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MODIFICATION OF WAŽEWSKI'S THEOREM FOR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract: This paper deals with a modification of Ważewski's Theorem as applied to initial value problems of integro-differential equations. It gives properly defined concepts of egress points, a left shadow for given equations and consequents of an initial point.

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1. Introduction

Ważewski's principle (see [18]) plays an important role in the study of qualitative properties of ordinary differential equations. Its applicability is largely due to the fact that, in a finite dimensional Euclidean space, the unit sphere is not a retract of the closed unit ball. After Ważewski's paper, several papers were written applying Ważewski's principle to the investigation of the asymptotic behavior of solutions of ordinary differential equations (e.g., [10]–[3], see [17] as well). In [14, 12] and [4], an extension of Ważewski's principle to differential equations with delay and partial differential equations is given.

Rybakowski [14], following an idea already used by Razumikhin [13], extended Ważewski's principle to delay equations defined on a space of continuous functions $C([-r,0],R^n)$, $r>0$. In addition, Rybakowski considered the Ważewski Theorem in its original form for what is called a system of curves in R^n. Conditions imposed on the boundary ∂W of an open set $W\subset R^n$ are weaker than those imposed by Onuchic [11] and the results can be applied to

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a larger class of delay equations. In [16, 15] a combination of fixed point theorems and retraction properties is applied to initial value problems of singular integro-differential equations.

The reason why many authors thought of the solutions of delayed equations as elements of Euclidean space is that they used the Ważewski Retract Theorem in its original form, which is not applicable in a natural way to infinite dimensions. In these cases Izé [9] used a method based on fixed point index properties instead of retraction properties.

In this paper we deal with a modification of Ważewski's Theorem as applied to initial value problems of integro-differential equations.

2. Preliminaries

Consider the initial value problem

$$x'(t) = f(t, x(t)) + \int_a^t g(t, s, x(s))ds, \quad x(a) = \alpha, \tag{1}$$

where

$$f = (f_1, \dots, f_n)\colon E \to R^n$$

and

$$g = (g_1, \dots, g_n)\colon R^+ \times E \to R^n$$

are continuous, $E := R^+ \times R^n$, $R^+ := [0, \infty)$, $x = (x_1, \dots, x_n)$, $a \in R^+$, $\alpha \in R^n$, and R^n is equipped with the Euclidean norm

$$|x| = \sqrt{\sum_{i=1}^n x_i^2}.$$

In [8, 16, 1, 7] conditions of existence and uniqueness of (1) were established and, moreover, the asymptotic behavior of solutions of (1) was described with respect to a certain open set $W \subset E$.

Now we recall some notions, definitions, and results taken from [1, 2] needed in the following constructions.

Hypothesis (A). For each compact $J \subset R^+$ and each compact $B \subset R^n$, there exist continuous functions

$$K_1\colon J \to R^+, \;\; K_2\colon J \times J \to R^+ ,$$

such that

$$|f(t, \overline{x}) - f(t, \overline{\overline{x}})| \le K_1(t)|\overline{x} - \overline{\overline{x}}|,$$

$$|g(t,s,\overline{x}) - g(t,s,\overline{\overline{x}})| \le K_2(t,s)|\overline{x} - \overline{\overline{x}}|$$

for every $t, s \in J$ and every $\overline{x}, \overline{\overline{x}} \in B$.

Theorem 1. *Let Hypothesis (A) hold. Then there exists a* $\delta > 0$ *such that the initial value problem* (1) *has a unique solution in* $[a, a+\delta]$.

The proof can be done by using Banach's contraction principle in the space of continuous functions $C(J,A)$ $A \subset E$ with the norm of $h \in C(J,A)$ given by

$$\|h(t)\| = \sup_{t \in J} |h(t)| .$$

Let O_1 be the set of points $P = (\tau, x_0) \in E$ such that exactly one solution of the system (1) passes through. Let O_2 be the set of points $P = (\tau, x_0) \in E$ in which more than one solution of the system (1) passes through. We denote by $\phi(t,P)$ the solution passing through P and by $\Delta(P)$ its maximal interval of definition. We put $L(t,P) = (t, \phi(t,P))$, $L(J,P) = \{(t,\phi(t,P)) \colon t \in J\}$, $L(\Delta(P), P) = L(P)$ and $L^+(P) = \{(t,\phi(t,P)) \colon t \ge \tau\}$ when $P = (\tau, x) \in E$. For a given $t \ge 0$, we call $E(t) = \{(t,x) \in E\}$ the section of E by t or, more briefly, the t-section of E.

Denote $\Gamma(P)$ the set of integrals $L(P)$ through P and $\Gamma^+(P)$ the set of integrals $L^+(P)$ through P.

Let $W \subset E$ be an open set and ∂W the boundary of W with respect to E. If $P \in W$, we say that $L(P)$ is asymptotic with respect to W if $L(P) \subset W$.

If $P = (\tau, x) \in W$ and $L(P)$ is not asymptotic with respect to W, then there exists the first point $Q = L(\tau, P)$ on which $L(P)$ reaches the boundary of W, Q is called a *consequent* of P and we write $Q = C(P)$.

Definition 2. The set of all points $P \in W$ for which there exists $C(P)$ is called the left shadow of the system (1) with respect to W and is denoted by G.

Now we show some particularities of integro-differential equations in the sense [6] which occur when solving initial value problems with singular points of the first and second order; i.e. such points that more than one solution passes through them or no solution exists at all.

Consider the following system of integro-differential equations

$$x_1'(t) = x_2(t), \tag{2}$$

$$x_2'(t) = x_1(t) - \int_0^t x_2(s)ds. \tag{3}$$

The general solution of (2), (3) has the form

$$x_1(t) = c_1 \left(\frac{t^2}{2} + 1 \right) + c_2 t,$$
$$x_2(t) = c_1 t + c_2$$

with arbitrary constants c_1, c_2. Now we consider an associated homogeneous system with respect to (2), (3)

$$x_1'(t) = x_2(t), \tag{4}$$
$$x_2'(t) = x_1(t). \tag{5}$$

The fundamental matrix of the solutions of (4), (5) has the form

$$Y_0(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix},$$

and

$$\det Y_0(t) = -2 \neq 0.$$

This is the basic property of systems of ordinary differential equations. The determinant of the fundamental matrix of solutions is either different from zero everywhere, or equals zero in an arbitrary point. But, in the case of system (2), (3), the matrix of linearly independent solutions has the form

$$Y(t) = \begin{pmatrix} \frac{t^2}{2} + 1 & t \\ t & 1 \end{pmatrix},$$

and

$$\det Y(t) = 1 - \frac{t^2}{2}.$$

Thus

$$\det\ Y(t) = 0 \Leftrightarrow t = \sqrt{2} \vee t = -\sqrt{2}.$$

Consider the system (2), (3) with the initial conditions

$$x_1(\sqrt{2}) = A, \quad x_2(\sqrt{2}) = B, \quad A, B \in R.$$

This initial value problem has an infinite number of solutions for $A = \sqrt{2}B$, while for $A \neq \sqrt{2}B$, it has no solution. In contrast, the system (4), (5) has exactly one solution at every point. Obviously, the singular points of (2), (3) are roots of the determinant of the matrix of solutions. Determining the matrix of solutions $Y(t)$ suffices to classify the system (2), (3). Nevertheless, the matrix $Y(t)$ can be obtained under a condition that the system of integro-differential equations is linear with constant coefficients. The problem how to determine singular points in the other cases or how to provide regions with no singular points has not yet been solved.

3. Formulation of the Initial Value Problem

The above results are typical for papers [1, 7, 6], but several important concepts such as the initial value problem for (1), egress and ingress points, are not properly defined. The reason is that the above-mentioned authors, essentially, just repeated the same definitions that Wažewski introduced for ordinary differential equations. But equation (1) has a behavior similar to functional differential equations and, therefore, our presentation here is close to the one given by Onuchic in [11] for differential equations with delay.

Definition 3. Let (τ, α) be a point in E. The initial value problem (i.v.p.) for (1) is to find an interval $I \subset R^+$ and a differentiable function

$$\phi = (\phi_1, \dots, \phi_n) \colon I \to R^n$$

satisfying

$$\phi'(t) = f(t, \phi(t)) + \int_a^t g(t, s, \phi(s)) ds$$

for every $t \in I$ and such that $\phi(\tau) = \alpha$.

In the what follows we shall (without loss of generality) fix $a = 0$.

Let $\mu \colon [0, \tau] \to R^n$ be any, but fixed, continuous function for which $\mu(\tau) = \alpha$. Consider (τ) - i.v.p.

$$y'(t) = F(t, y(t)) + \int_\tau^t G(t, s, y(s)) ds, \quad y(\tau) = \alpha\,,$$

where

$$F(t, y(t)) = f(t, y(t)) + \int_0^\tau g(t, s, \mu(s)) ds$$

and $G(t, s, y(s)) = g(t, s, y(s))$. By Theorem 1 the given (τ)-i.v.p. has a unique solution $\psi(t)$ defined on the interval $[\tau, \tau + \delta]$ for δ sufficiently small.

We put

$$\phi(t) = \begin{cases} \mu(t), & t \in [0, \tau], \\ \psi(t), & t \in [\tau, \tau + \delta]. \end{cases}$$

Then $\phi(t)$ is continuous and

$$\phi'(t) = f(t, \phi(t)) + \int_0^t g(t, s, \phi(s)) ds$$

for $t \in [\tau, \tau + \delta]$. From this we get the following result.

Theorem 4. *Let the Hypothesis (A) hold. If $P = (\tau, \alpha) \in E$, then there exists a solution of the (τ)-i.v.p. for (1) on the interval $[\tau, \tau + \delta]$, where $\delta > 0$ is sufficiently small.*

4. Ważewski's Topological Principle

In this section we assume:

(i) for every point $P \in E$ there exists at least one integral $L(P)$ of (1);

(ii) every integral $L(P)$ of (1) is continuable to the right up to the boundary of E;

(iii) $E(0) \subset O_1$.

By virtue of Section 3 we can introduce the following concepts.

Definition 5. $P = (\tau, x) \in \partial W$ is an egress point of the system (1) with respect to W if there exists at least one integral $L(P)$ of (1) continuable to the left, such that $L([\tau - \delta, \tau), P) \subset W$, where $\delta = \delta(l(P)) > 0$ is sufficiently small.

Definition 6. $P = (\tau, x) \in \partial W$ is a strict egress point of the system (1) with respect to W, if for every integral $L(P)$ of (1), we have $L((\tau, \tau + \delta], P) \subset E - \overline{W}$ and, if $L(P)$ is continuable to the left, then $L([\tau - \delta, \tau), P) \subset W$, where $\delta = \delta(L(P)) > 0$ is sufficiently small.

Let S be the set of egress points and S^* the set of strict egress points. It is easy to see that we have the following relations depending on the character of the system (1):

$$S \cap S^* = \emptyset, \quad S \subset S^*, \quad S^* \subset S.$$

Definition 7. Let A be a topological space and $B \subset A$. We say that B is a retract of A if there exists a continuous mapping $F\colon A \to B$ such that $F(P) = P$ for $P \in B$. In that case, F is called a retraction of A.

Now we recall the known results from topology to be used later.

Lemma 1. *If a set M is a retract of a set N, then M is a retract of a set T such that $M \subset T \subset N$.*

Lemma 2. *If a set M is a retract of a set N, and N is a retract of a set T, then M is a retract of T.*

Theorem 8. (see [2]) *If $S \subset S^*$, then the mapping*

$$K\colon G(0) \cup S \to S\,,$$

defined as $K(P) = C(P)$ *if* $P \in G(0)$ *and* $K(P) = P$ *if* $P \in S$*, is continuous.*

Now we can introduce a modification of Važewski's Theorem for integro-differential equations.

Theorem 9. *Let* $S \subset V \subset S^*$ *and there exist* $Z \subset W(0) \cup V$ *such that:*

1) $Z \cap V$ *is a retract of* V*,*

2) $Z \cap V$ *is not a retract of* Z*.*

Then there exists at least one point $P \in Z \setminus V$ *for which the integral* $L(P)$ *is asymptotic with respect to* W*.*

Proof. Suppose that, for each point $P \in Z \setminus V$, there exists at least one consequent $C(P) \in V$. We can define the mapping K on $Z \cup V$ in the following way:

$$K(P) = \begin{cases} C(P) \in V & \text{if} \quad P \in Z \setminus V, \\ K(P) = P & \text{if} \quad P \in V. \end{cases}$$

From Theorem 8 it follows that K is continuous on the set $Z \cup V$. Since $V \subset Z \cup V$, then V is a retract $Z \cup V$ and, according to Lemma 1, $Z \cap V$ is a retract of $Z \cup V$. It is obvious that

$$Z \cap V \subset V \subset Z \cup V.$$

Now, from Lemma 2 we get that $Z \cap V$ is a retract Z, which is contrary to condition 2. □

Remark 10. In the classical theory of Važewski's topological method, the condition $S = S^*$ is used, but in the case of integro-differential equations, the condition $S \subset S^*$ is fundamental.

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