International Journal of Pure and Applied Mathematics

Volume 41 No. 4 2007, 471-479

EINSTEIN TENSOR IN MANAGEMENT SCIENCE

Gregory L. Light

Providence College
Providence, Rhode Island, 02918, USA
e-mail: glight@providence.edu

Abstract: This paper introduces Einstein tensor E to a calculation of the energy contents of a general $k-$dimensional manifold by evaluating all the local curvatures. Since in management science the construct of input-output systems is fundamental and is amenable to a modeling by a $k-$manifold, we apply E to input-output systems, wherein we give a brief review of the connections from the metric g, to the Christoffel symbols, to the Riemann-Christoffel curvature tensor, to the Ricci curvature tensors, and finally to E and the stress-energy tensor T. As an illustration, we compute the Ricci curvature tensors for a Monge patch and apply them to a familiar production transformation function, with remarks on the relationships between curvatures and energies.

AMS Subject Classification: 53C21, 53A45, 58D17, 93B29
Key Words: manifold energy, system stress, curvatures, input-output

1. Introduction

This paper introduces Einstein tensor E to the field of management science. To begin with, E is a geometric construct that calculates the energy contents of a $k-$dimensional ($k \geq 2$) manifold M^k by evaluating the local curvatures – the greater the curvatures, the greater the energies (cf. [3, 5]). A basic theme in management science is that of an input-output system; equivalently, it is a transformation function f that maps k inputs to l outputs; as such, the graph of f can be made into a Riemannian manifold by equipping the graph with a metric g. Since it is a managerial task to monitor the extent of stress energies of a system, we present E as a tool to engage in the calculations; here we note the existence of standard software in differential geometry capable of

Received: September 13, 2007

$n-$dimensional tensor analyses.

The notion of stress energy originated from the field of continuum mechanics (for a recent survey on this topic dedicated to the three hundredth anniversary of Euler, see [4]); as a result, research in this area has been mostly focused on the mechanical stresses in manifolds of dimensions $k \leq 4$ (for $k \geq 4$, cf. e.g., [6, 9]), and applications of tensors that calculate the stress energies of a general manifold M^k due to its intrinsic non-flatness have not been found. At the same time, Einstein tensor E exactly serves the purpose; to wit, E is essentially the derivative of the total scalar curvature over M^k with respect to the metric g, and derivatives in this context are defined in general as the stress-energy tensors (see, e.g., [2]).

The construct of input-output systems has wide applications: e.g., it is a building block of network systems (see, e.g., [10]), it is a standard tool of analyzing multiple production lines (see, e.g., [1]), and mathematically if we treat parameters given in an optimization problem as inputs and the solutions as outputs, then we have an input-output system (consider, e.g., models of optimal network flows; cf. [8]) and E can calculate the intrinsic curvatures contained therein.

Section 2 below will first give a brief introduction of E, then show a calculation of the therein involved Ricci curvature scalar R for a Monge patch and provide an illustration with a familiar functional form – where one can gain an appreciation of how curvatures relate to the existence of energies. The section will then cite the well-known theorem that E is proportional to the stress-energy tensor T and remark on the implication that knowing E amounts to obtaining information about the various stress levels across all the ($2-$dimensional) sections within the manifold M^k. Finally Section 3 will conclude with a summary.

2. Einstein Tensor for Input-Output Systems

Definition 1. Let g be a Riemannian metric on a C^∞-manifold M^k of dimension $k \geq 2$; let $p \in M^k$ and $\mathbf{B} \equiv \left(\frac{\partial}{\partial x^i}\right)_{i=1}^k$ be a basis of the tangent space T_pM^k, with the matrix representation of $g = (g_{ij})_{k\times k,\mathbf{B}}$. Define Christoffel symbol of the second kind $\Gamma_{ij}^l \ \forall \{i,j,l\} \subset \{1, \cdots, k\}$ by

$$\begin{pmatrix} \Gamma_{ij}^1 \\ \vdots \\ \Gamma_{ij}^k \end{pmatrix}_{\mathbf{B}} := \frac{1}{2} (g_{ij})_{k\times k,\mathbf{B}}^{-1} \left(\frac{\partial}{\partial x^j} g_{i\cdot} + \frac{\partial}{\partial x^i} g_{j\cdot} - \text{grad } g_{ij} \right), \tag{2.1}$$

where

$$g_{i\cdot} \equiv \begin{pmatrix} g_{i1} \\ \vdots \\ g_{ik} \end{pmatrix} \text{ and grad } g_{ij} \equiv \begin{pmatrix} \frac{\partial g_{ij}}{\partial x^1} \\ \vdots \\ \frac{\partial g_{ij}}{\partial x^k} \end{pmatrix}. \tag{2.2}$$

Notation 1. The coefficients of the Riemann-Christoffel curvature $(1,3)$-tensor

$$\begin{pmatrix} R^1_{ilj} \\ \vdots \\ R^k_{ilj} \end{pmatrix}_{\mathbf{B}} \quad \forall \{i,j,l\} \subset \{1,\cdots,k\} \tag{2.3}$$

is given by

$$R^s_{ilj} = \frac{\partial \Gamma^s_{ij}}{\partial x^l} - \frac{\partial \Gamma^s_{il}}{\partial x^j} + \sum_{r=1}^{k} \left(\Gamma^r_{ij}\Gamma^s_{rl} - \Gamma^r_{il}\Gamma^s_{rj} \right), \; s = 1,\cdots,k. \tag{2.4}$$

Furthermore, $\forall \{i,j\} \subset \{1,\cdots,k\}$ set

$$R_{ij} \equiv \sum_{r=1}^{k} R^r_{irj}, \tag{2.5}$$

where

$$R_{ij} := \mathrm{Ric}\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \right) \tag{2.6}$$

is the Ricci curvature $(0,2)$ $-$tensor.

Notation 2. Denote by

$$R \equiv \sum_{l=1}^{k} \sum_{j=1}^{k} g^{lj} R_{jl}, \tag{2.7}$$

where $\left(g^{lj}\right)_{k\times k,\mathbf{B}} = g^{-1}$. Then R is the Ricci curvature $(0,0)$ $-$tensor.

Definition 2. Einstein tensor $E \equiv (E_{ij})_{k\times k}$, $k \geq 2$, is given by

$$E_{ij} := R_{ij} - \frac{1}{2} R \cdot g_{ij}. \tag{2.8}$$

Remark 1. By the Gauss-Bonnet Theorem, $\forall$ metric g we have

$$(E_{ij})_{2\times 2} \equiv \mathbf{O}_{2\times 2}. \tag{2.9}$$

Consider now a C^∞-manifold M^k parametrized by k (≥ 2) inputs, $(x^1,\cdots,$ $x^k)$,

$$\mathbf{F}\left(x^1,\cdots,x^k\right) = \left(x^1,\cdots,x^k,x^{k+1},\cdots,x^n\right), \tag{2.10}$$

where $\left(x^{k+1}, \cdots, x^n\right)$ represent quantities of the $(n-k) \in \mathbb{N}$ outputs. Define $\forall \{i,j\} \subset \{1, \cdots, k\}$

$$g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle, \tag{2.11}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. Then

$$g = (g_{ij})_{\left(\frac{\partial}{\partial x^i}\right)_{i=1}^{k}} \tag{2.12}$$

is a Riemannian metric and by the above constructions one can proceed to calculate R_{ij}, R, and E_{ij}.

Proposition 1. *Let $f \in C^3\left(\mathbb{R}^2, \mathbb{R}\right)$; define $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by*

$$\mathbf{F}\left(x^1, x^2\right) = \left(x^1, x^2,\ f\left(x^1, x^2\right)\right). \tag{2.13}$$

Then the Ricci curvature scalar R for the above Monge patch is given by

$$R = 2\left(1 + f_1^2 + f_2^2\right)^{-2}\left(f_{11} \cdot f_{22} - f_{12}^2\right), \tag{2.14}$$

where $f_i \equiv \frac{\partial f}{\partial x^i}$ and $f_{ij} \equiv \frac{\partial^2 f}{\partial x^j \partial x^i}$.

Proof. A direct calculation via the above definitions and notations leads to

$$R_{11} = \left(1 + f_1^2 + f_2^2\right)^{-2}\left(f_{11} \cdot f_{22} - f_{12}^2\right)\left(1 + f_1^2\right). \tag{2.15}$$

Applying the Gauss-Bonnet Theorem and observing that

$$g_{11} = 1 + f_1^2, \tag{2.16}$$

we have

$$R = \frac{2R_{11}}{g_{11}} = 2\left(1 + f_1^2 + f_2^2\right)^{-2}\left(f_{11} \cdot f_{22} - f_{12}^2\right). \tag{2.17}$$

Remark 2. A substitution of the above derived R into

$$R_{ij} = \frac{1}{2} R \cdot g_{ij} \tag{2.18}$$

leads to

$$R_{12} = R_{21} = \frac{1}{2} R \cdot g_{12} = \left(1 + f_1^2 + f_2^2\right)^{-2}\left(f_{11} \cdot f_{22} - f_{12}^2\right) \cdot f_1 f_2\,, \tag{2.19}$$

and

$$R_{22} = \frac{1}{2} R \cdot g_{22} = \left(1 + f_1^2 + f_2^2\right)^{-2}\left(f_{11} \cdot f_{22} - f_{12}^2\right)\left(1 + f_2^2\right); \tag{2.20}$$

here note that

$$g^{11} R_{11} + 2g^{12} R_{21} + g^{22} R_{22} = R\ , \tag{2.21}$$

as expected from equation (2.7). Also,

$$R = 2\mathbf{K}, \tag{2.22}$$

where $\mathbf{K}$:= the Gaussian curvature = $\det \mathfrak{L}$, with

$$\mathfrak{L} : \frac{\partial F}{\partial x^i} \longmapsto -\frac{\partial}{\partial x^i}\left(\frac{\frac{\partial F}{\partial x^1} \times \frac{\partial F}{\partial x^2}}{\left\| \frac{\partial F}{\partial x^1} \times \frac{\partial F}{\partial x^2} \right\|}\right) \equiv -\frac{\partial \nu}{\partial x^i} \in T_p M^2,\ i = 1, 2, \tag{2.23}$$

being the Weingarten map. Here,

$$\nu \equiv \left(\nu^1, \nu^2, \nu^3\right) = \frac{(-f_1, -f_2, 1)}{\sqrt{1 + f_1^2 + f_2^2}} \tag{2.24}$$

is the unit normal, and $\frac{\partial \nu}{\partial x^i}$, $\forall i = 1, 2$, are calculated as follows:

$$\begin{aligned}\frac{\partial \nu^3}{\partial x^1} &= -\frac{1}{2}\left(1 + f_1^2 + f_2^2\right)^{-\frac{3}{2}} \cdot 2\left(f_1 f_{11} + f_2 f_{12}\right) \\ &= -\left(1 + f_1^2 + f_2^2\right)^{-\frac{3}{2}}\left(f_1 f_{11} + f_2 f_{12}\right) \equiv -h \cdot \left(f_1 f_{11} + f_2 f_{12}\right),\end{aligned} \tag{2.25}$$

$$\frac{\partial \nu^3}{\partial x^2} = -h \cdot \left(f_1 f_{12} + f_2 f_{22}\right), \tag{2.26}$$

$$\begin{aligned}\frac{\partial \nu^1}{\partial x^1} &= f_1 \cdot h \cdot \left(f_1 f_{11} + f_2 f_{12}\right) - f_{11} \cdot h^{1/3} \\ &= h \cdot \left(f_1 f_2 f_{12} - f_{11} - f_{11} f_2^2\right),\end{aligned} \tag{2.27}$$

$$\frac{\partial \nu^1}{\partial x^2} = h \cdot \left(f_1 f_2 f_{22} - f_{12} - f_{12} f_2^2\right), \tag{2.28}$$

$$\frac{\partial \nu^2}{\partial x^1} = h \cdot \left(f_1 f_2 f_{11} - f_{12} - f_{12} f_1^2\right), \tag{2.29}$$

$$\frac{\partial \nu^2}{\partial x^2} = h \cdot \left(f_1 f_2 f_{12} - f_{22} - f_{22} f_1^2\right), \tag{2.30}$$

and thus,

$$\begin{aligned}\frac{\partial \nu}{\partial x^1} &= h \cdot \left(f_1 f_2 f_{12} - f_{11} - f_{11} f_2^2,\ f_1 f_2 f_{11} - f_{12} - f_{12} f_1^2,\ -f_1 f_{11} - f_2 f_{12}\right) \\ &= h \cdot \left(f_1 f_2 f_{12} - f_{11} - f_{11} f_2^2\right) \cdot (1, 0, f_1) \\ &\quad + h \cdot \left(f_1 f_2 f_{11} - f_{12} - f_{12} f_1^2\right) \cdot (0, 1, f_2),\end{aligned} \tag{2.31}$$

$$\begin{aligned}\frac{\partial \nu}{\partial x^2} &= h \cdot \left(f_1 f_2 f_{22} - f_{12} - f_{12} f_2^2\right) \cdot (1, 0, f_1) \\ &\quad + h \cdot \left(f_1 f_2 f_{12} - f_{22} - f_{22} f_1^2\right) \cdot (0, 1, f_2),\end{aligned} \tag{2.32}$$

and accordingly,

$$\begin{aligned}\mathbf{K} = \det \mathfrak{L} =& h^2 \cdot \left(f_1 f_2 f_{12} - f_{11} - f_{11} f_2^2\right) \cdot \left(f_1 f_2 f_{12} - f_{22} - f_{22} f_1^2\right) \\ & - h^2 \cdot \left(f_1 f_2 f_{22} - f_{12} - f_{12} f_2^2\right) \cdot \left(f_1 f_2 f_{11} - f_{12} - f_{12} f_1^2\right) = h^2 \cdot \left(1 + f_1^2 + f_2^2\right)\end{aligned}$$

$$\cdot \left(f_{11} \cdot f_{22} - f_{12}^2\right) \equiv \left(1 + f_1^2 + f_2^2\right)^{-2} \cdot \left(f_{11} f_{22} - f_{12}^2\right), \quad (2.33)$$

in agreement with the results in Proposition 1.

Remark 3. Since management science as in other social sciences has the problem of unit indeterminacy, it serves practical purposes to re-express the above Ricci curvature scalar R in relative derivatives and proportions (see [7]) as

$$R = 2 \cdot \frac{\left(\frac{\xi_1^2}{\eta} f_{11}\right)\left(\frac{\xi_2^2}{\eta} f_{22}\right) - \left(\frac{\xi_1 \xi_2}{\eta} f_{12}\right)^2}{\left[1 + \left(\frac{\xi_1}{\eta} f_1\right)^2 \left(\frac{\eta}{\xi_1}\right)^2 + \left(\frac{\xi_2}{\eta} f_2\right)^2 \left(\frac{\eta}{\xi_2}\right)^2\right]^2} \cdot \left(\frac{\eta}{\xi_1 \xi_2}\right)^2, \quad (2.34)$$

where $\xi_1, \xi_2, \eta > 0$ are the relative values of the derivatives, set to rescale and remove units. However, it is worth mentioning that R per se is not unit-free but rather carries a unit of the inverse of the radius of curvature squared, this is so since $\frac{1}{2}R =$ the Gaussian curvature $\mathbf{K}$.

Example 1. Consider the following transformation function

$$y = f(K, L) = AK^\alpha L^\beta, \quad (2.35)$$

where $y \equiv$ output, $K \equiv$ capital input, $L \equiv$ labor input, and $A, \alpha, \beta > 0$ are given parameters. Since

$$\begin{aligned} \frac{K^2}{y} f_{11} &= \frac{K^2}{y} \cdot \alpha(\alpha - 1) \cdot AK^{\alpha-2} L^\beta \\ &= \alpha(\alpha - 1), & (2.36) \\ \frac{L^2}{y} f_{22} &= \beta(\beta - 1), & (2.37) \\ \frac{KL}{y} f_{12} &= \alpha\beta, \text{ and} & (2.38) \\ \alpha(\alpha - 1)\beta(\beta - 1) - \alpha^2\beta^2 &= \alpha\beta(1 - \alpha - \beta), & (2.39) \end{aligned}$$

an application of the above equation (2.34) leads to the Ricci curvature scalar R for

$$F(K, L) = (K, L, f(K, L)) \quad (2.40)$$

equal to

$$R = 2 \cdot \frac{\alpha\beta(1 - \alpha - \beta)}{\left[1 + \left(\frac{\alpha y}{K}\right)^2 + \left(\frac{\beta y}{L}\right)^2\right]^2} \cdot \left(\frac{y}{KL}\right)^2. \quad (2.41)$$

Thus, if $\alpha + \beta = 1$, then $R = 0$; i.e., if f is homogeneous of degree 1, then the graph of f is intrinsically flat. If however $\alpha + \beta < 1$, then $R > 0$ and the graph is elliptic, with $f(mK, mL) = m^{\alpha+\beta} f(K, L) < mf(K, L)\ \forall m > 0$, indicating that the transformation process is under stress due to overuses of K and L. In contrast, if $\alpha + \beta > 1$, then $R < 0$ and the graph is hyperbolic, with $f(mK, mL) > mf(K, L)$, indicating that there exists a driving force in the transformation process to input more K and L. In either case, $R \neq 0$ is an indication that there exist stress energies in the input-output system. Furthermore, as a geometric invariant, a higher $|R|$ corresponds to a higher level of energies.

Example 2. As mentioned in the introduction, if parameters in an optimization problem are considered as inputs and solutions as outputs, then an input-output system arises. Consider now the following constrained minimization problem:

$$\begin{aligned} &\underset{(K,L)}{Min}\; rK + wL \quad \text{subject to} \\ &AK^{\alpha}L^{\beta} = y, \end{aligned} \tag{2.42}$$

where r and w are respectively the given unit costs of using quantities of capital K and labor L in the production of a prescribed output quantity y ($= AK^{\alpha}L^{\beta}$, which is the same transformation function as in the above Example 1). Then the value function of the minimized total production cost c as dependent on the parameters r and w is (see, e.g., [11], 15):

$$\begin{aligned} c(r, w) = \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}\right] \cdot r^{\frac{\alpha}{\alpha+\beta}} w^{\frac{\beta}{\alpha+\beta}} \\ \equiv c_o \cdot r^{\mu} w^{\nu}, \end{aligned} \tag{2.43}$$

which shares the same functional form as $y = AK^{\alpha}L^{\beta}$; consequently, we conclude from the previous Example 1 that for the manifold

$$\tilde{F}(r, w) := (r, w, c(r, w)) \tag{2.44}$$

in view of

$$\mu + \nu \equiv \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} = 1 \tag{2.45}$$

we have the Ricci curvature scalar

$$R = 0. \tag{2.46}$$

Theorem 1. *$E = \kappa T$, where $\kappa \in \mathbb{R}$, and $T \equiv (T_{ij})_{k \times k}$, with $k \geq 2$, is the stress-energy tensor (cf. [2]).*

Remark 4. A typical proof of the above theorem follows the logic of:

(1) E is the linear variation of the total scalar curvature S over M^k with

respect to the metric g, where

$$S := \int_{M^k} R\,(g)\, dv\,(g)\,, \tag{2.47}$$

(2) T is the linear variation of the integral of any Hamiltonian H of M^k with respect to g, and thus (3) an identification of R with H leads to $E = \kappa T$. A corollary here is that if $E = \mathbf{O}$, then the underlying metric g is optimal in the sense of a vanishing linear variation of S with respect to g.

Example 3. Consider the manifold M^k due to the transformation process as introduced by equations (2.10) and (2.11)

$$\begin{aligned} \mathbf{F}\left(x^1, \cdots, x^k\right) &= \left(x^1, \cdots, x^k, x^{k+1}, \cdots, x^n\right), \text{ with} \\ g_{ij} &= \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle. \end{aligned}$$

As remarked earlier, $g := (g_{ij})_{k\times k}$ determines $(R_{ij})_{k\times k}$, R, and $E \equiv (E_{ij})_{k\times k}$, but by Theorem 1, one has $E = \kappa T$; thus, g also determines T.

Remark 5. Whereas in Einstein General Relativity $(T_{ij})_{4\times 4}$ determines $(g_{ij})_{4\times 4}$ by a solution of ten second-order partial differential equations, for the above input-output system M^k it appears more reasonable to obtain first the parametrization $\mathbf{F}\left(x^1, \cdots, x^k\right) = \left(x^1, \cdots, x^k, x^{k+1}, \cdots, x^n\right)$ through econometrics, and then apply tensor software to a computation of $g = (g_{ij})_{k\times k}$, $(R_{ij})_{k\times k}$, R, and

$$E \equiv (E_{ij})_{k\times k} = (R_{ij})_{k\times k} - \frac{1}{2} R \cdot (g_{ij})_{k\times k} = \kappa \cdot (T_{ij})_{k\times k}\,, \tag{2.48}$$

where $(T_{ij})_{k\times k}$ yield information about the relative stress levels across all sections, $\{(i,j) \mid i,j = 1, \cdots, k\}$.

3. Conclusion

In this paper we have provided a methodology to calculate stress energies in input-output systems through the formula,

$$(E_{ij})_{k\times k} = (R_{ij})_{k\times k} - \frac{1}{2} R \cdot (g_{ij})_{k\times k} = \kappa \cdot (T_{ij})_{k\times k}\,, \tag{3.1}$$

which yields two notable results: (1) R as an indicator of the overall stress level of the system, and (2) $(T_{ij})_{k\times k}$ as identifiers of the relative stress levels across all sections, $\{(i,j) \mid i,j = 1, \cdots, k\}$. As such, Einstein tensor serves as a distinct tool in management science. To the extent that system stresses, mechanical or otherwise, are major concerns to managers and yet Einstein tensor has eluded managerial recognition, this introductory note serves as a useful reference to management.

References

[1] A. Alfieri, G. Nicosia, Minimum cost multi-product flow lines, *Ann. Oper. Res.*, **150** (2007), 31-46.

[2] O. Calin, D.-C. Chang, *Geometric Mechanics on Riemannian Manifolds – Applications to Partial Differential Equations*, Birkhäuser, Boston (2005).

[3] S.-Y. Cheng, L.-F. Tam, T. Y.-H. Wan, Harmonic maps with finite total energy, *Proc. Amer. Math. Soc.*, **124** (1996), 275-284.

[4] D. Christodoulou, The Euler equations of compressible fluid flow, *Bull. Amer. Math. Soc.*, on-line (June 18, 2007).

[5] P. DaSilva, C.C. Filho, Relative flatness and flatness of implicit systems, *SIAM J. Control Optim.*, **39**, No. 6 (2001), 1929-1951.

[6] B.C. Hall, Harmonic analysis with respect to heat kernel measure, *Bull. Amer. Math. Soc.*, **38** (2001), 43-78.

[7] G.L. Light, An introductory note on relative derivative and proportionality, *International Journal of Contemporary Mathematical Sciences*, **1**, No. 7 (2006), 327-332.

[8] M.J. Lopez-Herrero, A maximum entropy approach for the busy period of the "M/G/1" retrial queue, *Ann. Oper. Res.*, **141** (2006), 271-281.

[9] C. Ortner, E. Süli, Discontinuous Galerkin finite element approximation of nonlinear second-order elliptic and hyperbolic systems, *SIAM J. Numer. Anal.*, **45**, No. 4 (2007), 1370-1397.

[10] N. Pippenger, The linking probability of deep spider-web networks, *SIAM J. Discrete Math.*, **20**, No. 1 (2006), 143-159.

[11] H.R. Varian, *Microeconomic Analysis*, Norton, New York (1978).