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INEQUALITIES FOR THE JACOBI POLYNOMIALS AND THEIR DERIVATIVES

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Abstract: Some inequalities for the Jacobi polynomials and their derivatives are presented in this paper. An upper bound is also given for rational terms containing the Jacobi polynomials and the first derivative $P_n^{(\alpha,\beta)\prime}(x)$, $\alpha,\beta > -1$ at the roots $x_{kn} = \cos(\theta_{kn})$, $k = 1, 2, \dots, n$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$.

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1. Introduction

The orthogonal polynomials are strongly connected with other branches of mathematics. This connection is specially impressive with moment problems, continued fractions, operator theory, interpolation and approximation, quadrature, special functions, and numerical analysis. For further background on orthogonal polynomials, the reader is invited to see the books [3, 4, 6, 8, 10].

The most important class of orthogonal polynomials is the Jacobi polynomials. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha,\beta > -1$ are orthogonal with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta, \alpha, \beta > -1$ on the interval $[-1, 1]$. The leading coefficient of the Jacobi polynomial plays a special and important role, many properties depend on their behavior. The normalization of $P_n^{(\alpha,\beta)}(x)$ is effected by

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$$P_n^{(\alpha,\beta)}(1) = \left(\begin{array}{c} n+\alpha \\ n \end{array} \right).$$

In this paper, we are interested in finding an upper bound for the following rational term

$$\left| \frac{P_n^{\alpha,\beta}(\cos\theta)}{\sin\theta_{kn} P_n^{(\alpha,\beta)'}(\cos\theta_{kn})} \right|, \tag{1}$$

where $P_n^{(\alpha,\beta)}(\cos\theta)$ is the Jacobi polynomial, and $P_n^{(\alpha,\beta)'}(\cos\theta_{kn})$ is the first derivative of $P_n^{\alpha,\beta}(\cos\theta)$ evaluated at the roots $x_{kn} = \cos(\theta_{kn}), \quad k = 1, 2, \dots, n$ of the Jacobi polynomial. There are many interesting properties for the Jacobi polynomials, and many applications in several fields of science. For more about the Jacobi polynomials, see [1, 2, 7, 9, 10].

2. Preliminaries

The Jacobi polynomials have very interesting properties. These properties of the Jacobi polynomials are powerful tools to study their behavior.

The first derivative of $P_n^{(\alpha,\beta)}(x)$ is a constant multiple of $P_{n-1}^{(\alpha+1,\beta+1)}(x)$, and in general the k-th derivative of $P_n^{(\alpha,\beta)}(x)$ is a constant multiple of $P_{n-k}^{(\alpha+k,\beta+k)}(x)$ and it is given by the following formula

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+n+1+k)}{2^k \Gamma(\alpha+\beta+n+1)} P_{n-k}^{(\alpha+k,\beta+k)}(x).$$

We also have the following formula

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

If $\alpha = \beta$ then $P_n^{(\alpha,\alpha)}(x)$ is called the ultraspherical polynomial. From the last formula it follows that the ultraspherical polynomial $P_n^{(\alpha,\alpha)}(x)$ is odd if n is odd, and it is even if n is even.

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x), \quad \alpha, \beta > -1$ possess the following recurrence relation

$$\begin{aligned} 2n(\alpha+\beta+n)(\alpha+\beta+2n-2) P_n^{(\alpha,\beta)}(x) &= (\alpha+\beta+2n-1)[\alpha^2 - \beta^2 \\ &\quad + x(\alpha+\beta+2n)(\alpha+\beta+2n-2)] P_{n-1}^{(\alpha,\beta)}(x) \\ &\quad - 2(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n) P_{n-2}^{(\alpha,\beta)}(x). \end{aligned}$$

The Jacobi polynomial has the following Rodrigue's type formula

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}(\frac{d}{dx})^n\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}.$$

The Jacobi polynomial $y=P_n^{(\alpha,\beta)}(x)$ satisfies the following second order differential equation

$$(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y=0.$$

The orthonormal Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ on $[-1,1]$ has the following formula

$$P_n^{(\alpha,\beta)}(x)=\{h_n^{(\alpha,\beta)}\}^{\frac{1}{2}}\sum_{\nu=0}^{n}\left(\begin{array}{c} n+\alpha \\ n-\nu \end{array}\right)\left(\begin{array}{c} n+\beta \\ \nu \end{array}\right)(\frac{x-1}{2})^{\nu}(\frac{x+1}{2})^{n-\nu}, \tag{2}$$

where

$$h_n^{(\alpha,\beta)}=\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}.$$

If $n=0$, then

$$h_0^{(\alpha,\beta)}=\frac{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}=\frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)},$$

where $B(p,q)$ is the beta function.

The roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x),\quad \alpha,\beta>-1$ are all real, simple and belong to the interior of the interval $[-1,1]$. The roots of $P_n^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta)}(x)$ interlace, this means that in between any two roots of $P_{n+1}^{(\alpha,\beta)}(x)$ there is a root of $P_n^{(\alpha,\beta)}(x)$. Moreover $P_n^{(\alpha,\beta)}(x)$ changes its sign n times. In case it had only $m<n$ roots at the points $x_1,\ldots,x_m$ then it is not orthogonal to the polynomial $\prod_{i=1}^m(x-x_i)$ of degree $m<n$.

In discussing the roots $x_{kn}=\cos(\theta_{kn}),\quad k=1,2,\ldots,n$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ we use the enumeration:

$$0<\theta_{1n}<\theta_{2n}<\cdots<\theta_{nn}<\pi,$$

and

$$1>x_{1n}>x_{2n}>\cdots>x_{nn}>-1.$$

3. Upper Bound

Let $x_{kn} = \cos\theta_{kn}, \quad k = 1, 2, \ldots, n$ be the roots of the Jacobi polynomial, $P_n^{(\alpha,\beta)}(x)$, and $x \in [-1, 1]$, and $|\alpha| \le \frac{1}{2}, |\beta| \le \frac{1}{2}$. As in [5], we set

$$U_n^{(\alpha,\beta)}(\theta) = (\sin(\frac{\theta}{2}))^{\alpha+\frac{1}{2}}(\cos(\frac{\theta}{2}))^{\beta+\frac{1}{2}} P_n^{(\alpha,\beta)}(\cos\theta), \quad 0 < \theta < \pi. \tag{3}$$

Then there exist a positive constant $c_1(\alpha, \beta)$ depends on α, β such that the following inequality holds:

$$|U_n^{(\alpha,\beta)}(\theta)| \le \frac{c_1(\alpha, \beta)}{\sqrt{n}}, \quad n = 1, 2, \ldots \tag{4}$$

And for $0 < \theta_{kn} < \pi$ there exist positive constants $c_2(\alpha, \beta), n_0(\alpha, \beta)$ dependent on α, β such that the following inequality holds:

$$|U_n^{(\alpha,\beta)'}(\theta_{kn})| \ge c_2(\alpha, \beta)\sqrt{n}, \quad \forall n \ge n_0(\alpha, \beta). \tag{5}$$

Thus we have

$$\left| \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{\sin\theta_{kn} P_n^{(\alpha,\beta)'}(\cos\theta_{kn})} \right| = \left| \frac{\sqrt{\sin\theta_{kn}}}{\sqrt{\sin\theta}} \frac{U_n^{(\alpha,\beta)}(\cos\theta)}{U_n^{(\alpha,\beta)'}(\cos\theta_{kn})} \right|. \tag{6}$$

This is simplified to

$$\left| \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{\sin\theta_{kn} P_n^{(\alpha,\beta)'}(\cos\theta_{kn})} \right| = \frac{\sqrt{\sin\theta_{kn}}}{\sin\theta} \left| \frac{U_n^{(\alpha,\beta)}(\cos\theta)}{U_n^{(\alpha,\beta)'}(\cos\theta_{kn})} \right|. \tag{7}$$

And thus, we have

$$\left| \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{\sin\theta_{kn} P_n^{(\alpha,\beta)'}(\cos\theta_{kn})} \right| \le \frac{\sqrt{\sin\theta_{kn}}}{\sqrt{\sin\theta}} \frac{c_1(\alpha, \beta)}{\sqrt{n}} \frac{c_2(\alpha, \beta)}{\sqrt{n}} \tag{8}$$

$$\le \frac{c(\theta, \alpha, \beta)}{n}. \tag{9}$$

This discussion leads to the proof of the following theorem.

Theorem. *Let $x \in [-1, 1]$ and $x_{kn} = \cos\theta_{kn}$ be the roots of the Jacobi polynomial, $P_n^{(\alpha,\beta)}(x)$, then for $|\alpha| \le \frac{1}{2}, |\beta| \le \frac{1}{2}$, we have*

$$\left| \frac{P_n^{\alpha,\beta}(\cos\theta)}{\sin\theta_{kn} P_n^{(\alpha,\beta)'}(\cos\theta_{kn})} \right| \le \frac{c(\theta, \alpha, \beta)}{n}, \tag{10}$$

where $c(\theta, \alpha, \beta)$ is a positive constant which depends on θ, α, and β.

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