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OPTIMALITY CONDITIONS AND DUALITY FOR NONLINEAR PROGRAMMING PROBLEMS INVOLVING LOCALLY (H_p, r, α)-PRE-INVEX FUNCTIONS AND H_p-INVEX SETS

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Abstract: In this paper, we introduce new types of generalized convex functions and sets which are called locally (H_p, r, α)-pre-invex functions and locally H_p-invex sets, respectively. Some properties of these new classes of functions and sets are established. We also obtain necessary optimality conditions, sufficient optimality conditions and duality theorems for nonlinear programming problems introducing new concepts of generalized differential and gradient sets of functions. Furthermore, we give some examples to illustrate that the new generalized convexity is an extension of some generalized convexities introduced in literature.

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1. Introduction

Convexity plays a central role in many aspects of mathematical programming (see [22, 5, 8]) including analysis of stability [9, 16], sufficient optimality conditions and duality [12, 15, 11]. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to tackle many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1, 2, 19, 27, 28, 25]. One of these concepts, invexity, was introduced by Hanson in [11]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush Kuhn Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [7] introduced the concept of pre-invex functions which is a special case of invexity.

Following Hanson [11] and Liu [7], many authors have introduced concepts of generalized invexity and pre-invexity including strictly pseudoinvex functions and quasiinvex functions [13], prepseudoinvex and prequasiinvex functions [21] and r-pre-pseudoinvex functions [1]. The relationships between some of these generalized invex functions were studied in [20, 21].

Recently, Antczak [3] introduced new definitions of p-invex sets and (p, r)-invex functions which can be seen as generalizations of invex and pre-invex functions. He also discussed nonlinear programming problems involving the (p, r)-invexity-type functions in [2, 4].

On the other hand, Kaul et al [14] introduced the classes of locally connected sets which generalize the arcwise connected sets [6] and locally star-shaped sets [10]. Based on the new class of sets, they [14, 15] introduced a new class of functions called locally connected functions. They (see [15]) also defined the directional derivative (with respect to a vector function) of a real valued function, and also defined locally P-connected functions in terms of its right differential.

Motivated by [7, 14, 1, 2, 3, 4, 15, 18], we introduce definition of a new class of sets, locally H_p-invex sets, and definitions of classes of generalized convex functions called locally (H_p, r, α)-pre-invex functions, locally (H_p, r, α)-pre-quasiinvex functions and locally (H_p, r, α)-pre-pseudoinvex functions in this paper. Based on the definitions of classes of generalized convex functions, we have managed to deal with nonlinear programming problems under some assumptions.

The rest of the paper is organized as follows: In Section 2, we give some preliminary concepts and properties regarding generalized convex functions. In Section 3, we deal with the nonlinear programming problems, obtain necessary conditions, sufficient optimality conditions for the problems, and also establish duality results to the nonlinear programming problems. In Section 4, we give some examples to illustrate this new type of generalized convex functions.

2. Definitions and Preliminary Results

In this section, we give definitions of locally H_p-invex set, locally (H_p, r, α)-pre-invex function, (H_p, r, α)-pre-quasiinvex functions and locally (H_p, r, α)-pre-pseudoinvex functions. Therefore, some properties of the new sets and functions are discussed. Before introducing them, we need to present the following preliminary definition.

Definition 1. (see [3]) Let $a_1, a_2 > 0$, $\lambda \in (0, 1)$ and $r \in \mathbb{R}$. Then the weighted r-mean of a_1 and a_2 is given by

$$M_r(a_1, a_2; \lambda) := \begin{cases} (\lambda a_1^r + (1-\lambda) a_2^r)^{\frac{1}{r}} & \text{for } r \neq 0, \\ a_1^\lambda a_2^{1-\lambda} & \text{for } r = 0. \end{cases}$$

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x \geq 0\}$ and $\dot{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n | x > 0\}$. As we mentioned before, the notion of p-invex set was introduced by Antczak in [3] and the notion of a locally connected set was introduced by Kaul et al in [14]. Based on these concepts, we now introduce the concept of a locally H_p-invex set as follows.

Definition 2. $S \subset \mathbb{R}^n$ is a locally H_p-invex set if and only if, for any x, $u \in S$, there exist a maximum positive number $a(x, u) \leq 1$ and a vector function $H_p : S \times S \times [0, 1] \to \mathbb{R}^n$, such that

$$H_p(x, u; 0) = \mathbf{e}^u, \ H_p(x, u; \lambda) \in \dot{\mathbb{R}}^n_+,$$
$$\ln (H_p(x, u; \lambda)) \in S, \ \forall \ 0 < \lambda < a(x, u) \ for \ p \in \mathbb{R},$$

and $H_p(x, u; \lambda)$ is continuous on the interval $[0, a(x, u))$, where the logarithm and the exponentials appearing in the relation are understood to be taken componentwise.

Remark 1. Obviously, a locally H_p-invex set is p-invex (see [3]) when $H_p(x, u; \lambda) = M_p(\mathbf{e}^{\eta(x,u)+u}, \mathbf{e}^u; \lambda)$ and $a(x, u) = 1$ for all x and u in S. In general case, there exist locally H_p-invex sets which are not p-invex. Let us consider, for example, the set $S \subset \mathbb{R}^2$ given by

$$S = \{x = (x_1, x_2) : x_2 = f(x_1)\},$$

where $f : [0,1] \to \mathbb{R}$ is defined as

$$f(t) = \begin{cases} 0, & t = 0, \\ t \sin \frac{1}{t}, & t \in (0,1]. \end{cases}$$

For any given $x, u \in S$, the function H_p is given by

$$H_p(x, u; \lambda) = \begin{cases} h_p(x; \lambda), & u = (0,0)^T, \ \lambda \in [0,1], \\ \left(\mathbf{e}^{u_1}, \mathbf{e}^{u_1 \sin \frac{1}{u_1}}\right)^T, & u \neq (0,0)^T, \ \lambda \in [0,1], \end{cases}$$

where $x = (x_1, x_2)^T$, $u = (u_1, u_2)^T$, $0 < p < 1$ and

$$h_p(x; \lambda) = \begin{cases} (1,1)^T, & \lambda = 0, \\ \left(\mathbf{e}^{\lambda}, \mathbf{e}^{\lambda \sin \frac{1}{\lambda}}\right)^T, & \lambda \in (0,1]. \end{cases}$$

It is easy to verify that this set is locally H_p-invex, but not p-invex: We sketch the proof as follows. Suppose that S is a p-invex set. Then, according to the definition, $\ln\left(M_p(\mathbf{e}^{\eta(x,u)+u}, \mathbf{e}^{u}; \lambda)\right) \in S$, that is $\ln(\lambda \mathbf{e}^{p\eta(x,u)+u} + (1-\lambda)\mathbf{e}^{pu}) \in S$ for all $\lambda \in [0,1]$. However, we can easily show that this is not true.

For the convenience, we use the following notations.

Definition 3. Let $\xi, x, u \in \mathbb{R}^n$ and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and let p be an arbitrary real number. Then the p-inner product of ξ and $\eta(x,u)$ denoted by $\langle \xi, \eta(x,u) \rangle^*$ can be defined as follows:

$$\langle \xi, \eta(x,u) \rangle^* = \begin{cases} \frac{1}{p}\, \xi^T \left(\mathbf{e}^{p\eta(x,u)} - \mathbf{1}\right) & \text{for } p \neq 0, \\ \xi^T \eta(x,u) & \text{for } p = 0, \end{cases}$$

where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$, $\mathbf{e}^{p\eta(x,u)} = \left(e^{p\eta_1(x,u)}, \ldots, e^{p\eta_n(x,u)}\right)^T$.

Definition 4. Let a, b and r be real numbers. Then the generalized difference of a and b with respect to r, $\nabla_r(a,b)$, is defined by

$$\nabla_r(a,b) = \begin{cases} e^{ra} - e^{rb} & \text{for } r \neq 0, \\ a - b & \text{for } r = 0. \end{cases}$$

We now extend the notion of a (p,r)-preinvex function (see [3]) using the notion of an H_p-invex set.

Definition 5. A function $f : S \to \mathbb{R}$ defined on a locally H_p-invex set $S \subset \mathbb{R}^n$ is said to be locally (H_p, r, α)-pre-invex on S if, for any $x, u \in S$, there exists a maximum positive number $a(x,u) \leq 1$ such that

$$f\left(\ln\left(H_p(x,u;\lambda)\right)\right) \leq \ln\left(M_r(e^{f(x)}, e^{f(u)}; \lambda^{\alpha})\right), \forall\ 0 < \lambda < a(x,u) \text{ for } p \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$ and the logarithm and the exponentials appearing on the left-hand side of the inequality are understood to be taken componentwise. If u is fixed, then f is said to be locally (H_p, r, α)-pre-invex at u.

In order to establish different optimality conditions in the next section, we need to present the following definitions.

Definition 6. A function $f : S \to \mathbb{R}$ defined on a locally H_p-invex set $S \subset \mathbb{R}^n$ is said to be locally (H_p, r, α)-pre-quasiinvex on S if, for any $x, u \in S$, there exists a maximum positive number $a(x, u) \leq 1$ such that

$$f\left(\ln\left(H_p(x, u; \lambda)\right)\right) \leq \max\{f(x), f(u)\}, \forall\ 0 < \lambda < a(x, u) \text{ for } p \in \mathbb{R}\,,$$

where the logarithm and the exponentials appearing on the left-hand side of the inequality are understood to be taken componentwise. If u is fixed, then f is said to be locally (H_p, r, α)-pre-quasiinvex at u.

According to our observations, we now introduce the notions of α-(p, r) differentiability and gradient sets of a function which enable us to define a new class of functions, express the properties of locally (H_p, r, α)-pre-invex functions and formulate optimality conditions and duality theorems in the next section.

Definition 7. Let p, r and α be real numbers such that $0 < \alpha \leq 1$, and let $f : \mathbb{R}^n \to \mathbb{R}$. Then the right super order α-(p, r) differential, $d^{\,\alpha}_{(p,r)} f(u; H_p(x, u; 0+))$, and the right super order α-(p, r) gradient set of f at u, $\partial^{\alpha}_{(p,r)} f(u; \eta(x, u))$, are defined by

$$d^{\,\alpha}_{(p\,,\,r)} f(u; H_p(x, u; 0+)) = \begin{cases} \limsup\limits_{\lambda \downarrow 0} \frac{\nabla_r(f(\ln(H_p(x,u;\lambda))),\, f(u))}{\lambda^{\alpha}}, \; r \geq 0 \\ \liminf\limits_{\lambda \downarrow 0} \frac{\nabla_r(f(\ln(H_p(x,u;\lambda))),\, f(u))}{\lambda^{\alpha}}, \; r < 0 \end{cases},$$

$$\partial^{\,\alpha}_{(p\,,\,r)} f(u; \eta(x, u)) = \begin{cases} \{\xi | \langle \xi, \eta(x, u)\rangle^* \leq d^{\,\alpha}_{(p\,,\,r)} f(u; \eta(x,\ u)), \forall\, x \in S\},\; r \geq 0 \\ \{\xi | \langle \xi, \eta(x, u)\rangle^* \geq d^{\,\alpha}_{(p\,,\,r)} f(u; \eta(x,\ u)),\ \forall\, x \in S\},\; r < 0 \end{cases},$$

where $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector function. We denote the differential $d^{\,\alpha}_{(p,r)} f(u; H_p(x, u; 0+))$ by $d^{\alpha}_{(p,r)} f(u; \eta(x, u))$ when $H_p(x, u; \lambda) = M_p(\mathbf{e}^{\eta(x,u)+u}, \mathbf{e}^u; \lambda)$.

Remark 2. If f is differentiable on $\mathbb{R}^n$ and $\alpha = 1$, then we have

$$d^{\,1}_{(p,r)} f(u; \eta(x, u)) = \begin{cases} re^{rf(u)}\langle \xi, \eta(x, u)\rangle^*, & \text{for } r \neq 0 \\ \langle \xi, \eta(x, u)\rangle^*, & \text{for } r = 0 \end{cases}.$$

If $r = 0$ and $p = 0$, it follows that $d^{\,1}_{(0,0)} f(u; \eta(x, u) = \nabla f(u)^T \eta(x, u)$. Therefore, $d^{\,1}_{(0,0)} f(u; \eta(x, u)) = \nabla f(u)^T (x - u)$ when $\eta(x, u) = x - u$.

Definition 8. Let $S \subset \mathbb{R}^n$ be a locally H_p-invex set and $f\colon S \to \mathbb{R}$. Then

f is said to be locally (locally strong) (H_p, r, α)-pre-pseudoinvex at u if

$$\left.\begin{array}{l} d^{\alpha}_{(p,r)} f(u; H_p(x,u;0+)) \geq 0, \text{ for } r \geq 0, \\ d^{\alpha}_{(p,r)} f(u; H_p(x,u;0+)) \leq 0, \text{ for } r < 0, \end{array}\right\} \Rightarrow f(x) \geq (>) f(u).$$

The relationship between a locally (H_p, r, α)-pre-invex function and a locally (H_p, r, α)-pre-quasiinvex function can be given by the following theorem.

Theorem 1. *If f is a locally (H_p, r, α)-pre-invex function on S, then f is locally (H_p, r, α)-pre-quasiinvex on S.*

Proof. From Definition 5, we have

$$f\left(\ln\left(H_p(x,u;\lambda)\right)\right) \leq \ln\left(M_r(e^{f(x)}, e^{f(u)}; \lambda^{\alpha})\right), \forall\ 0 < \lambda < a(x,u) \text{ for } p \in \mathbb{R}.$$

On the other hand, it is easy to see that, for all r, the inequality

$$\ln\left(M_r(e^{f(x)}, e^{f(u)}; \lambda^{\alpha})\right) \leq \max\{f(x), f(u)\}$$

holds. Hence, we have

$$f\left(\ln\left(H_p(x,u;\lambda)\right)\right) \leq \max\{f(x), f(u)\}, \forall\ 0 < \lambda < a(x,u) \text{ for } p \in \mathbb{R}. \qquad \square$$

There are some interesting properties of the generalized convex functions.

Theorem 2. *Let $S \subset \mathbb{R}^n$ be a locally H_p-invex set, and let $f : S \to \mathbb{R}$ be a locally (H_p, r, α)-pre-invex function at u. Then*

$$\nabla_r\left(f(x), f(u)\right) \geq d^{\alpha}_{(p,r)} f(u; H_p(x,u;0+)),\ \forall\ x, \in S \ \textit{ for } \ r \geq 0,$$

$$\nabla_r(f(x), f(u)) \leq d^{\alpha}_{(p,r)} f(u; H_p(x,u;0+)),\ \forall\ x, \in S \ \textit{ for } \ r < 0.$$

Proof. We prove only the part "$r \geq 0$" since the part $r < 0$ can be proven in the similar way. By Definition 5, we have

$$f\left(\ln\left(H_p(x,u;\lambda)\right)\right) \leq \ln\left(M_r(e^{f(x)}, e^{f(u)}; \lambda^{\alpha})\right), \forall\ 0 < \lambda < a(x,u) \text{ for } p \in \mathbb{R};$$

that is,

$$e^{rf(\ln(H_p(x,u;\lambda)))} - e^{rf(u)} \leq \lambda^{\alpha}\left(e^{rf(x)} - e^{rf(u)}\right), \forall\ 0 < \lambda < a(x,u)$$

when $r > 0$ and

$$f\left(\ln\left(H_p(x,u;\lambda)\right)\right) - f(u) \leq \lambda^{\alpha}\left(f(x) - f(u)\right), \forall\ 0 < \lambda < a(x,u)$$

when $r = 0$.

Hence,

$$\frac{e^{rf(\ln(H_p(x,u;\lambda)))} - e^{rf(u)}}{\lambda^{\alpha}} \leq e^{rf(x)} - e^{rf(u)}, \forall\, 0 < \lambda < a(x,u),\ r > 0,$$

$$\frac{f\left(\ln\left(H_p(x,u;\lambda)\right)\right) - f(u)}{\lambda^{\alpha}} \leq f(x) - f(u), \forall\ 0 < \lambda < a(x,u),\ r = 0.$$

From Definition 4 and Definition 7, we have

$$\nabla_r\left(f(x), f(u)\right) \geq d_{(p,r)}^{\alpha} f(u; H_p(x,u;0+)),\ \forall\, x \in S, \text{ for } \ r \geq 0. \qquad \square$$

Corollary 1. *Let $S \subset \mathbb{R}^n$ be a locally H_p-invex set and $f : S \to \mathbb{R}$ be a locally (p, r, α)-pre-invex function at u. If $H_p(x, u; \lambda) = M_p(\mathbf{e}^{\eta(x,u)+u}, \mathbf{e}^u; \lambda)$, then*

$$\nabla_r\left(f(x), f(u)\right) \geq \langle \xi, \eta(x,u)\rangle^*,\ \forall\, \xi \in \partial_{(p,r)}^{\alpha} f(u; \eta(x,u)),\ \forall\, x \in S, \ \ r \geq 0,$$

$$\nabla_r(f(x), f(u)) \leq \langle \xi, \eta(x,u)\rangle^*,\ \forall\, \xi \in \partial_{(p,r)}^{\alpha} f(u; \eta(x,u)),\ \forall\, x \in S,\ r < 0.$$

Proof. Using Definition 7 and Theorem 2, we can derive the result. □

Corollary 2. *Let $S \subset \mathbb{R}^n$ be a locally H_p-invex set and $f : S \to \mathbb{R}$ be a locally (p, r, α)-pre-invex function at u. Then f is said to be locally (H_p, r, α)-pre-pseudoinvex at u.*

Proof. Using Definition 8 and Theorem2, we can derive the result. □

Theorem 3. *Let $S \subset \mathbb{R}^n$ be a locally H_p-invex set and $f : S \to \mathbb{R}$ be a locally (H_p, r, α)-pre-quasiinvex function at u, and let $f(x) \leq f(u)$. Then*

$$0 \geq d_{(p,r)}^{\alpha} f(u; H_p(x,u;0+)),\ \forall\, x \in S, \text{ for } \ r \geq 0,$$

$$0 \leq d_{(p,r)}^{\alpha} f(u; H_p(x,u;0+)),\ \forall\, x \in S, \text{ for } \ r < 0.$$

Proof. Note that $\nabla_r(f(x), f(u)) \leq 0$ when $r \geq 0$ and $\nabla_r(f(x), f(u)) \geq 0$ when $r < 0$ since $f(x) \leq f(u)$. Now the desired result follows Theorem 2. □

3. Optimality and Duality

In this section, we consider the nonlinear programming problem

$$(P) \qquad \begin{array}{l} \min f(x)\,, \\ \text{s.t.} x \in X, \end{array}$$

where $X = \{x \in S \mid g_j(x) \leq 0,\ j = 1, \ldots, m\}$, $f : S \to \mathbb{R}$ and $g_j : S \to \mathbb{R}$ $(j = 1, \ldots, m)$ are real valued functions defined on $S \subset \mathbb{R}^n$. Here S is a locally H_p-invex set. In order to introduce necessary optimality conditions for Problem (P), we need to present the following lemma [17]:

Lemma 1. *Let $C \subset \mathbb{R}^n$ be a nonempty set, and let $F : C \to \mathbb{R}^p$ and $G : C \to \mathbb{R}^q$ be convex functions. Then the following statements are true.*

(i) If the system $F(x) < 0, G(x) \leq 0$ has no solution in C, then there exist $u \in \mathbb{R}^p_+$, $v \in \mathbb{R}^q_+$ and $(u, v) \neq 0$ such that

$$u^T F(x) + v^T G(x) \geq 0, x \in C; \tag{3.1}$$

(ii) Suppose that there exists $\bar{x} \in intC$ such that $G(\bar{x}) < 0$. Then the system $F(x) < 0$, $G(x) \leq 0$ has no solution in C iff there exist $u \in \mathbb{R}^p_+$, $v \in \mathbb{R}^q_+$ and $(u, v) \neq 0$ satisfying (3.1).

Theorem 4. (Necessary Optimality Conditions) *Let S be convex (or $S = \mathbb{R}^n$), and x^* be an optimal solution of (P). If $d^{\,\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+))$ and $d^{\,\alpha}_{(p,r)} g_I(x^*; H_p(x, x^*; 0+))$ are convex functions with respect to variable x, and g_j $(j \in J)$ are super semi-continuous at x^* in S, then there exist $\mu^*_0 \in \mathbb{R}$ and $\mu^* = (\mu^*_1, \ldots, \mu^*_m) \in \mathbb{R}^m$ such that, for $\forall\ x \in S$,*

$$\mu^*_0 d^{\,\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+)) + \sum_{j=1}^{m} \mu^*_j d^{\,\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+)) \geq 0,\ \ r \geq 0\,,$$

$$\mu^*_0 d^{\,\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+)) + \sum_{j=1}^{m} \mu^*_j d^{\,\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+)) \leq 0,\ \ r < 0\,,$$

$$\sum_{j=1}^{m} \mu^*_j g_j(x^*) = 0,$$

$$(\mu^*_0, \mu^*) \geq 0, (\mu^*_0, \mu^*) \neq 0,$$

where $I = I(x^*) = \{j | g_j(x^*) = 0\}$ and $J = J(x^*) = \{j | g_j(x^*) < 0\}$.

Proof. First we show that the system

$$\begin{cases} d^{\,\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+)) < 0\,, \\ d^{\,\alpha}_{(p,r)} g_I(x^*; H_p(x, x^*; 0+)) \leq 0\,, \end{cases} \tag{3.2}$$

has no solution in S when $r \geq 0$ (the proof in the case of $r < 0$ is analogous except the directions of the inequalities should be changed to the opposite ones in the system (3.2) for given x^*.

Suppose to the contrary that $x \in S$ is a solution of the system (3.2). Since

$$d^{\,\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+)) < 0$$

and

$$d^{\,\alpha}_{(p,r)} g_I(x^*; H_p(x, x^*; 0+)) \leq 0,$$

there exist $\delta_0, \delta_j, j \in I$, $0 < \delta_0 < a(x, x^*)$, $0 < \delta_j < a(x, x^*)$ such that

$$\nabla_r \left(f(\ln(H_p(x, x^*; \lambda))), f(x^*) \right) < 0, \forall \lambda \in (0, \delta_0),$$

$$\nabla_r \left(g_j(\ln(H_p(x, x^*; \lambda)), g_j(x^*) \right) \leq 0, \forall \lambda \in (0, \delta_j).$$

In the case of $r > 0$, we have

$$e^{rf(\ln(H_p(x,x^*;\lambda)))} - e^{rf(x^*)} < 0, \forall \lambda \in (0, \delta_0), \tag{3.3}$$

$$e^{rg_j(\ln(H_p(x,x^*;\lambda)))} - e^{rg_j(x^*)} \leq 0, \forall \lambda \in (0, \delta_j). \tag{3.4}$$

In the case of $r = 0$, we have

$$f(\ln(H_p(x,x^*;\lambda))) - f(x^*) < 0, \forall \lambda \in (0, \delta_0), \tag{3.5}$$

$$g_j(\ln(H_p(x,x^*;\lambda))) - g_j(x^*) \leq 0, \forall \lambda \in (0, \delta_j). \tag{3.6}$$

Using (3.3)-(3.6), we can write the following.

$$f(\ln(H_p(x,x^*;\lambda))) < f(x^*), \forall \lambda \in (0, \delta_0), \tag{3.7}$$

$$g_j(\ln(H_p(x,x^*;\lambda))) \leq 0, \forall \lambda \in (0, \delta_j). \tag{3.8}$$

We note that $\ln(H_p(x,x^*;\lambda))$ is a continuous function with respect to λ and g_j $(j \in J)$ are super semi-continuous at x^*. Therefore, for $\varepsilon = -\frac{1}{2}g_j(x^*)$, there exist δ_j, $0 < \delta_j < a(x,x^*)$ $(j \in J)$, such that

$$g_j(\ln(H_p(x,x^*;\lambda))) < g_j(x^*) + \varepsilon \; < 0, \forall \lambda \in (0, \delta_j), \text{for } \; j \in J. \tag{3.9}$$

Let $\delta = \min\{\delta_0, \delta_j, j = 1, \cdots, m\}$. By (3.7)-(3.8) and (3.9), it follows that

$$\ln(H_p(x,x^*;\lambda)) \in S \text{ and } f(\ln(H_p(x,x^*;\lambda))) < f(x^*)$$

for $0 < \lambda < \delta$. This contradicts the assumption that x^* is an optimal solution of (P). Since $d^{\,\alpha}_{(p,r)}f(x^*; H_p(x,x^*;0+))$ and $d^{\,\alpha}_{(p,r)}g_I(x^*; H_p(x,x^*;0+))$ are convex functions with respect to the variable x, by Lemma 1, there exist $\mu_0^*, \mu_j^* \in \mathbb{R}_+ (j \in I)$, and $(\mu_0^*, \mu_I^*) \neq 0$ such that

$$\mu_0^* d^{\,\alpha}_{(p,r)}f(x^*; H_p(x,x^*;0+)) + \mu^{*T} d^{\,\alpha}_{(p,r)}g_I(x^*; H_p(x,x^*;0+)) \geq 0,$$
$$\forall \; x \in S, \; r \geq 0.$$

Defining $\mu_j^* = 0, j \in J$, we have the required result. □

Corollary 3. *Let S be convex (or $S = \mathbb{R}^n$), and x^* be an optimal solution of (P). If $d^{\,\alpha}_{(p,r)}f(x^*;\eta(x,x^*))$ and $d^{\,\alpha}_{(p,r)}g_I(x^*;\eta(x,x^*))$ are convex functions with respect to x and g_j $(j \in J)$ are super semi-continuous at x^*, then there exist $\mu_0^* \in \mathbb{R}$, $\mu^* = (\mu_1^*, \ldots, \mu_m^*) \in \mathbb{R}^m$ such that, for all $x \in S$,*

$$\mu_0^* d^{\,\alpha}_{(p,r)}f(x^*;\eta(x,x^*)) + \sum_{j=1}^{m} \mu_j^* d^{\,\alpha}_{(p,r)}g_j(x^*;\eta(x,x^*)) \geq 0, \; r \geq 0, \tag{3.10}$$

$$\mu_0^* d^{\,\alpha}_{(p,r)}f(x^*;\eta(x,x^*)) + \sum_{j=1}^{m} \mu_j^* d^{\,\alpha}_{(p,r)}g_j(x^*;\eta(x,x^*)) \leq 0, \; r < 0, \tag{3.11}$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0, \tag{3.12}$$

$$(\mu_0^*, \mu^*) \geq 0, (\mu_0^*, \mu^*) \neq 0, \tag{3.13}$$

where $I = I(x^*) = \{j | g_j(x^*) = 0\}$ and $J = J(x^*) = \{j | g_j(x^*) < 0\}$.

Theorem 5. (Necessary Optimality Conditions) *Let x^* be an optimal solution of (P) and g_j $(j \in I)$ be locally (H_p, r, α)-pre-pseudoinvex functions at x^*. If $d^{\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+))$ and $d^{\alpha}_{(p,r)} g_I(x^*; H_p(x, x^*; 0+))$ are convex functions with respect to the variable x, and g_j $(j \in J)$ are super semi-continuous at x^* in the convex set S (or $S = \mathbb{R}^n$) and there exists $\bar{x}$ such that $g_j(\bar{x}) < 0$, $j \in I$, then there exists $\mu^* = (\mu_1^*, \cdots, \mu_m^*) \in \mathbb{R}^m$ and $\mu^* \geq 0$ such that, for all $x \in S$,*

$$d^{\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+)) + \sum_{j=1}^{m} \mu_j^* d^{\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+)) \geq 0, \quad \forall r \geq 0, \tag{3.14}$$

$$d^{\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+)) + \sum_{j=1}^{m} \mu_j^* d^{\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+)) \leq 0, \quad \forall r < 0, \tag{3.15}$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0, \tag{3.16}$$

where $I = I(x^*) = \{j | g_j(x^*) = 0\}$ and $J = J(x^*) = \{j | g_j(x^*) < 0\}$.

Proof. According to Theorem 4, there exist $\mu_0^* \geq 0$ and $\mu^* = (\mu_1^*, \ldots, \mu_m^*) \geq 0$ such that (3.10) and (3.11) hold. Next, we show that $\mu_0^* > 0$. Suppose to the contrary that $\mu_0^* = 0$. Then, from (3.10) and (3.11), we have

$$\sum_{j=1}^{m} \mu_j^* d^{\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+)) \geq 0, \forall\, x \in S, \quad \text{for } r \geq 0\,, \tag{3.17}$$

$$\sum_{j=1}^{m} \mu_j^* d^{\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+)) \leq 0, \forall\, x \in S, \quad \text{for } r < 0\,. \tag{3.18}$$

If there exists $\bar{x}$ such that $g_j(\bar{x}) < 0$, since g_j is locally (p, r, α)-pre-pseudoinvex at x^* $(i \in I)$, it follows that

$$\sum_{j \in I} \mu_j^* d^{\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+)) < 0, \text{ for } r \geq 0, \tag{3.19}$$

$$\sum_{j \in I} \mu_j^* d^{\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+)) > 0, \text{ for } r < 0. \tag{3.20}$$

By (3.12), we have $\mu_j^* = 0$ $(j \in J)$. So there exists $j_0 \in I$ such that $\mu_{j_0}^* > 0$.

Using (3.19) and (3.20), we can write

$$\sum_{j=1}^{m} \mu_j^* d_{(p,r)}^{\alpha} g_j(x^*; H_p(x, x^*; 0+)) < 0, \text{ for } r \geq 0,$$

$$\sum_{j=1}^{m} \mu_j^* d_{(p,r)}^{\alpha} g_j(x^*; H_p(x, x^*; 0+)) > 0, \text{ for } r < 0.$$

These contradict (3.17) and (3.18), respectively. Now letting $\mu_j^* = \frac{\mu_j^*}{\mu_0^*}$ $(j = 1, \ldots, m)$, we can have the required result. □

Theorem 6. (Sufficient Optimality Conditions) *Suppose that there exist a feasible solution x^* of (P) and $\mu_0^* \geq 0$, $\mu^* = (\mu_1^*, \ldots, \mu_m^*) \in \mathbb{R}^m$, $\mu^* \geq 0$ such that (3.10), (3.11), (3.12) and (3.13) hold. If f is locally (H_p, r, α)-pre-invex and g_j $(j \in I)$ are locally strong (H_p, r, α)-pre-pseudoinvex at x^*, then x^* is an optimal solution of (P).*

Proof. We prove the theorem only when $r \geq 0$ (the proof for the case $r < 0$ is analogous). Suppose to the contrary that x^* is not an optimal solution. Then there exists $x \in S$ such that $f(x) < f(x^*)$. Since f is a locally (H_p, r, α)-pre-invex function at x^*, we can write

$$d_{(p,r)}^{\alpha} f(x^*; H_p(x, x^*; 0+)) < 0. \tag{3.21}$$

Also, we have $g(x) \leq g(x^*)$, $j \in I$. According to the condition of the theorem, g_j $(j \in I)$ are locally strong (p, r, α)-pre-pseudoinvex function at x^*. Thus we have

$$d_{(p,r)}^{\alpha} g_j(x^*; H_p(x, x^*; 0+)) < 0, \;\; j \in I. \tag{3.22}$$

Using (3.12), (3.13), (3.21) and (3.22), we come up with a contradiction to (3.10) or (3.11). □

Theorem 7. (Sufficient Optimality Conditions) *Suppose that there exist a feasible solution x^* of (P) and $\mu^* = (\mu_1^*, \ldots, \mu_m^*) \in \mathbb{R}^m$, $\mu^* \geq 0$ such that (3.14), (3.15) and (3.16) hold. If f is locally (H_p, r, α)-pre-invex and g_j $(j \in I)$ are locally (p, r, α)-pre-pseudoinvex at x^*, then x^* is an optimal solution of (P).*

Proof. The proof is analogous to Theorem 6. □

We associate the following Mond-Weir dual (D) to the problem (P):

(D) max $f(u)$

s.t. $\mu_0 d_{(p,r)}^{\alpha} f(u; H_p(u; (x, u; 0+))$

$$+\sum_{j=1}^{m}\mu_j d^{\alpha}_{(p,r)} g_j(u; H_p(u;(x,u;0+)) \geq 0, \text{ for } r \geq 0 \quad (3.23)$$

$$\mu_0 d^{\alpha}_{(p,r)} f(u; H_p(u;(x,u;0+))$$

$$+\sum_{j=1}^{m}\mu_j d^{\alpha}_{(p,r)} g_j(u; H_p(u;(x,u;0+)) \leq 0, \text{ for } r < 0, \quad (3.24)$$

$$g_j(u) \geq 0,\ j = 1, \cdots, m, \quad (3.25)$$

$$u \in S,\ (\mu_0,\ \mu) \geq 0,\ \mu_0 \in \mathbb{R},\ \mu \in \mathbb{R}^m \quad (3.26)$$

Theorem 8. (Weak Duality) *Let x be feasible for (P) and (u, μ_0, μ) be feasible for* (D). *If f is locally (H_p, r, α)-pre-invex and g_j $(j \in I)$ are locally strong (H_p, r, α)-pre-pseudoinvex at u, then $f(x) \geq f(u)$.*

Proof. Suppose to the contrary that $f(x) < f(u)$. Since f is a locally (p, r)-pre-invex function at u, it follows that the inequality (3.21) is true. We also have the system (3.22) since g_j $(j \in I)$ are locally strong (p, r)-pre-pseudoinvex at u. Combining (3.21), (3.22) and (3.26), we have contradictions to (3.23) and (3.24). □

Theorem 9. (Strong Duality) *Let x^* be an optimal solution of (P), and $d^{\alpha}_{(p,r)} f(x^*; H_p(x, x^*; 0+)$ and $d^{\alpha}_{(p,r)} g_j(x^*; H_p(x, x^*; 0+))$ $(j \in I)$ be convex functions with respect to the variable x. Suppose that g_j $(j \in J)$ are super semi-continuous at x^* in the convex set S (or $S = \mathbb{R}^n$). Then there exist $\mu_0^* \in \mathbb{R}$, $\mu^* = (\mu_1^*, \ldots, \mu_m^*) \in \mathbb{R}^m$ such that (x^*, μ_0^*, μ^*) is feasible for (D), and the values of the objective functions for* (P) *and* (D) *are the same at x^*. Moreover, if, for each feasible solution (u, μ_0, μ) of (D), f is locally (H_p, r, α)-pre-invex and g_j $(j \in I)$ is locally strong (H_p, r, α)-pre-pseudoinvex at u, then (x^*, μ_0^*, μ^*) is optimal for (D).*

Proof. Since x^* is an optimal solution of (P), by Theorem 4, there exist $\mu_0^* \in \mathbb{R}$, $\mu^* \in \mathbb{R}^m$ such that (x^*, μ_0^*, μ^*) is feasible for (D). Equality of the objective functions for (P) and (D) is obvious. Moreover, if (x^*, μ_0^*, μ^*) is not optimal for (D), then there exists (u, μ_0, μ) such that $u \in S$, $(\mu_0, \mu) \geq 0$ and $f(u) > f(x^*)$. This contradicts Theorem 8. □

4. Examples

In this section, we give some examples of the classes of new generalized convex functions in the case of $p = 0$, $r = 0$ and $\alpha = 1$. For the convenience, we first recall the following two concepts of generalized convexity.

Definition 9. (see [15]) Let $S \in \mathbb{R}^n$, the right differential of f at x^* with respect to the arc $H_{x^*,x}(\lambda)$, denoted by $df^+(x^*, H_{x^*,x}(0+))$, is given by

$$\lim_{\lambda \to 0+} \frac{f(H_{x^*,x}(\lambda)) - f(x^*)}{\lambda}$$

provided the limit exists. Here $H_{x^*,x}(\lambda)$ is a vector valued function defined on $[0,1]$ and $H_{x^*,x}(0) = x^*$, $H_{x^*,x}(1) = x$. Further, for each pair of points $x^*, x \in S$, there exists a maximum positive number $a(x^*, x) \leq 1$ such that $H_{x^*,x}(\lambda)$ is continuous in the interval $(0, a(x^*, x))$ and $H_{x^*,x}(\lambda) \in S$ when $\lambda \in (0, a(x^*, x))$.

Definition 10. (see [15]) If the right differential of f with respect to $H_{x^*,x}(\lambda)$ at x^* exists and, for each $x \in X$,

$$df^+(x^*, H_{x^*,x}(0+)) \geq 0 \Rightarrow f(x) \geq f(x^*),$$

then f is said to be a locally P-connected (LPCN) function at x^*.

Definition 11. Let X be a nonempty set in $\mathbb{R}^n$ and $f : X \to \mathbb{R}$. Let $x^* \in X$ and d be a no zero vector such that $x^* + \lambda d \in X$ for $\lambda > 0$ and sufficiently small. The directional derivative of f at x^* along with the vector d, denoted by $f'(x^*, d)$, is give by the following limit if it exists

$$\lim_{\lambda \to 0+} \frac{f(x^* + \lambda d) - f(x^*)}{\lambda}$$

Definition 12. (see [17]) Let $x^* \in \text{int}C$. If, for each $x \in X$,

$$f'(x^*, x - x^*) \geq 0 \Rightarrow f(x) \geq f(x^*),$$

then f is said to be a pseudo-convex function at x^*.

Example 1. We consider the function

$$f(x) = \begin{cases} |x| \sin^2 \dfrac{1}{x}, & x \in (-1,0) \bigcup (0,1) \\ 0, & x = 0 \end{cases} .$$

For $x^* = 0$ and $H_p(x, x^*; t) = tx$, we can check that

$$d^1_{(0,0)} f(0; H_p(x, 0; 0+)) = f'(0; x - 0) = |x| \geq 0 \Rightarrow f(x) \geq f(0).$$

However, $df^+(x^*, H_{x^*,x}(0+))$ does not exist. Hence, f is pseudo-convex and locally (p, r, α)-pre-pseudoinvex, but not LPCN.

Example 2. We consider the function

$$f(x) = \begin{cases} -x^3, & x \in [-1, 2) \\ 2x^2 - 16, & x \in [2, 4] \end{cases} .$$

For $x^* = 2$ and $H_p(x, x^*; t) = 2 + \dfrac{1}{2} x^2 (x - 2) t$, we can check that

1) If $x \in [-1, 2)$, then $d^1_{(0,0)} f(0; H_p(x, 0; 0+)) = -6x^2(x - 2) \geq 0 \Rightarrow f(x) \geq$

$f(2)$;

2) If $x \in [-1, 2)$, then $d^1_{(0,0)} f(0; H_p(x, 0; 0+)) = 4x^2(x-2) \geq 0 \Rightarrow f(x) \geq f(2)$.

By Definition 8, f is locally (p, r, α)-pre-pseudoinvex, but not pseudo-convex and LPCN, since both limits do not exist. Therefore the class of locally (p, r, α)-pre-pseudoinvex functions contains the classes of pseudo-convex and LPCN functions.

Example 3. Let the set $S \subset \mathbb{R}^2$ given by

$$S = \{x = (x_1, x_2) : x_1^2 + x_2^2 \geq 1, x_1 \neq 2x_2, x_1 > 0, x_2 > 0\}$$

and the function $H_p : S \times S \times [0, 1] \to \mathbb{R}^2$ given by

$$H_p(x, u; \lambda) = \left(\mathbf{e}^{\{(1-\lambda)u_1^2 + \lambda x_1^2\}^{\frac{1}{2}}}, \mathbf{e}^{\{(1-\lambda)u_2^2 + \lambda x_2^2\}^{\frac{1}{2}}} \right)^T,$$

where $x = (x_1, x_2)^T$, $u = (u_1, u_2)^T$ and $p = 0$. It is easy to verify that this set is a locally H_p-invex set. We consider the function

$$f(x) = \begin{cases} (x_1^2 + 2x_2^2 - 3) \sin^2 \frac{1}{x_1^2 + 2x_2^2 - 3}, & x_1 > 1, x_2 > 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can check that

$$d^1_{(0,0)} f(0; H_p(x, 0; 0+)) = \begin{cases} x_1^2 + 2x_2^2 - 3, & x_1 > 1, x_2 > 1 \\ 0, & \text{otherwise} \end{cases}.$$

Hence, $d^1_{(0,0)} f(0; H_p(x, 0; 0+)) \geq 0 \Rightarrow f(x) \geq f(1, 1)$, that is f is locally (p, r, α)-pre-pseudoinvex at $(1, 1)$. However, f is neither pseudo-convex, LPCN nor differentiable (p, r)-pre-invex with respect to η, see [3].

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