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VECTOR-VALUED SINGULAR INTEGRAL OPERATORS ON THE PRODUCT SPACES H^1 AND BMO

Luiz Antonio Pereira Gomes[1], Eduardo Brandani da Silva[2 §]
[1,2]Department of Mathematics
Maringa State University - UEM
Av. Colombo 5790, Maringá, PR 87020-900, BRAZIL
[1]e-mail: lapgomes@uem.br
[2]e-mail: ebsilva@wnet.com.br

Abstract: Conditions for boundedness of singular vector integral operators on Benedek-Panzone spaces $L^P = L^{p_2}(L^{p_1})$ with mixed norms for $1 < P$, p_1, $p_2 < \infty$ are know in the literature. We shall concern here with the limiting cases of integral operators in (H^1, L^1) and (L^∞, BMO), where H^1 and BMO are spaces in the product case.

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1. Introduction

The study of characterizations and properties of H^p and BMO spaces in product spaces has called the attention of researches since the seventies and originated several works until recent years (see, for instance, Ferguson and Lacey [7] and Ferguson and Sadosky [8]). In this work we concern with one of the main aspect of this theory, namely the boundedness of singular integral operators.

In [9] Fernandez reached a result which gives conditions for boundedness of singular vector integral operators on Benedek-Panzone spaces $L^P = L^{p_2}(L^{p_1})$ with mixed norms (Benedek and Panzone [1]). This result is a generalization

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§Correspondence author

of a version of Benedek, Calderón and Panzone's Theorem obtained by Rubio de Francia et al in [12] and [13]. Rubio de Francia-Ruiz-Torrea's result covers not only the case of singular integral operators of convolution type but even the case of singular integral operators with variable kernels. It also garantees boundedness of these operators on L^p for $1 < p < \infty$.

In the limit cases $p = 1$ and $p = \infty$ boundedness ceases. This may be verified by the Hilbert transform. Actually this is the classical example of operators considered by Rubio de Francia-Ruiz-Torrea. In [12] and [13] the same authors proved these singular integral operators are of the strong type (H^1, L^1) and of the strong type (L^∞, BMO), or rather, they are bounded from H^1 to L^1 and from L^∞ to BMO.

In this work we are concerning with the limiting cases of singular integral operators in (H^1, L^1) and (L^∞, BMO) (where H^1 and BMO are spaces in the product case).

2. $H^1(\mathbb{R} \times \mathbb{R})$ and $BMO(\mathbb{R} \times \mathbb{R})$ Spaces of Vectorial Functions

We denote by $\mathcal{S}(\mathbb{R}^n)$ the class of rapidly decreasing functions (at infinity). If E is a Banach space, $\mathcal{S}'(\mathbb{R}^n, E)$ is the class of all linear applications T defined on $\mathcal{S}(\mathbb{R}^n)$ with values in E, which are continuous (namely, if $\varphi_j \longrightarrow \varphi$ in $\mathcal{S}(\mathbb{R}^n)$, then $T(\varphi_j) \longrightarrow T(\varphi)$ in E). If $T \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $T(\varphi)$ or $\langle T, \varphi \rangle$ is the action of T in φ. If $E = \mathbb{C}$, we have only the space $\mathcal{S}'(\mathbb{R}^n)$ of mixed distributions.

A diadic rectangle is a rectangle $R = I \times J$, where I and J are diadic intervals. Let $\Omega \subset \mathbb{R}^2$ a open subset with finite measure, $M = M(\Omega)$ is the family of all maximal diadic rectangle contained in Ω.

Definition 1. Let E be a Hilbert space. A function a defined on $\mathbb{R}^2$ with values in space E is an $E-$atom of Chang-Fefferman if there is a finite measure open set $\Omega \in \mathbb{R}^2$ satisfying:

(1) $||a||_{L^2(\mathbb{R}^2,E)} \leq |\Omega|^{-1/2}$;

(2) $a = \sum_{R \in M(\Omega)} a_R$, where each function a_R, called elementary particle, verifies:

(i) supp$a_R \subset \tilde{R}$, with R a diadic rectangle, and $\tilde{R}$ the quadruple of R;

(ii) $\int a_R(x,y)dx = 0$ a.e $y \in \mathbb{R}$ and $\int a_R(x,y)dy$ a.e $x \in \mathbb{R}$;

(iii) $(\sum_{R \in M(\Omega)} ||a_R||^2_{L^2(\mathbb{R}^2,E)})^{1/2} \leq |\Omega|^{-1/2}$.

If $(a_k)_{k\in\mathbb{Z}}$ is a sequence of E-atoms of Chang-Fefferman and $(\lambda_k)_{k\in\mathbb{Z}}$ is a sequence of real numbers such that $\sum_{k=-\infty}^{\infty}|\lambda_k| < \infty$, then $\sum_{k=-\infty}^{\infty}\lambda_k a_k$ converges in $L^1(\mathbb{R}^2, E)$. From this observation the following definition is given.

Definition 2. Let E be a Hilbert space. Then $H^1_{at}(\mathbb{R}\times\mathbb{R}, E)$ is the space of all functions f in $L^1(\mathbb{R}^2, E)$ which have a representation in the form $f = \sum_{k=-\infty}^{\infty}\lambda_k a_k$, where $\{a_k\}_{k\in\mathbb{Z}}$ is a family of E-atoms of Chang-Fefferman and $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a family of scalars satisfying $\sum_{k=-\infty}^{\infty}|\lambda_k| < \infty$. Space $H^1_{at}(\mathbb{R}\times\mathbb{R}, E)$ is equipped with the norm

$$||f||_{H^1_{at}(\mathbb{R}\times\mathbb{R},E)} = \inf\{\sum|\lambda_k|\},$$

where the infimum is taken over all representations of f in the form $f = \sum_{k=-\infty}^{\infty}\lambda_k a_k$.

The following notation is given:

$$\square = \{(0,0),(1,0),(0,1),(1,1)\}.$$

Definition 3. Let E be a Hilbert space and f in $L^1(\mathbb{R}^2, E)$. The Hilbert transform of f, $H_k f$, $k\in\square$ are the elements in $\mathcal{S}'(\mathbb{R}^2, E)$, defined by:

(1) $\mathcal{F}(H_{10}f) = -\mathrm{i}\ sgx\ \hat{f}(x,y)$,

(2) $\mathcal{F}(H_{01}f) = -\mathrm{i}\ sgy\ \hat{f}(x,y)$,

(3) $\mathcal{F}(H_{11}f) = (-\mathrm{i}\ sgx)(-\mathrm{i}\ sgy)\hat{f}(x,y)$,

(4) $H_{00}f = f$.

Now we may define the space $BMO(\mathbb{R}\times\mathbb{R}, E)$, which is a generalization of space $BMO(\mathbb{R}\times\mathbb{R})$ of functions with real values.

Definition 4. Let E be a Hilbert space. A function g defined in $\mathbb{R}^2$, with values in E, belongs to $BMO(\mathbb{R}\times\mathbb{R}, E)$ if it may represented as

$$g = \sum_{k\in\square} H_k g_k \tag{1}$$

with $H_{00}g_{00} = g_{00}$ and $\sum_{k\in\square}||g_k||_{L^\infty(\mathbb{R}^2,E)} < \infty$.

The space $BMO(\mathbb{R}\times\mathbb{R}, E)$ is equipped with the norm

$$||g||_{BMO(\mathbb{R}\times\mathbb{R},E)} = \inf\{\sum_{k\in\square}||g_k||_{L^\infty(\mathbb{R}^2,E)}\},$$

where the infimum takes over all representations of g in form (1).

Chang-Fefferman proved (see [2]) that the product space $BMO(\mathbb{R}\times\mathbb{R})$ is the dual of product space $H^1_{at}(\mathbb{R}\times\mathbb{R})$ when we have real value functions. This result is valid also for spaces $BMO(\mathbb{R}\times\mathbb{R}, E)$, where E is a Hilbert space;

therefore, $BMO(I\!R \times I\!R, E)$ is the dual of space $H^1_{at}(I\!R \times I\!R, E)$.

Carleson's Regions. $I\!R^2_+ \times I\!R^2_+$ comprise all elements in the form $(x, s; y, t)$, where x, y, s, t are in $I\!R$ with positive s and t. Given an open set Ω in $I\!R^2$, the product Carleson's region $S(\Omega)$ is defined by elements $(x, s; y, t)$ in $I\!R^2_+ \times I\!R^2_+$, such that the product of intervals $(x - s, x + s) \times (y - t, y + t)$ is in Ω.

Definition 5. A positive Borel measure μ defined on $I\!R^2_+ \times I\!R^2_+$ is a Carleson measure if there exists a positive constant C satisfying:

$$\mu(S(\Omega)) = \int\int_{S(\Omega)} d\mu \le C|\Omega|, \tag{2}$$

for all open set $\Omega \in I\!R^2$ with finite measure.

The infimum over the set of all constants satisfying (2) will be denoted by $||\mu||$.

Chang-Fefferman proved (see [2]) that a locally integrated function g is in $BMO(I\!R \times I\!R)$ if, and only if

$$d\mu(x, s; y, t) = |\Psi_s \Psi_t * g(x, y)|^2 dxdy \frac{ds}{s}\frac{dt}{t}$$

is a Carleson measure on $I\!R^2_+ \times I\!R^2_+$, where $\Psi_s(x) = s^{-1}\Psi(s^{-1}x)$ and $\Psi_t(y) = t^{-1}\Psi(t^{-1}y)$, for a convenient $\Psi \in S(I\!R)$. This result is also valid when $g \in BMO(I\!R \times I\!R, E)$, where E is a Hilbert space, that is, we have the following result.

Theorem 6. *Let $\Psi \in \mathcal{S}(I\!R)$ an even function with $\operatorname{supp}\Psi \subset [-1, 1]$, $0 \notin \operatorname{supp}\widehat{\Psi}$ and $\int_0^\infty |\widehat{\Psi}(t)|^2 \frac{dt}{t} = 1$. A locally integrated function g is in $BMO(I\!R \times I\!R, E)$ if, and only if, there exist a constant $C_g > 0$ with*

$$\int\int_{S(\Omega)} ||\Psi_s \Psi_t * g(x, y)||^2_E dxdy \frac{ds}{s}\frac{dt}{t} \le C_g|\Omega|,$$

for all open set $\Omega \subset I\!R^2$ with finite measure. Moreover,

$$\sup_\Omega \{\frac{1}{|\Omega|}\int\int_{S(\Omega)} ||\Psi_s \Psi_t * g(x, y)||^2_E dxdy \frac{ds}{s}\frac{dt}{t}\}^{1/2} \sim ||g||_{BMO(I\!R\times I\!R, E)},$$

where supreme takes over all open sets with finite measure and $a \sim b$ means that there are positive constants C_1 and C_2 with

$$C_1 \le \frac{a}{b} \le C_2.$$

3. Vector-Valued Singular Integral Operators on Spaces $H^1(\mathbb{R} \times \mathbb{R})$ and $BMO(\mathbb{R} \times \mathbb{R})$ of Vectorial Functions

Let E and F be Banach spaces and $\Delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$. $L(E, F)$ is the set of continous linear maps from E to F and $L^1_{loc}(\mathcal{U}, E)$ is the set of maps defined on the open set $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^n$ with values in E, which are measurable and localy integrable in $\mathcal{U}$. $L^\infty_c(\mathbb{R}^n, E)$ is the subspace of $L^\infty(\mathbb{R}^n, E)$ which consists of functions with compact support, whereas $M(\mathbb{R}^n, E)$ is the space of the measurable functions $f : \mathbb{R}^n \to E$.

Let k in $L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n - \Delta, L(E, F))$, k verifies condition (α) if

$$\int_{|x-y'|>2|y-y'|} \|k(x,y) - k(x,y')\|_{L(E,F)} dx \leq C\,.$$

The k verifies condition (β) if

$$\int_{|x'-y|>2|x-x'|} \|k(x,y) - k(x',y)\|_{L(E,F)} dy \leq C\,.$$

Let T be a linear operator from $L^\infty_c(\mathbb{R}^n, E)$ to $L^1_{loc}(\mathbb{R}^n, F)$, such that

(i) T may be extended for a linear bounded operator from $L^r(\mathbb{R}^n, E)$ to $L^r(\mathbb{R}^n, F)$, for r, with $1 < r < \infty$;

(ii) the kernel k exists in $L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n - \Delta, L(E, F))$, such that

$$Tf(x) = \int k(x,y)\ f(y)\ dy$$

for all f in $L^\infty_c(\mathbb{R}^n, E)$ and $x \notin \operatorname{supp} f$.

Then, the operator T is called a *singular integral operator* (with variable kernel). If k is in $L^1_{loc}(\mathbb{R}^n - \{0\}, L(E, F))$ and $k(x, y) = k(x - y)$, the singular integral operator is of convolution type.

The main result about these operators is (see [11], [12]):

Theorem 7. *Let T be a linear operator from $L^\infty_c(\mathbb{R}^n, E)$ to $L^1_{loc}(\mathbb{R}^n, F)$, defined as in Definition 3(2).*

(a) If k satisfies (α) and (β), then T is a bounded operator from $L^p(\mathbb{R}^n, E)$ to $L^p(\mathbb{R}^n, F)$, for all p with $1 < p < \infty$;

(b) If k satisfies (α), then T is a bounded operator from $H^1(\mathbb{R}^n, E)$ to $L^1(\mathbb{R}^n, F)$;

(c) If k satisfies (β), then T is a bounded operator from $L^\infty_c(\mathbb{R}^n, E)$ to $BMO(\mathbb{R}^n, F)$.

Spaces $H^1(\mathbb{R}^n, E)$ and $BMO(\mathbb{R}^n, F)$ in Theorem 7 are the spaces in one parameter setting.

In [9] Fernandez presents a partial generalization of Theorem 7 for product kernel and L^p spaces with mixed norms of Benedek-Panzone [1]. The objective of this section is to obtain a generalization of (b) and (c) of Theorem 7 for the product spaces $H^1(I\!R \times I\!R)$ and $BMO(I\!R \times I\!R)$. A slightly stronger condition on the kernels than the conditions given by (α) and (β) is assumed.

The proof deploys the following geometric lemma of Soria [14], in which a geometric lemma from Journé is combined with an interesting observation of R. Fefferman in [6].

Lemma 8. *Let Ω be an open set of $I\!R^2$. Then, for each $R_\alpha = I_\alpha \times J_\alpha$ in $\mathcal{M} = \mathcal{M}(\Omega)$ there exists a rectangle $R'_\alpha = I'_\alpha \times J'_\alpha$ containing R_α, such that*

$$|\bigcup_{R_\alpha \in \mathcal{M}} R'_\alpha| \leq C \cdot |\Omega| \tag{3}$$

and, for all $\varepsilon > 0$,

$$\sum_{R_\alpha \in \mathcal{M}} |R_\alpha| \left[\left(\frac{|I_\alpha|}{|I'_\alpha|} \right)^\varepsilon + \left(\frac{|J_\alpha|}{|J'_\alpha|} \right)^\varepsilon \right] \leq C_\varepsilon \cdot |\Omega|, \tag{4}$$

where the constant C_ε depends only on ε.

Given interval $I \subset I\!R$, let $S_1(I) = I \times (0, |I|)$ and if $R = I \times J$, then $S_1(R) = S_1(I) \times S_1(J) \subset I\!R^2_+ \times I\!R^2_+$.

If $c > 0$ is a real number, cI denotes the interval $cI = \{cx : x \in I\}$ and, if $R = I \times J$, then $cR = cI \times cJ$.

Let $\psi \in C_c^\infty(I\!R)$ be an even function such that $\operatorname{supp} \psi \subset [-1, 1]$, $\int_{-\infty}^{\infty} \psi(s)$ $ds = 0$ and

$$\int_0^\infty |\widehat{\psi}(s)|^2 \, \frac{ds}{s} < \infty\,. \tag{5}$$

For $t > 0$, $\psi_t(s) = t^{-1}\,\psi(t^{-1}\,s)$. Thus we have the following useful lemma.

Lemma 9. *Let E be a Banach space, F and G Hilbert spaces and k_1 and k_2 kernels in $L^1_{loc}(I\!R^2, L(E, F))$ and $L^1_{loc}(I\!R^2, L(F, G))$, respectively, satisfying*

$$\int_{|x'-y| > \gamma|x-x'|} \|k_j(x, y) - k_j(x', y)\|_{L_j} dy \leq C \cdot \gamma^{-\delta} \quad , \quad j = 1, 2\,, \tag{6}$$

for all $\gamma \geq 2$ and some $\delta > 0$, where $L_1 = L(E, F)$ and $L_2 = L(F, G)$. Let T_1 be a bounded linear operator from $L^2(I\!R, E)$ to $L^2(I\!R, F)$ and T_2 be a bounded linear operator from $L^2(I\!R, F)$ to $L^2(I\!R, G)$, satisfying

$$T_1 f(x) = \int k_1(x, u)\, f(u)\, du\,,$$

for all $f \in L_c^\infty(I\!R, E)$, *and*

$$T_2 f(y) = \int k_2(y,v)\, f(v)\, dv\,,$$

for all $f \in L_c^\infty(I\!R, F)$. T *is a linear operator from* $L_c^\infty(I\!R^2, E)$ *to* $M(I\!R^2, G)$, *satisfying*

$$Tf(x,y) = \int\int k_2(y,v)\, k_1(x,u)\, f(u,v)\, du\, dv\,,$$

for all $f \in L_c^\infty(I\!R^2, E)$ *and* $(x,y) \notin \operatorname{supp} f$. *Let* $R = I \times J$ *and* $\widetilde{R} = \widetilde{I} \times \widetilde{J}$ *be rectangles in* $I\!R^2$ *with* $\widetilde{R} \supset 6\, R$. *Then, there are constants* $C > 0$ *and* $\varepsilon > 0$ *such that*

$$\int\int_{S_1(R)} \|\psi_s \psi_t * Tf(x,y)\|_G^2\, dx\, dy\, \frac{ds}{s}\, \frac{dt}{t}$$
$$\leq C \cdot \|f\|^2_{L^\infty(I\!R^2,E)} |R| \left[\left(\frac{|I|}{|\widetilde{I}|}\right)^{\varepsilon} + \left(\frac{|J|}{|\widetilde{J}|}\right)^{\varepsilon} \right],$$

for all $f \in L_c^\infty(I\!R^2, E)$ *with* $\operatorname{supp} f \cap 6\, \widetilde{R} = \emptyset$.

Proof. Let us define

$$A_1 = \{(z,w) \in I\!R^2 \;:\; z \in 6\, I \;,\;\; w \notin 6\, \widetilde{J}\}\,,$$
$$A_2 = \{(z,w) \in I\!R^2 \;:\; z \notin 6\, \widetilde{I} \;,\;\; w \in 6\, J\}\,, \quad A_3 = A_4 \cup A_5\,,$$

where

$$A_4 = \{(z,w) \in I\!R^2 \;:\; z \notin 6\, I \;,\;\; w \notin 6\, \widetilde{J}\}$$

and

$$A_5 = \{(z,w) \in I\!R^2 \;:\; z \notin 6\, \widetilde{I} \;,\;\; w \notin 6\, J\}.$$

These definitions demonstrate that A_1, A_2 and A_3 are mutualy disjunct sets and $\operatorname{supp} f \subset A_1 \cup A_2 \cup A_3 = (6\, \widetilde{R})^c$, since by hypothesis, $\operatorname{supp} f \cap (6\, \widetilde{R}) = \emptyset$. Thus, if $f_i = f \chi_{A_i}$, $i = 1,2,3$, then $f = f_1 + f_2 + f_3$ and consequently

$$\int\int_{S_1(R)} \|\psi_s \psi_t * Tf(x,y)\|_G^2\, dx\, dy\, \frac{ds}{s}\, \frac{dt}{t} \tag{7}$$
$$\leq \int\int_{S_1(I\!R)} \|\psi_s \psi_t * Tf_1(x,y)\|_G^2\, dx\, dy\, \frac{ds}{s}\, \frac{dt}{t}$$
$$+ \int\int_{S_1(R)} \|\psi_s \psi_t * Tf_2(x,y)\|_G^2\, dx\, dy\, \frac{ds}{s}\, \frac{dt}{t}$$
$$+ \int\int_{S_1(R)} \|\psi_s \psi_t * Tf_3(x,y)\|_G^2\, dx\, dy\, \frac{ds}{s}\, \frac{dt}{t}.$$

Let us estimate each integral in the second member of (7). The former is estimated by observing that if $(x,s;y,t) \in S_1(R)$ and $(u,v) \in I\!R^2$ such that $|x-u| \le s$ and $|y-v| \le t$, then $(u,v) \in 3\,R \subset 6\widetilde{R}$, obtaining $(u,v) \notin \operatorname{supp} f_1$. Therefore, if $f_{1w}(.) = f_1(.,w)$ and taking in account that $\int \psi(s)ds = 0$, then

$$\begin{aligned}
&\psi_s\psi_t * Tf_1(x,y)\\
&= \int\int \psi_s(x-u)\psi_t(y-v)\ \left(\int\int k_2(v,w).k_1(u,z).f_1(z,w)\ dz\ dw\right) du\ dv\\
&= \int_{(6\ \widetilde{J})^c} \left(\int \psi_t(y-v)k_2(v,w)dv\right) (\int \psi_s(x-u)(\int k_1(u,z)f_{1w}(z)dz)du)dw\\
&= \int_{(6\ \widetilde{J})^c} Q_tT_2(y,w)\ (\psi_s * T_1f_{1w})(x)dw\,,
\end{aligned}$$

where

$$Q_t\ T_2(y,w) = \int\ \psi_t(y-v)\ (k_2(v,w) - k_2(y,w))\ dv.$$

Thus, from Minkowsky inequality,

$$\begin{aligned}
&\int\int_{S_1(R)} \|\psi_s\psi_t * Tf_1(x,y)\|_G^2\ dxdy\frac{ds}{s}\frac{dt}{t}\\
&\le \int_{S_1(J)}\int_{S_1(I)} \left[\int_{(6\ \widetilde{J})^c} \|Q_tT_2(y,w)\|_{L(F,G)} \|(\psi_s * T_1f_{1w})(x)\|_F\ dw\right]^2 dx\frac{ds}{s}dy\frac{dt}{t}\\
&\le \int_{S_1(J)} \left[\int_{(6\ \widetilde{J})^c} \|Q_tT_2(y,w)\|_{L(F,G)} \left(\int_{S_1(I)} \|(\psi_s * T_1f_{1w})(x)\|_F^2 dx\frac{ds}{s}\right)^{1/2} dw\right]^2\\
&\qquad dy\frac{dt}{t}. \qquad (8)
\end{aligned}$$

Now, applying Plancherel's Theorem, condition (5) and limitation of the operator T_1 from $L^2(I\!R,E)$ to $L^2(I\!R,F)$, one has

$$\begin{aligned}
&\int_{S_1(I)} \|(\psi_s * T_1f_{1w})(x)\|_F^2\ dx\frac{ds}{s}\\
&\le \int_0^\infty \int_{-\infty}^\infty |\widehat{\psi}_s(\xi)|^2 \|\mathcal{F}(T_1f_{1w})(\xi)\|_F^2\ d\xi\ \frac{ds}{s} \le C\cdot\int_{-\infty}^\infty \|T_1f_{1w}(x)\|_F^2\ dx\\
&\le C\cdot\int_{6\ I} \|f_{1w}(x)\|_E^2\ dx \le C\cdot \|f\|^2_{L^\infty(I\!R^2,E)}\ |I|\,.
\end{aligned}$$

Therefore, from (8), it follows

$$\int\int_{S_1(R)} \|\psi_s\psi_t * Tf_1(x,y)\|_G^2 \, dx \, dy \, \frac{ds}{s} \, \frac{dt}{t}$$
$$\leq \; C \cdot \|f\|^2_{L^\infty(I\!R^2,E)} \, |I| \int_0^{|J|} \int_J \left[\int_{(6\,\widetilde{J})^c} \|Q_t T_2(y,w)\|_{L(F,G)} dw \right]^2 dy \frac{dt}{t} \, .$$

Now, if $\bar{y}$ is the center of the interval J and since $y \in J$, $|y-v| \leq t$ and $\widetilde{J} \supset 6\,J$, then for all $w \in (6\,\widetilde{J})^c$,

$$|y-w| \geq |\bar{y}-w| \; - \; |y-\bar{y}|$$
$$\geq \frac{5}{2}|\widetilde{J}| + 3\,|J| \; - \; \frac{1}{2}\,|J| > t\,\frac{|\widetilde{J}|}{t} \; \geq \; \frac{|\widetilde{J}|}{t}\,|y-v|\,,$$

where $\frac{|\widetilde{J}|}{t} > 2$, since $2t < 2|J| \leq |\widetilde{J}|$. Applying condition (6) for $j=2$,

$$\int_{(6\,\widetilde{J})^c} \|Q_t T_2(y,w)\|_{L(F,G)}\, dw$$
$$\leq \int_{|y-w|>\frac{|\widetilde{J}|}{t}|y-v|} \int |\psi_t(y-v)| \|k_2(v,w) - k_2(y,w)\|_{L(F,G)}\, dv dw$$
$$= \int_{|y-v|\leq t} |\psi_t(y-v)| \left(\int_{|y-w|>\frac{|\widetilde{J}|}{t}|y-v|} \|k_2(v,w)-k_2(y,w)\|_{L(F,G)} dw \right) dv$$
$$\leq C \cdot \left(\frac{|\widetilde{J}|}{t} \right)^{-\delta} \int_{|y-v|\leq t} |\psi_t(y-v)|\, dv \leq C \cdot \left(\frac{|\widetilde{J}|}{t} \right)^{-\delta} .$$

Consequently,

$$\int\int_{S_1(R)} \|\psi_s\psi_t * Tf_1(x,y)\|_G^2 \, dx \, dy \, \frac{ds}{s} \, \frac{dt}{t} \leq c \cdot \|f\|^2_{L^\infty(I\!R^2,E)} |I| \; |J| \; |\widetilde{J}|^{-2\delta}$$
$$\times \int_0^{|J|} t^{2\delta-1}\, dt = c \cdot \|f\|^2_{L^\infty(I\!R^2,E)} |R| \left(\frac{|J|}{|\widetilde{J}|} \right)^{2\delta} \, . \quad (9)$$

The estimation of the first integral is complete. In a analogous way, we obtain

$$\int\int_{S_1(R)} \|\psi_s\psi_t * Tf_2(x,y)\|_G^2 dx dy \frac{ds}{s} \frac{dt}{t} \leq C\,||f||^2_{L^\infty(I\!R^2,E)}\,|R| \left(\frac{I}{|\widetilde{I}|} \right)^{2\delta} . \quad (10)$$

For the estimation of the last integral in the second member of (7), since $\int \; \psi(x)\, dx = 0$, it follows

$$\psi_s\psi_t * Tf_3(x,y)$$

$$= \int\int \psi_s(x-u)\psi_t(y-v)\left(\int\int_{A_4\cup A_5} k_2(v,w)k_1(u,z)f_3(z,w)dzdw\right) dudv$$
$$= \int\int_{A_4\cup A_5} \left(\int \psi_t(y-v)k_2(v,w)dv\right)\left(\int \psi_s(x-u)k_1(u,z)du\right) f_3(z,w)$$
$$dzdw = \int\int_{A_4\cup A_5} Q_t\, T_2(y,w).Q_s\, T_1(x,z).f_3(z,w)\, dzdw.$$

Thus,

$$\int\int_{S_1(R)} \|\psi_s\psi_t * Tf_3(x,y)\|_G^2 dxdy\frac{ds}{s}\frac{dt}{t} \tag{11}$$
$$\leq \int\int_{S_1(R)} (\int\int_{A_4\cup A_5} \|Q_t\, T_2(y,w)\|_{L(F,G)}\|Q_s\, T_1(x,z)\|_{L(E,F)}$$
$$\times \|f_3(z,w)\|_E dzdw)^2\; dxdy\frac{ds}{s}\frac{dt}{t}$$
$$\leq C\,\|f\|^2_{L^\infty(I\!R^2,E)} \int\int_{S_1(R)} (\int_{(6\,\widetilde{J})^c} \|Q_t\, T_2(y,w)\|_{L(F,G)}dw)^2$$
$$\times (\int_{(6\,I)^c} \|Q_s\, T_1(x,z)\|_{L(E,F)}dz)^2\; dxdy\frac{ds}{s}\frac{dt}{t}$$
$$+ C\,\|f\|^2_{L^\infty(I\!R^2,E)} \int\int_{S_1(R)} (\int_{(6\,J)^c} \|Q_t\, T_2(y,w)\|_{L(F,G)}dw)^2$$
$$\times (\int_{(6\,\widetilde{I})^c} \|Q_s\, T_1(x,z)\|_{L(E,F)}dz)^2\; dxdy\frac{ds}{s}\frac{dt}{t}.$$

But, if $\bar{y}$ is the center of interval J and, since $y \in J$, $w \in (6\,J)^c$ and $|y-v| \leq t$, then

$$|y-w| \geq |\bar{y}-w| - |y-\bar{y}| \geq 3|J| - \frac{1}{2}|J| > 2|J| \geq \frac{2|J|}{t}|y-v|\,,$$

where $\frac{2|J|}{t} > 2$, since $t \in (0,|J|)$. Applying condition (6) for $j=2$, one has

$$\int_{(6\,J)^c} \|Q_t\, T_2(y,w)\|_{L(F,G)}\, dw$$
$$\leq \int_{(6\,J)^c}\int |\psi_t(y-v)|\|k_2(v,w)-k_2(y,w)\|_{L(F,G)}\, dv\, dw$$
$$\leq \int\left(\int_{|y-w|>\frac{2|J|}{t}|y-v|} \|k_2(v,w)-k_2(y,w)\|_{L(F,G)}dw\right)|\psi_t(y-v)|\, dv$$
$$\leq C\cdot\left(\frac{2|J|}{t}\right)^{-\delta} = C\cdot\left(\frac{|J|}{t}\right)^{-\delta}\,.$$

In a analogous way we obtain

$$\int_{(6\,I)^c} \|Q_s\,T_1(x,z)\|_{L(E,F)}\,dz \le C\cdot\left(\frac{|I|}{s}\right)^{-\delta}.$$

Consequentely, from (11) one has

$$\int\!\!\int_{S_1(R)} \|\psi_s\psi_t * Tf_3(x,y)\|_G^2\,dxdy\frac{ds}{s}\frac{dt}{t} \tag{12}$$

$$\begin{aligned}
&\le C\cdot\|f\|^2_{L^\infty(I\!R^2,E)}\int_0^{|J|}\int_0^{|I|}\int_J\int_I |\widetilde{J}|^{-2\delta}\,t^{2\delta-1}\,|I|^{-2\delta}\,s^{2\delta-1}\,dx\,dy\,ds\,dt\\
&\quad +C\cdot\|f\|^2_{L^\infty(I\!R^2,E)}\int_0^{|J|}\int_0^{|I|}\int_J\int_I |J|^{-2\delta}\,t^{2\delta-1}\,|\widetilde{I}|^{-2\delta}\,s^{2\delta-1}\,dx\,dy\,ds\,dt\\
&\le C\cdot\|f\|^2_{L^\infty(I\!R^2,E)}|R|\left[\left(\frac{|I|}{|\widetilde{I}|}\right)^{2\delta}+\left(\frac{|J|}{|\widetilde{J}|}\right)^{2\delta}\right].
\end{aligned}$$

The Estimation of the last integral in the second member of (7) is thus complete. Finally, since $\varepsilon = 2\delta$, from (7), (9), (10) and (12) we have

$$\begin{aligned}
&\int\!\!\int_{S_1(R)} \|\psi_s\psi_t * Tf(x,y)\|_G^2\,dxdy\frac{ds}{s}\frac{dt}{t} \le\\
&\le C\cdot\|f\|^2_{L^\infty(I\!R^2,E)}|R|\left[\left(\frac{|I|}{|\widetilde{I}|}\right)^{\varepsilon}+\left(\frac{|J|}{|\widetilde{J}|}\right)^{\varepsilon}\right],
\end{aligned}$$

which completes the proof of the lemma. □

One of our main results may be proved now.

Theorem 10. *Let k_1, k_2 be kernels and T_1, T_2 and T be linear operators as in Lemma 9. If T admits a bounded extension from $L^2(I\!R^2, E)$ to $L^2(I\!R^2, G)$, then T is a bounded operator from $L^\infty_c(I\!R^2, E)$ to $BMO(I\!R\times I\!R, G)$; that is, there exists a constant $C > 0$ with*

$$\|Tf\|_{BMO(I\!R\times I\!R,G)} \le C\cdot\|f\|_{L^\infty(I\!R^2,E)}$$

for all $f \in L^\infty_c(I\!R^2, E)$.

Proof. We take $f \in L^\infty_c(I\!R^2, E)$. From Theorem 6 it is enough to show the existence of a constant $C > 0$ with

$$\int\!\!\int_{S(\Omega)} \|\psi_s\psi_t * Tf(x,y)\|_G^2\,dxdy\frac{ds}{s}\frac{dt}{t} \le C\cdot\|f\|^2_{L^\infty(I\!R^2,E)}|\Omega|\,, \tag{13}$$

for every open set $\Omega\subset I\!R^2$ with finite measure. Using the notation of the Lemma 8, let $\widehat{\Omega} = \bigcup_{R_\alpha\in\mathcal{M}} 180R'_\alpha$ and $f = f_0 + f_1$, where $f_0 = f\chi_{\widehat{\Omega}}$ and

$f_1 = f\chi_{\widehat{\Omega}^c}$. Then, by Plancherel's Theorem, by the boundedness of operator T from $L^2(I\!R^2, E)$ to $L^2(I\!R^2, G)$ and by (4), we have

$$\begin{aligned}\int\!\!\int_{S(\Omega)} &\|\psi_s\psi_t * Tf_0(x,y)\|_G^2 \, dxdy\frac{ds}{s}\frac{dt}{t} \\ &\le \int\!\!\int_{I\!R^2_+\times I\!R^2_+} \|\psi_s\psi_t * Tf_0(x,y)\|_G^2 \, dxdy\frac{ds}{s}\frac{dt}{t} \\ &\le \int_{I\!R^2_+}\int_{I\!R^2} |\widehat{\psi}_s(\xi)|^2 \, |\widehat{\psi}_t(\eta)|^2 \|\mathcal{F}(Tf_0)(\xi,\eta)\|_G^2 \, d\xi d\eta \, \frac{ds}{s}\frac{dt}{t} \\ &\le C\cdot\|f_0\|^2_{L^2(I\!R^2,E)} \le C\cdot\|f\|^2_{L^\infty(I\!R^2,E)}|\widehat{\Omega}| \le C\cdot\|f\|^2_{L^\infty(I\!R^2,E)}|\Omega| \,. \quad (14)\end{aligned}$$

Now,

$$S(\Omega) \subset \bigcup_\alpha S_1(5R_\alpha) \ . \quad (15)$$

Indeed, if $(x,s;y,t) \in S(\Omega)$, then $(x-s,x+s)\times(y-t,y+t) \subset \Omega$. Thus, there are diadic intervals I' and J' with

$$I' \subset (x-s,x+s) \subset 5\,I' \quad (16)$$

and

$$J' \subset (y-t,y+t) \subset 5\,J' \ . \quad (17)$$

Now, let $R = I\times J \in \mathcal{M}(\Omega)$ such that $I'\times J' \subset R$. Then, (16) and (17) imply $s < 2s \le 5|I|$ and $t < 2t \le 5|J|$ and one has $(x,s;y,t) \in S_1(5I)\times S_1(5J) = S_1(5R)$, following (15). Therefore, applying Lemma 9 to $R = 5R_\alpha$ and $\widetilde{R} = 30R'_\alpha$ and the Lemma 8, we have

$$\begin{aligned}\int\!\!\int_{S(\Omega)} &\|\psi_s\psi_t * Tf_1(x,y)\|_G^2 \, dxdy\frac{ds}{s}\frac{dt}{t} \quad (18)\\ \le \ & \sum_\alpha \int\!\!\int_{S_1(5R_\alpha)} \|\psi_s\psi_t * Tf_1(x,y)\|_G^2 \, dxdy\frac{ds}{s}\frac{dt}{t} \\ \le \ & \sum_\alpha C\,\|f_1\|^2_{L^\infty(I\!R^2,E)}\,|5R_\alpha| \left[\left(\frac{|5I_\alpha|}{|30I'_\alpha|}\right)^\varepsilon + \left(\frac{|5J_\alpha|}{|30J'_\alpha|}\right)^\varepsilon\right] \\ \le \ & C\cdot\|f\|^2_{L^\infty(I\!R^2,E)}\,|\Omega| \ .\end{aligned}$$

Now, by (14) and (18), (13) is obtained and the proof of the theorem is complete. □

Definition 11. Let E be a Hilbert space. A function a defined on $I\!R^2$ with values on E is a *rectangle E-atom* if it verifies that:

(1) there are bounded intervals $I, J \subset I\!R$ with supp $a \subset I\times J$;

(2) $\|a\|_{L^2(I\!R^2,E)} \le |I\times J|^{-1/2}$;

(3) $\int a(x,y)dx = 0$, q.t.p. $y \in I\!R$ e $\int a(x,y)dy = 0$, a.e. $x \in I\!R$.

Lemma 12. *Let E be a Hilbert space, F and G Banach spaces and k_1 and k_2 kernels in $L^2_{loc}(I\!R^2, L(E,F))$ and $L^2_{loc}(I\!R^2, L(F,G))$, respectively, satisfying*

$$\int_{|x-y'|>\gamma|y-y'|} \|k_j(x,y) - k_j(x,y')\|_{L_j}\ dx \leq C \cdot \gamma^{-\delta}\ ,\ j = 1,2, \tag{19}$$

for every $\gamma \geq 2$ and some $\delta > 0$, where $L_1 = L(E,F)$ and $L_2 = L(F,G)$. Let T_1 and T_2 be bounded linear operators from $L^2(I\!R,E)$ to $L^2(I\!R,F)$ and from $L^2(I\!R,F)$ to $L^2(I\!R,G)$, respectively, satisfying

$$T_1 f(x) = \int k_1(x,u)\ f(u)\ du\ , \tag{20}$$

for every $f \in L^2_c(I\!R,E)$, and

$$T_2 f(y) = \int k_2(y,v)\ f(v)\ dv\ , \tag{21}$$

for every $f \in L^2_c(I\!R,F)$. If T is a linear operator from $L^2_c(I\!R^2,E)$ to $M(I\!R^2,G)$ satisfying

$$Tf(x,y) = \int\int k_2(y,v)\ k_1(x,u)\ f(u,v)\ du\ dv\ , \tag{22}$$

for every $f \in L^2_c(I\!R^2,E)$ and $(x,y) \notin \operatorname{supp} f$, then for all rectangle E-atom, with support in a rectangle $R = I \times J$, we have

$$\int\int_{R^c_\gamma} \|Ta(x,y)\|_G\ dx\ dy \leq C \cdot \gamma^{-\delta}\ , \tag{23}$$

for every $\gamma \geq 2$ and some $\delta > 0$, where $R_\gamma = 2\gamma R$.

Proof. Let a be a rectangle E-atom with supp $a \subset R = I \times J$. For $\gamma \geq 2$, let

$$\begin{aligned} A_1 &= \{(x,y) \in I\!R^2\ :\ x \in 4I\ ,\ \ y \notin 2\gamma J\}\,, \\ A_2 &= \{(x,y) \in I\!R^2\ :\ x \notin 2\gamma I\ ,\ \ y \in 4J\}\ , \\ A_3 &= \{x,y) \in I\!R^2\ :\ x \notin 2\gamma I\ ,\ \ y \notin 4J\}\ , \\ A_4 &= \{(x,y) \in I\!R^2\ :\ x \notin 4I\ ,\ \ y \notin 2\gamma J\}\ . \end{aligned}$$

Since $R^c_\gamma = (2\gamma R)^c = A_1 \cup A_2 \cup A_3 \cup A_4$, then

$$\int\int_{R^c_\gamma} \|Ta(x,y)\|_G\, dx\, dy \leq V_1 + V_2 + V_3 + V_4\ ,$$

where

$$V_i = \int\int_{A_i} \|Ta(x,y)\|_G\, dx\, dy\ \ ,\ \ i = 1,2,3,4.$$

Let us estimate each one of these integrals. $\bar{v}$ is the center of the interval J.

If $(x,y) \in A_1$, then $(x,y) \notin$ supp a and by (22), Definition 11(3) and (20) we have

$$\begin{aligned} Ta(x,y) &= \int_J \int_I k_2(y,v)\, k_1(x,u)\, a(u,v)\, du\, dv \\ &= \int_J \int_I (k_2(y,v) - k_2(y,\bar{v}))\, k_1(x,u)\, a(u,v)\, du\, dv \\ &= \int_J (k_2(y,v) - k_2(y,\bar{v}))\, (\int_I k_1(x,u)\, a_v(u)\, du)\, dv \\ &= \int_J (k_2(y,v) - k_2(y,\bar{v}))\, T_1 a_v(x)\, dv\,, \end{aligned}$$

where $a_v(\cdot) = a(\cdot, v)$. Thus,

$$\begin{aligned} V_1 &\le \int_{(2\gamma J)^c} \int_{4I} \int_J \|k_2(y,v) - k_2(y,\bar{v})\|_{L(F,G)} \|T_1 a_v(x)\|_F\, dv\, dx\, dy \\ &= \int_J \left(\int_{(2\gamma J)^c} \|k_2(y,v) - k_2(y,\bar{v})\|_{L(F,G)}\, dy \right) \left(\int_{4I} \|T_1 a_v(x)\|_F\, dx \right) dv\ . \end{aligned}$$

But, if $y \notin 2\gamma J$ and $v \in J$, then

$$|y - \bar{v}| \ge 2\, \gamma\, \frac{|J|}{2} > \gamma\, |v - \bar{v}|\ \ .$$

Taking into account these inequalities, plus (19) with $j = 2$ and applying the Cauchy-Schwarz inequality in relation to the x variable, then

$$\begin{aligned} V_1 &\le \\ &\int_J \left(\int_{|y-\bar{v}|>\gamma|v-\bar{v}|} \|k_2(y,v) - k_2(y,\bar{v})\|_{L(F,G)}\, dy \right) \left(\int_{4I} \|T_1 a_v(x)\|_F\, dx \right) dv \\ &\qquad \le C \cdot \gamma^{-\delta}\, |I|^{1/2} \int_J \left(\int \|T_1 a_v(x)\|_F^2\, dx \right)^{1/2} dv\ . \end{aligned}$$

Since T_1 is a bounded operator from $L^2(I\!R, E)$ to $L^2(I\!R, F)$, the application of the Cauchy-Schwarz inequality to the v variable and from Definition 11(2) produces

$$\begin{aligned} V_1 &\le C \cdot\ \gamma^{-\delta}\, |I|^{1/2} \int_J \left(\int_I \|a_v(u)\|_E^2\, du \right)^{1/2} dv \\ &\le C \cdot\ \gamma^{-\delta}\, |I|^{1/2}\, |J|^{1/2} \left(\int_J \int_I \|a(u,v)\|_E^2\, du\, dv \right)^{1/2} \le C \cdot \gamma^{-\delta}\, . \end{aligned}$$

For a estimation of V_2, let $\bar{u}$ be the center of the interval I. If $(x,y) \in A_2$, then $(x,y) \notin$ supp a and by (22) and Definition 11(3) we have

$$Ta(x,y) = \int\int \; k_2(y,v).k_1(x,u).a(u,v)\,du\,dv$$
$$= \int_I \int_J \; k_2(y,v).[k_1(x,u) - k_1(x,\bar{u})] \; a^u(v)\,du\,dv\,, \quad (24)$$

where $a^u(\cdot) = a(u,\cdot)$. Now, for each x, u, the function $g_{x,u} \; : \; I\!R \longrightarrow F$ is defined by

$$g_{x,u}(v) = [k_1(x,u) - k_1(x,\bar{u})].a^u(v)\ .$$

Function $g_{x,u}$ is in $L^2(I\!R, F)$ because

$$\|g_{x,u}\|_{L^2(I\!R,F)} \le \|k_1(x,u) - k_1(x,\bar{u})\|_{L(E,F)} \|a^u\|_{L^2(I\!R,E)}\ . \qquad (25)$$

Since supp $a^u \subset J$, this implies supp $g_{x,u} \subset J$. Then, by (20) and (24) one has

$$Ta(x,y) = \int_I \; T_2\; g_{x,u}(y)\; du\ .$$

Thus, by the Cauchy-Schwarz inequality (in variable y), by (25) and since T_2 is a bounded operator from $L^2(I\!R, F)$ to $L^2(I\!R, G)$, then

$$V_2 \le \int_{4J} \int_{(2\gamma I)^c} \int_I \|T_2 g_{x,u}(y)\|_G\,du\,dx\,dy$$
$$\le \int_{(2\gamma I)^c} \int_I 2|J|^{1/2} \left(\int \|T_2 g_{x,u}(y)\|_G^2 dy \right)^{1/2} du\,dx$$
$$\le C \cdot |J|^{1/2} \int_{(2\gamma I)^c} \int_I \left(\int \|g_{x,u}(v)\|_F^2 dv \right)^{1/2} du\,dx$$
$$\le C{\cdot}|J|^{1/2} \int_I \left(\int_{(2\gamma I)^c} \|k_1(x,u) - k_1(x,\bar{u})\|_{L(E,F)} dx \right) \left(\int \|a^u(v)\|_E^2 dv \right)^{1/2} du\,.$$

But, if $x \notin 2\,\gamma\, I$ and $u \in I$, we have

$$|x - \bar{u}| \ge 2\gamma \frac{|I|}{2} > \gamma |u - \bar{u}|.$$

Therefore, applying (19) with $j = 1$, the Cauchy-Schwarz inequality (in variable u) and by Definition 11(2), we obtain

$$V_2 \le C \cdot |J|^{1/2} \int_I \left(\int_{|x-\bar{u}|>\gamma|u-\bar{u}|} \|k_1(x,u) - k_1(x,\bar{u})\|_{L(E,F)}\, dx \right)$$
$$\times \left(\int \|a^u(v)\|_E^2 dv \right)^{1/2} du \le C \cdot |J|^{1/2} \gamma^{-\delta} \int_I \left(\int \|a^u(v)\|_E^2\, dv \right)^{1/2} du$$
$$\le C \cdot \gamma^{-\delta} |R|^{1/2} \|a\|_{L^2(I\!R^2,E)} \le C \cdot \gamma^{-\delta}.$$

Finally, let us estimate V_3. If $(x,y) \in A_3$ then $(x,y) \notin$ supp$a$ and by (22) and Definition 11(3) we have

$$Ta(x,y) = \int\int (k_2(y,v) - k_2(y,\bar{v}))(k_1(x,u) - k_1(x,\bar{u})).a(u,v)dudv.$$

Thus,

$$\begin{aligned} V_3 &\leq \int\int_{A_3}\int\int_R \|k_2(y,v) - k_2(y,\bar{v})\|_{L(F,G)} \\ &\quad \times \|k_1(x,u) - k_1(x,\bar{u})\|_{L(E,F)}\|a(u,v)\|_E \, du\, dv\, dx\, dy \\ &= \int\int_R \left(\int_{(4J)^c} \|k_2(y,v) - k_2(y,\bar{v})\|_{L(F,G)}\, dy\right) \\ &\quad \times \left(\int_{(2\gamma I)^c} \|k_1(x,u) - k_1(x,\bar{u})\|_{L(E,F)} dx\right) \|a(u,v)\|_E \, du\, dv\,. \end{aligned}$$

Now, if $y \notin 4J$ and $v \in J$, one has

$$|y - \bar{v}| \geq 4\frac{|J|}{2} > 2|v - \bar{v}|\,,$$

from estimation of V_2, if $x \notin 2\,\gamma\, I$ and $u \in I$, then $|x - \bar{u}| > \gamma|u - \bar{u}|$.

Consequentely, by (19), by the Cauchy-Schwarz inequality and by Definition 11(2), we have

$$\begin{aligned} V_3 &\leq \int\int_R \left(\int_{|y-\bar{v}|>2|v-\bar{v}|} \|k_2(y,v) - k_2(y,\bar{v})\|_{L_2}\, dy\right) \\ &\quad \times \left(\int_{|x-\bar{u}|>\gamma|u-\bar{u}|} \|k_1(x,u) - k_1(x,\bar{u})\|_{L_1} dx\right) \|a(u,v)\|_E \, du\, dv \\ &\leq C\cdot 2^{-\delta}\gamma^{-\delta}\int\int_R \|a(u,v)\|_E \, du\, dv \leq C\cdot 2^{-\delta}\gamma^{-\delta}|R|^{1/2}\|a\|_{L^2(\mathbb{R}^2,E)} \leq C\cdot\gamma^{-\delta}. \end{aligned}$$

The estimation of V_4 is analogous to V_3, then

$$V_4 \leq C\cdot\gamma^{-\delta}.$$

The proof is thus complete. □

Our second main result can be established now.

Theorem 13. *Let E be a Hilbert space, F and G Banach spaces, k_1 and k_2 kernels and T_1, T_2 and T linear operators as in Lemma 12. If T has an bounded extension from $L^2(\mathbb{R}^2,E)$ to $L^2(\mathbb{R}^2,G)$, then T has a bounded extension from $H^1_{at}(\mathbb{R}\times\mathbb{R},E)$ to $L^1(\mathbb{R}^2,G)$, that is, there exists a constant $C>0$, such that*

$$\|Tf\|_{L^1(\mathbb{R}^2,G)} \leq C\cdot\|f\|_{H^1_{at}(\mathbb{R}\times\mathbb{R},E)}\,,$$

for every $f \in H^1_{at}(\mathbb{R} \times \mathbb{R}, E)$.

Proof. It is enough to prove that there exists constant $C > 0$, such that

$$\|Ta\|_{L^1(\mathbb{R}^2,G)} \leq C\,, \tag{26}$$

for every E-atom of Chang-Fefferman a. From Definition 11(2) we have

$$a = \sum_{R\in\mathcal{M}(\Omega)} a_R\,, \tag{27}$$

where supp$a_R \subset 4R$. Using the notation of the Lemma 8, let $\widehat{\Omega} = \bigcup_{R\in\mathcal{M}} 48R'$. Thus,

$$\|Ta\|_{L^1(\mathbb{R}^2,G)} = \int\int_{\widehat{\Omega}} \|Ta(x,y)\|_G dxdy + \int\int_{\widehat{\Omega}^c} \|Ta(x,y)\|_G dxdy.$$

Now, let us estimate each integral separately. To estimate the first one, let us observe that by the inequality of Cauchy-Schwarz, by the fact that T be a bounded operator from $L^2(\mathbb{R}^2, E)$ to $L^2(\mathbb{R}^2, G)$ and by (3) and Definition 1(1), one has

$$\int\int_{\widehat{\Omega}} \|Ta(x,y)\|_G dxdy \leq |\widehat{\Omega}|^{1/2}\|Ta\|_{L^2(\mathbb{R}^2,G)} \leq C\cdot|\widehat{\Omega}|^{1/2}\|a\|_{L^2(\mathbb{R}^2,E)}$$
$$\leq C\cdot|\Omega|^{1/2}|\Omega|^{-1/2} = C.$$

To estimate the second integral, initially we observe that for each $R \in \mathcal{M}$, the function $\tilde{a}_R$, defined by

$$\tilde{a}_R(x,y) = \|a_R\|^{-1}_{L^2(\mathbb{R}^2,E)}|4R|^{-1/2}a_R(x,y)\,,$$

is a rectangle E-atom with supp$\tilde{a}_R \subset 4R = \widetilde{R}$. Then by (27), we obtain

$$\int\int_{\widehat{\Omega}^c} \|Ta(x,y)\|_G dxdy \leq \sum_{R\in\mathcal{M}} \int\int_{\widehat{\Omega}^c} \|Ta_R(x,y)\|_G dxdy$$
$$\leq \quad C\cdot\sum_{R\in\mathcal{M}} \|a_R\|_{L^2(\mathbb{R}^2,E)}|R|^{1/2}\int\int_{(48\widetilde{R})^c} \|T\tilde{a}_R(x,y)\|_G\,dx\,dy.$$

Now let

$$\gamma = \gamma(R) = \min\{2\frac{|I'|}{|I|}\,, 2\frac{|J'|}{|J|}\}.$$

Hence, $\gamma \geq 2$ and $2\gamma\widetilde{R} \subset 48R'$; whence $(48R')^c \subset (2\gamma\widetilde{R})^c = \widetilde{R}^c_\gamma$. Therefore, applying Lemma 3.9 and the Cauchy-Schwarz inequality, we have

$$\int\int_{\widehat{\Omega}^c} \|Ta(x,y)\|_G dxdy$$
$$\leq \quad C\cdot\sum_{R\in\mathcal{M}} \|a_R\|_{L^2(\mathbb{R}^2,E)}|R|^{1/2}\int\int_{\widetilde{R}^c_\gamma} \|T\tilde{a}_R(x,y)\|_G\,dx\,dy$$

$$
\begin{aligned}
&\le C\cdot\sum_{R\in\mathcal{M}}\|a_R\|_{L^2(\mathbb{R}^2,E)}|R|^{1/2}\gamma^{-\delta}\\
&\le C\cdot\sum_{R\in\mathcal{M}}\|a_R\|_{L^2(\mathbb{R}^2,E)}|R|^{1/2}\left[\left(\frac{|I|}{|I'|}\right)^{\delta}+\left(\frac{|J|}{|J'|}\right)^{\delta}\right]\\
&\le C\cdot\left(\sum_{R\in\mathcal{M}}\|a_R\|^2_{L^2(\mathbb{R}^2,E)}\right)^{1/2}\left(\sum_{R\in\mathcal{M}}|R|\left[\left(\frac{|I|}{|I'|}\right)^{2\delta}+\left(\frac{|J|}{|J'|}\right)^{2\delta}\right]\right)^{1/2}.
\end{aligned}
$$

Finally, using (4) and Definition 1(2iii), we obtain

$$
\int\int_{\widehat{\Omega}^c}\|Ta(x,y)\|_G\,dx\,dy\le C\cdot|\Omega|^{-1/2}|\Omega|^{1/2}=C\,,
$$

and this proves (26). □

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