partial y_1^2} \\ &- y_1\,y_2\,\frac{\partial^3}{\partial y_1^2\partial y_2} + 2\,y_1\,\frac{\partial^2}{\partial x_{2,3}^2} - y_1\,y_2\,\frac{\partial^3}{\partial x_{2,3}^2\partial y_2} + 4\,x_{1,2}\,y_1\,\frac{\partial^2}{\partial x_{1,3}\partial x_{2,3}} \\ &- 2\,x_{1,2}\,y_1\,y_2\,\frac{\partial^3}{\partial x_{1,3}\partial x_{2,3}\partial y_2} + 2\,{x_{1,2}}^2\,y_1\,\frac{\partial^2}{\partial x_{1,3}^2} + 2\,y_1\,{y_2}^2\,\frac{\partial^2}{\partial x_{1,3}^2} \\ &- {x_{1,2}}^2\,y_1\,y_2\,\frac{\partial^3}{\partial x_{1,3}^2\partial y_2} + y_1\,{y_2}^3\,\frac{\partial^3}{\partial x_{1,3}^2\partial y_2} - {y_1}^2\,{y_2}^2\,\frac{\partial^3}{\partial x_{1,3}^2\partial y_1} \\ &+ 2\,y_1\,{y_2}^2\,\frac{\partial^3}{\partial x_{1,2}\partial x_{1,3}\partial x_{2,3}} + 2\,x_{1,2}\,y_1\,{y_2}^2\,\frac{\partial^3}{\partial x_{1,2}\partial x_{1,3}^2} + {y_2}^2\,\frac{\partial^3}{\partial x_{1,2}^2\partial y_1})\,,\end{aligned}$$

$$\Delta_{2,4} = 2\,({y_3}^2\,\frac{\partial^2}{\partial y_3^2} - y_2\,y_3\,\frac{\partial^2}{\partial y_2\partial y_3} + {y_2}^2\,\frac{\partial^2}{\partial y_2^2} - y_1\,y_2\,\frac{\partial^2}{\partial y_1\partial y_2} + {y_1}^2\,\frac{\partial^2}{\partial y_1^2} + {y_1}^2\,\frac{\partial^2}{\partial x_{3,4}^2}$$

$$+ 2\, x_{2,3}\, {y_1}^2 \frac{\partial^2}{\partial x_{2,4} \partial x_{3,4}} + {x_{2,3}}^2\, {y_1}^2 \frac{\partial^2}{\partial x_{2,4}^2} + {y_1}^2\, {y_2}^2 \frac{\partial^2}{\partial x_{2,4}^2}$$
$$+ {y_2}^2 \frac{\partial^2}{\partial x_{2,3}^2} + 2\, x_{1,3}\, {y_1}^2 \frac{\partial^2}{\partial x_{1,4} \partial x_{3,4}} + 2\, x_{1,3}\, x_{2,3}\, {y_1}^2 \frac{\partial^2}{\partial x_{1,4} \partial x_{2,4}}$$
$$+ 2\, x_{1,2}\, {y_1}^2\, {y_2}^2 \frac{\partial^2}{\partial x_{1,4} \partial x_{2,4}} + {x_{1,3}}^2\, {y_1}^2 \frac{\partial^2}{\partial x_{1,4}^2} + {x_{1,2}}^2\, {y_1}^2\, {y_2}^2 \frac{\partial^2}{\partial x_{1,4}^2}$$
$$+ {y_1}^2\, {y_2}^2\, {y_3}^2 \frac{\partial^2}{\partial x_{1,4}^2} + 2\, x_{1,2}\, {y_2}^2 \frac{\partial^2}{\partial x_{1,3} \partial x_{2,3}} + {x_{1,2}}^2\, {y_2}^2 \frac{\partial^2}{\partial x_{1,3}^2}$$
$$+ {y_2}^2\, {y_3}^2 \frac{\partial^2}{\partial x_{1,3}^2} + {y_3}^2 \frac{\partial^2}{\partial x_{1,2}^2}) \,,$$

$\Delta_{3,4} = A_1 + A_2$, where:

$$\begin{aligned}
A_1 \;=\; & -2\, {y_3}^2\, \partial^{(0,0,0,0,0,0,0,0,2)} - y_2\, y_3\, \partial^{(0,0,0,0,0,0,0,1,1)} \\
+ \;& 3\, y_2\, {y_3}^2\, \partial^{(0,0,0,0,0,0,0,1,2)} + 4\, {y_2}^2\, \partial^{(0,0,0,0,0,0,0,2,0)} \\
- \;& 3\, {y_2}^2\, y_3\, \partial^{(0,0,0,0,0,0,0,2,1)} - 7\, y_1\, y_2\, \partial^{(0,0,0,0,0,0,1,1,0)} \\
+ \;& 3\, y_1\, {y_2}^2\, \partial^{(0,0,0,0,0,0,1,2,0)} + 10\, {y_1}^2\, \partial^{(0,0,0,0,0,0,2,0,0)} \\
- \;& 3\, {y_1}^2\, y_2\, \partial^{(0,0,0,0,0,0,2,1,0)} + 10\, {y_1}^2\, \partial^{(0,0,0,0,0,2,0,0,0)} \\
- \;& 3\, {y_1}^2\, y_2\, \partial^{(0,0,0,0,0,2,0,1,0)} + 20\, x_{2,3}\, {y_1}^2\, \partial^{(0,0,0,0,1,1,0,0,0)} \\
- \;& 6\, x_{2,3}\, {y_1}^2\, y_2\, \partial^{(0,0,0,0,1,1,0,1,0)} + 10\, {x_{2,3}}^2\, {y_1}^2\, \partial^{(0,0,0,0,2,0,0,0,0)} \\
+ \;& 10\, {y_1}^2\, {y_2}^2\, \partial^{(0,0,0,0,2,0,0,0,0)} - 3\, {y_1}^2\, {y_2}^2\, y_3\, \partial^{(0,0,0,0,2,0,0,0,1)} \\
- \;& 3\, {x_{2,3}}^2\, {y_1}^2\, y_2\, \partial^{(0,0,0,0,2,0,0,1,0)} + 3\, {y_1}^2\, {y_2}^3\, \partial^{(0,0,0,0,2,0,0,1,0)} \\
- \;& 3\, {y_1}^3\, {y_2}^2\, \partial^{(0,0,0,0,2,0,1,0,0)} + 6\, {y_1}^2\, {y_2}^2\, \partial^{(0,0,0,1,1,1,0,0,0)} \\
+ \;& 6\, x_{2,3}\, {y_1}^2\, {y_2}^2\, \partial^{(0,0,0,1,2,0,0,0,0)} + 4\, {y_2}^2\, \partial^{(0,0,0,2,0,0,0,0,0)} \\
- \;& 3\, {y_2}^2\, y_3\, \partial^{(0,0,0,2,0,0,0,0,1)} + 3\, y_1\, {y_2}^2\, \partial^{(0,0,0,2,0,0,1,0,0)} \\
+ \;& 20\, x_{1,3}\, {y_1}^2\, \partial^{(0,0,1,0,0,1,0,0,0)} - 6\, x_{1,3}\, {y_1}^2\, y_2\, \partial^{(0,0,1,0,0,1,0,1,0)} \\
+ \;& 20\, x_{1,3}\, x_{2,3}\, {y_1}^2\, \partial^{(0,0,1,0,1,0,0,0,0)} + 20\, x_{1,2}\, {y_1}^2\, {y_2}^2\, \partial^{(0,0,1,0,1,0,0,0,0)} \\
- \;& 6\, x_{1,2}\, {y_1}^2\, {y_2}^2\, y_3\, \partial^{(0,0,1,0,1,0,0,0,1)} - 6\, x_{1,3}\, x_{2,3}\, {y_1}^2\, y_2\, \partial^{(0,0,1,0,1,0,0,1,0)} \\
+ \;& 6\, x_{1,2}\, {y_1}^2\, {y_2}^3\, \partial^{(0,0,1,0,1,0,0,1,0)} - 6\, x_{1,2}\, {y_1}^3\, {y_2}^2\, \partial^{(0,0,1,0,1,0,1,0,0)} \\
+ \;& 6\, x_{1,2}\, {y_1}^2\, {y_2}^2\, \partial^{(0,0,1,1,0,1,0,0,0)} + 6\, x_{1,3}\, {y_1}^2\, {y_2}^2\, \partial^{(0,0,1,1,1,0,0,0,0)} \\
+ \;& 6\, x_{1,2}\, x_{2,3}\, {y_1}^2\, {y_2}^2\, \partial^{(0,0,1,1,1,0,0,0,0)} + 10\, {x_{1,3}}^2\, {y_1}^2\, \partial^{(0,0,2,0,0,0,0,0,0)} \\
+ \;& 10\, {x_{1,2}}^2\, {y_1}^2\, {y_2}^2\, \partial^{(0,0,2,0,0,0,0,0,0)} + 10\, {y_1}^2\, {y_2}^2\, {y_3}^2\, \partial^{(0,0,2,0,0,0,0,0,0)} \\
- \;& 3\, {x_{1,2}}^2\, {y_1}^2\, {y_2}^2\, y_3\, \partial^{(0,0,2,0,0,0,0,0,1)} \,,
\end{aligned}$$

$$
\begin{aligned}
A_2 \;=\;& 3\,{y_1}^2\,{y_2}^2\,{y_3}^3\,\partial^{(0,0,2,0,0,0,0,0,1)} - 3\,{x_{1,3}}^2\,{y_1}^2\,y_2\,\partial^{(0,0,2,0,0,0,0,1,0)} \\
+\;& 3\,{x_{1,2}}^2\,{y_1}^2\,{y_2}^3\,\partial^{(0,0,2,0,0,0,0,1,0)} - 3\,{x_{1,2}}^2\,{y_1}^3\,{y_2}^2\,\partial^{(0,0,2,0,0,0,1,0,0)} \\
-\;& 3\,{y_1}^3\,{y_2}^2\,{y_3}^2\,\partial^{(0,0,2,0,0,0,1,0,0)} + 6\,x_{1,2}\,x_{1,3}\,{y_1}^2\,{y_2}^2\,\partial^{(0,0,2,1,0,0,0,0,0)} \\
+\;& 6\,x_{1,2}\,{y_1}^2\,{y_2}^2\,\partial^{(0,1,0,0,1,1,0,0,0)} + 6\,x_{1,2}\,x_{2,3}\,{y_1}^2\,{y_2}^2\,\partial^{(0,1,0,0,2,0,0,0,0)} \\
+\;& 8\,x_{1,2}\,{y_2}^2\,\partial^{(0,1,0,1,0,0,0,0,0)} - 6\,x_{1,2}\,{y_2}^2\,y_3\,\partial^{(0,1,0,1,0,0,0,0,1)} \\
+\;& 6\,x_{1,2}\,y_1\,{y_2}^2\,\partial^{(0,1,0,1,0,0,1,0,0)} + 6\,{x_{1,2}}^2\,{y_1}^2\,{y_2}^2\,\partial^{(0,1,1,0,0,1,0,0,0)} \\
+\;& 6\,{y_1}^2\,{y_2}^2\,{y_3}^2\,\partial^{(0,1,1,0,0,1,0,0,0)} + 6\,x_{1,2}\,x_{1,3}\,{y_1}^2\,{y_2}^2\,\partial^{(0,1,1,0,1,0,0,0,0)} \\
+\;& 6\,{x_{1,2}}^2\,x_{2,3}\,{y_1}^2\,{y_2}^2\,\partial^{(0,1,1,0,1,0,0,0,0)} + 6\,x_{2,3}\,{y_1}^2\,{y_2}^2\,{y_3}^2\,\partial^{(0,1,1,0,1,0,0,0,0)} \\
+\;& 6\,{x_{1,2}}^2\,x_{1,3}\,{y_1}^2\,{y_2}^2\,\partial^{(0,1,2,0,0,0,0,0,0)} + 6\,x_{1,3}\,{y_1}^2\,{y_2}^2\,{y_3}^2\,\partial^{(0,1,2,0,0,0,0,0,0)} \\
+\;& 4\,{x_{1,2}}^2\,{y_2}^2\,\partial^{(0,2,0,0,0,0,0,0,0)} + 4\,{y_2}^2\,{y_3}^2\,\partial^{(0,2,0,0,0,0,0,0,0)} \\
-\;& 3\,{x_{1,2}}^2\,{y_2}^2\,y_3\,\partial^{(0,2,0,0,0,0,0,0,1)} + 3\,{y_2}^2\,{y_3}^3\,\partial^{(0,2,0,0,0,0,0,0,1)} \\
-\;& 3\,{y_2}^3\,{y_3}^2\,\partial^{(0,2,0,0,0,0,0,1,0)} + 3\,{x_{1,2}}^2\,y_1\,{y_2}^2\,\partial^{(0,2,0,0,0,0,1,0,0)} \\
+\;& 3\,y_1\,{y_2}^2\,{y_3}^2\,\partial^{(0,2,0,0,0,0,1,0,0)} + 6\,{y_1}^2\,{y_2}^2\,{y_3}^2\,\partial^{(1,0,1,0,1,0,0,0,0)} \\
+\;& 6\,x_{1,2}\,{y_1}^2\,{y_2}^2\,{y_3}^2\,\partial^{(1,0,2,0,0,0,0,0,0)} + 6\,{y_2}^2\,{y_3}^2\,\partial^{(1,1,0,1,0,0,0,0,0)} \\
+\;& 6\,x_{1,2}\,{y_2}^2\,{y_3}^2\,\partial^{(1,2,0,0,0,0,0,0,0)} - 2\,{y_3}^2\,\partial^{(2,0,0,0,0,0,0,0,0)} \\
+\;& 3\,y_2\,{y_3}^2\,\partial^{(2,0,0,0,0,0,0,1,0)}\,.
\end{aligned}
$$

The operator $\Delta_{4,4}$ is needed to have a complete set of generators in dimension $n = 4$. Since it would take 4 printed pages to include, it has been omitted. Of course it can be computed using GL(n)pack .

Observe that for $n = 2$ each coefficient is in $\mathbb{Z}[y]$ with highest degree $2[\frac{m}{2}]$ which is also the order of the operator so the order d of the derivatives in each term satisfies $2 \leq d \leq 2[\frac{m}{2}]$.

For $n > 2$ the order of the derivatives satisfies $2 \leq d \leq m$ with the order of the operator being m. It is more difficult to determine any pattern for the degree of the coefficients from this limited data set, since in case $n = 3$ is $m+2$ and in $n = 4$, $m + 4$. The number of monomial terms in a selection of the operators for those that could be computed is given in the table below:

$m : n$	2	3	4
2	2	8	20
3	2	16	70
4	7	277	294
5	7	?	?

4. Eigenvalues

If for $n \geq 2$, $\nu = (\nu_1, \cdots, \nu_{n-1})$ is a vector of complex parameters and $z \in GL(n, \mathbb{R})$ with Iwasawa form $x.y$ let

$$I_\nu(z) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j}\nu_j},$$

where $b_{i,j} = ij$ if $i + j \leq n$ and $(n-i)(n-j)$ otherwise. This is the so-called power function. Then for all elements of the group W of $n \times n$ permutation matrices w, and all Casimir operators of dimension n,

$$DI_\nu(w.z) = \lambda_D I_\nu(w.z).$$

In other words the $I_\nu(w.z)$ are eigenfunctions with the same eigenvalue. Below are some examples of power functions and eigenvalues as computed by the GL (n)pack program.

$$\begin{aligned}
n = 2 \quad &\rightarrow \quad I_\nu(z) = y_1^{\nu_1}, \\
n = 3 \quad &\rightarrow \quad I_\nu(z) = y_1^{\nu_1+2\nu_2} \, y_2^{2\nu_1+\nu_2}, \\
n = 4 \quad &\rightarrow \quad I_\nu(z) = y_1^{\nu_1+2\nu_2+3\nu_3} y_2^{2\nu_1+4\nu_2+2\nu_3} y_3^{3\nu_1+2\nu_2+\nu_3},
\end{aligned}$$

$$\begin{aligned}
\lambda_{2,2} &= 2v_1(1-v_1), \\
\lambda_{3,2} &= 2v_1(1-v_1), \\
\lambda_{4,2} &= 2(v_1-1)v_1(1-v_1+v_1^2), \\
\lambda_{2,3} &= 6(v_1^2+v_1v_2+v_2^2-v_1-v_2), \\
\lambda_{3,3} &= 3(v_2-v_1+1)(2v_1+v_2-2)(v_1+2v_2), \\
\lambda_{4,3} &= 3\,(-2\,v_1+2\,{v_1}^2-6\,{v_1}^3+6\,{v_1}^4-8\,v_2+17\,v_1\,v_2-15\,{v_1}^2\,v_2 \\
&\quad +12\,{v_1}^3\,v_2+20\,{v_2}^2-33\,v_1\,{v_2}^2+18\,{v_1}^2\,{v_2}^2-18\,{v_2}^3+12\,v_1\,{v_2}^3+6\,{v_2}^4), \\
\lambda_{2,4} &= 4\,(3\,{v_1}^2+4\,{v_2}^2+4\,v_2\,(-1+v_3)+3\,(-1+v_3)\,v_3 \\
&\quad +v_1\,(-3+4\,v_2+2\,v_3)), \\
\lambda_{3,4} &= 4\,(6\,{v_1}^3+8\,{v_2}^2+3\,{v_1}^2\,(-1+4\,v_2+2\,v_3) \\
&\quad +v_1\,(-3+2\,v_2+4\,v_3-6\,{v_3}^2)-3\,v_3\,(3-5\,v_3+2\,{v_3}^2) \\
&\quad -2\,v_2\,(4-7\,v_3+6\,{v_3}^2)), \\
\lambda_{4,4} &= 4\,(21\,{v_1}^4+16\,{v_2}^4+32\,{v_2}^3\,(-1+v_3)+2\,{v_1}^3\,(-9+28\,v_2+14\,v_3) \\
&\quad +8\,{v_2}^2\,(4-9\,v_3+9\,{v_3}^2)+3\,v_3\,(-9+24\,v_3-22\,{v_3}^2+7\,{v_3}^3)
\end{aligned}$$

```
In[1]:= G = WeylGroup[3]

Out[1]= {{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}, {{1, 0, 0}, {0, 0, 1}, {0, 1, 0}},
         {{0, 1, 0}, {1, 0, 0}, {0, 0, 1}}, {{0, 1, 0}, {0, 0, 1}, {1, 0, 0}},
         {{0, 0, 1}, {1, 0, 0}, {0, 1, 0}}, {{0, 0, 1}, {0, 1, 0}, {1, 0, 0}}}

In[2]:= z = MakeZMatrix[3, "x", "y"]

Out[2]= {{y[1] y[2], x[1, 2] y[1], x[1, 3]}, {0, y[1], x[2, 3]}, {0, 0, 1}}

In[3]:= Do[Print[Factor[ApplyCasimirOperator[3, a = IFun[{v1, v2}, G[[i]].z], z] / a]],
         {i, 1, 6}]
        3 (1 + v1 - v2) (-2 + 2 v1 + v2) (v1 + 2 v2)

        3 (1 + v1 - v2) (-2 + 2 v1 + v2) (v1 + 2 v2)

        3 (1 + v1 - v2) (-2 + 2 v1 + v2) (v1 + 2 v2)

        3 (1 + v1 - v2) (-2 + 2 v1 + v2) (v1 + 2 v2)

        3 (1 + v1 - v2) (-2 + 2 v1 + v2) (v1 + 2 v2)

        3 (1 + v1 - v2) (-2 + 2 v1 + v2) (v1 + 2 v2)
```

Figure 1: Validating the operator $\Delta_{3,3}$

$$
\begin{aligned}
&+ 8\,v_2\,(-2 + 10\,v_3 - 18\,{v_3}^2 + 7\,{v_3}^3) + v_1\,(-3 + 32\,{v_2}^3 + 8\,v_2\,(2 - 3\,v_3)^2 \\
&+ 40\,v_3 - 78\,{v_3}^2 + 28\,{v_3}^3 + 24\,{v_2}^2\,(-3 + 2\,v_3)) \\
&+ 6\,{v_1}^2\,(12\,{v_2}^2 + 5\,(-1 + v_3)\,v_3 + 4\,v_2\,(-2 + 3\,v_3))).
\end{aligned}
$$

Observe that each eigenvalue is in $\mathbb{Z}[v_1, \cdots, v_{n-1}]$, that the degree $\partial\lambda_{m,n} = m$ for $n > 2$ and that $n \mid \lambda_{m,n}$ for all $n \geq 2$. Observe also that these eigenvalues can be obtained by setting all partial derivatives with respect to the variables $x_{i,j}$ to zero in the operators $\Delta_{m,n}$, equivalent to starting from the subset of $\mathfrak{h}^n$ with $x = id$ in the Iwasawa form.

5. Comparison with Bump's Operator

Since reference [2, p. 34] contained the same expression for the first operator in dimension 3 (his $\Delta_1 = \Delta_{2,3}/2$), but a different, but similar, expression for the operator $\Delta_{3,3}$, the GL(n)pack operator was tested using the 6 elements of the appropriate group W and found to return the same eigenvalue as $\lambda_{3,3}$ given above. This is the stringent test suggested by Dorian Goldfeld. A transcript of the corresponding *Mathematica* session is set out in Figure 1.

```
Out[149]=
  -y2^2 f^(0,0,0,0,2)[x1,2, x1,3, x2,3, y1, y2] +
   y1 y2^2 f^(0,0,0,1,2)[x1,2, x1,3, x2,3, y1, y2] +
   y1^2 f^(0,0,0,2,0)[x1,2, x1,3, x2,3, y1, y2] -
   y1^2 y2 f^(0,0,0,2,1)[x1,2, x1,3, x2,3, y1, y2] +
   y1^2 f^(0,0,2,0,0)[x1,2, x1,3, x2,3, y1, y2] -
   y1^2 y2 f^(0,0,2,0,1)[x1,2, x1,3, x2,3, y1, y2] +
   2 y1^2 x1,2 f^(0,1,1,0,0)[x1,2, x1,3, x2,3, y1, y2] -
   2 y1^2 y2 x1,2 f^(0,1,1,0,1)[x1,2, x1,3, x2,3, y1, y2] +
   y1^2 (y2^2 + x1,2^2) f^(0,2,0,0,0)[x1,2, x1,3, x2,3, y1, y2] +
   y1^2 y2 (y2^2 - x1,2^2) f^(0,2,0,0,1)[x1,2, x1,3, x2,3, y1, y2] -
   y1^3 y2^2 f^(0,2,0,1,0)[x1,2, x1,3, x2,3, y1, y2] +
   2 y1^2 y2^2 f^(1,1,1,0,0)[x1,2, x1,3, x2,3, y1, y2] +
   2 y1^2 y2^2 x1,2 f^(1,2,0,0,0)[x1,2, x1,3, x2,3, y1, y2] -
   y2^2 f^(2,0,0,0,0)[x1,2, x1,3, x2,3, y1, y2] +
   y1 y2^2 f^(2,0,0,1,0)[x1,2, x1,3, x2,3, y1, y2]
```

Figure 2: The amended form of Bump's operator Δ_2

The relationship between the second operators in dimension 3 is observed to be

$$\Delta_2 = \Delta_{3,3}/3 - \Delta_{2,3}/2$$

provided that the term $2y_1^2 y_2 x_2 \frac{\partial^3}{\partial x_2 \partial x_3^2}$ is replaced by $2y_1^2 y_2^2 x_2 \frac{\partial^3}{\partial x_2 \partial x_3^2}$ so Bump's $\{\Delta_1, \Delta_2\}$, with this amendment, still generate the center (note that the variable correspondence is $x_{1,2} \to x_2, x_{1,3} \to x_3, x_{2,3} \to x_1$). The *Mathematica* form of Δ_2 used is set out in Figure 2.

Because of the importance of this operator a detailed examination of the discrepancy was carried out. Before simplification six terms in the Casimir operator $\Delta_{3,3}$ had coefficients $y_1^2 y_2^2 x_2$ multiplying the operator $\frac{\partial^3}{\partial x_2 \partial x_3^2}$ and no other coefficients appeared. These terms corresponded to the matrix multipliers (see Section 1 and the matrix M in the
sl Mathematica program):

$$\begin{pmatrix} 1 & 0 & t_3 \\ t_2 & 1 & t_2+t_3 \\ t_1t_2 & t_1 & 1+t_1t_2t_3 \end{pmatrix}, \begin{pmatrix} 1 & t_3 & 0 \\ t_1t_2 & 1+t_1t_2t_3 & t_1 \\ t_2 & t_2t_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_2 & t_2t_3 \\ 0 & 1 & t_3 \\ t_1 & t_1t_2 & 1+t_1t_2t_3 \end{pmatrix},$$

$$\begin{pmatrix} 1+t_1t_2t_3 & t_1t_2 & t_1 \\ t_3 & 1 & 0 \\ t_2t_3 & t_2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_2t_3 & t_2 \\ t_1 & 1+t_1t_2t_3 & t_1t_2 \\ 0 & t_3 & 1 \end{pmatrix}, \begin{pmatrix} 1+t_1t_2t_3 & t_1 & t_1t_2 \\ t_2t_3 & 1 & t_2 \\ t_3 & 0 & 1 \end{pmatrix}.$$

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